

2. The Theorem of Shephard and Todd

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **24 (1978)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **27.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Since

$$\frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k}$$

is homogeneous of degree $d_i - 1$, we conclude from Lemma 2.1 that

$$(2.2) \quad \frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k} = \sum_{j=1}^r B_j I_j, \quad 1 \leq i \leq s,$$

where the B_j 's are homogeneous and each term in (2.2) is homogeneous of degree $d_i - 1$. This forces $B_i = 0$. Multiply both sides of (2.2) by x_k and sum over k . We conclude, by Euler's identity for homogeneous polynomials,

$$(2.3) \quad d_i I_i + \sum_{l=1}^{r-s} V_{li} d_{s+l} I_{s+l} = \sum_{j=1}^r A_j I_j,$$

the A_j 's being homogeneous with $A_i = 0$.

(2.3) shows that $I_i \in (I_1, \dots, I_{i-1}, I_{i+1}, \dots, I_r)$, contradicting the minimality of the basis I_1, \dots, I_r . Hence I_1, \dots, I_r are algebraically independent and $r = n$.

2. THE THEOREM OF SHEPHARD AND TODD

We obtain in this section a converse to Chevalley's Theorem, thereby obtaining an invariant theoretical characterization of finite reflection groups. We first prove several preliminary results.

LEMMA 2.2. Let H be a finite group of linear transformations acting on the n -dimensional space V and fixing the $n - 1$ dimensional hyperplane π . The elements of H have a common eigenvector $v \in V - \pi$. Let $\sigma(v) = \zeta(\sigma)v$, $\sigma \in H$. $\zeta(\sigma)$ is an isomorphism from H into the multiplicative group of the roots of unity in k . It follows that H is a cyclic group.

REMARK. The above lemma is a consequence of Maschke's Theorem proven in section 2.3. We provide another proof below.

Proof. Let $\sigma_1 \in H$, $\sigma_1 \neq e$ (the identity of H). By the remark following Definition 2.1, there exists $v \in V - \pi$ such that $\sigma_1(v) = \zeta_1 v$, ζ_1 being a root of unity $\neq 1$. For $\sigma \in H$, let $\sigma(v) = \zeta(\sigma)v + p(\sigma)$, $\zeta(\sigma) \in k$ and $p(\sigma) \in \pi$. Let $\sigma^* = \sigma_1^{-1} \sigma^{-1} \sigma_1 \sigma$. Then $\sigma^*(v) = v + (1 - \zeta_1)p(\sigma)$. Since σ^* is of finite order, $(1 - \zeta_1)p(\sigma) = 0 \Rightarrow p(\sigma) = 0$. Hence $\sigma(v) = \zeta(\sigma)v$. $\zeta(\sigma)$ is clearly an isomorphism from H into U , the multiplicative group of

the roots of unity in k . U is known to be cyclic ([22], Vol. 1, p. 112). It follows that $\zeta(H)$, a subgroup of U , is cyclic and so H is cyclic.

THEOREM 2.2. *Let G be a finite group acting on the n -dimensional space V . Let I_1, \dots, I_n be homogeneous polynomials forming a basis for the invariants of G . Let d_1, \dots, d_n be the respective degrees of I_1, \dots, I_n . Then*

$$(2.4) \quad \prod_{i=1}^n d_i = |G|, \quad \sum_{i=1}^n (d_i - 1) = r$$

where $r = \text{number of reflections in } G$.

Proof. By Theorem 1.2, I_1, \dots, I_n are algebraically independent. Let $I(x)$ be a homogeneous invariant of degree m . Then I is a linear combination of the monomials $I_1^{a_1} \dots I_n^{a_n}$ where $a_1 d_1 + \dots + a_n d_n = m$. Furthermore, these monomials are linearly independent over k , as I_1, \dots, I_n are algebraically independent over k . It follows that the dimension δ_m of homogeneous invariants of degree m = number of non-negative integer solutions to $a_1 d_1 + \dots + a_n d_n = m$. Hence

$$(2.5) \quad \sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{(1-t^{d_1}) \dots (1-t^{d_n})}.$$

(1.9) and (2.5) yield

$$(2.6) \quad \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1-\omega_1(\sigma)t) \dots (1-\omega_n(\sigma)t)} = \frac{1}{(1-t^{d_1}) \dots (1-t^{d_n})}$$

Expand both sides of (2.6) in powers of $(1-t)$. Let \mathcal{R} = set of reflections in G and $\zeta(\sigma)$ = eigenvalue of the reflection σ which $\neq 1$. We have

$$(2.7) \quad \begin{aligned} & \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1-\omega_1(\sigma)t) \dots (1-\omega_n(\sigma)t)} \\ &= \frac{1}{|G|} \frac{1}{(1-t)^n} + \frac{1}{|G|} \sum_{\sigma \in \mathcal{R}} \frac{1}{1-\zeta(\sigma)} \frac{1}{(1-t)^{n-1}} + \dots \end{aligned}$$

$$(2.8) \quad \begin{aligned} & \frac{1}{(1-t^{d_1}) \dots (1-t^{d_n})} = \prod_{i=1}^n \frac{1}{d_i(1-t) - \binom{d_i}{2}(1-t)^2 + \dots \pm (1-t)^{d_i}} \\ &= \frac{1}{\prod_{i=1}^n d_i (1-t)^n} + \frac{\frac{1}{2} \sum_{i=1}^n (d_i - 1)}{\prod_{i=1}^n d_i} \frac{1}{(1-t)^{n-1}} + \dots \end{aligned}$$

Equating coefficients of (2.7), (2.8), we get

$$(2.9) \quad \prod_{i=1}^n d_i = |G|, \quad \sum_{i=1}^n (d_i - 1) = 2 \sum_{\sigma \in \mathcal{R}} \frac{1}{1 - \zeta(\sigma)}.$$

We evaluate the sum

$$\sum_{\sigma \in \mathcal{R}} \frac{1}{1 - \zeta(\sigma)} :$$

Let π be any r.h. Let $H_\pi = \{\sigma \mid \sigma \in G \text{ and } \sigma \text{ fixes } \pi\}$. Thus H_π is the subgroup of G consisting of the identity and those reflections in G with r.h. π . Applying Lemma 2.2 to H_π , we conclude that there exists $v \notin \pi$ such that $\sigma(v) = \zeta(\sigma)v$ for $\sigma \in H_\pi$. Let $H'_\pi = H_\pi - \{e\}$. Since $\zeta(\sigma^{-1}) = (\zeta(\sigma))^{-1}$, we obtain

$$(2.10) \quad \begin{aligned} \sum_{\sigma \in H'_\pi} \frac{1}{1 - \zeta(\sigma)} &= \sum_{\sigma \in H'_\pi} \frac{1}{1 - \zeta(\sigma^{-1})} \\ &= \sum_{\sigma \in H'_\pi} \left(1 - \frac{1}{1 - \zeta(\sigma)}\right) = |H'_\pi| - \sum_{\sigma \in H'_\pi} \frac{1}{1 - \zeta(\sigma)}. \end{aligned}$$

Hence

$$(2.11) \quad \sum_{\sigma \in H'_\pi} \frac{1}{1 - \zeta(\sigma)} = \frac{|H'_\pi|}{2}.$$

Summing both sides of (2.11) over all r.h. π , we get

$$(2.12) \quad \sum_{\sigma \in \mathcal{R}} \frac{1}{1 - \zeta(\sigma)} = \frac{r}{2}.$$

(2.9), (2.12) yield Theorem 2.2.

THEOREM 2.3. *Let f_1, \dots, f_n be polynomials in the variables x_1, \dots, x_n . f_1, \dots, f_n are algebraically independent over $k \Leftrightarrow$*

$$\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \neq 0.$$

Proof. Suppose that f_1, \dots, f_n are algebraically independent. Then $G(f_1, \dots, f_n) = 0$ for some polynomial $G = G(y_1, \dots, y_n)$. Assume that $G(y_1, \dots, y_n)$ is of minimal positive degree. Differentiating this relation with respect to x_j , we get

$$(2.13) \quad \sum_{i=1}^n \frac{\partial G}{\partial y_i} (f_1, \dots, f_n) \frac{\partial f_i}{\partial x_j} = 0, \quad 1 \leq j \leq n.$$

(2.13) is a system of linear equations (with coefficients in $k(x_1, \dots, x_n)$) in the unknowns $H_i(x) = \frac{\partial G}{\partial y_i}(f_1, \dots, f_n)$, $1 \leq i \leq n$. $\frac{\partial G}{\partial y_i} \neq 0$ for some i , as G is not constant, and $\deg \frac{\partial G}{\partial y_i} < \deg G$. It follows that the corresponding $H_i(x) \neq 0$. Thus the linear system (2.13) has a non-zero solution, so that its determinant

$$\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \neq 0.$$

Conversely, let f_1, \dots, f_n be algebraically independent. For each i , x_i, f_1, \dots, f_n are algebraically dependent. Hence there exists a polynomial $G_i(x_i, y_1, \dots, y_n)$ of minimal positive degree in x_i such that $G_i(x_i, f_1, \dots, f_n) = 0$. Differentiating these relations with respect to x_k , we get

$$(2.14) \quad \begin{aligned} & \sum_{j=1}^n \frac{\partial G_i}{\partial y_j}(x_i, f_1, \dots, f_n) \frac{\partial f_j}{\partial x_k} \\ & + \frac{\partial G_i}{\partial x_k}(x_i, f_1, \dots, f_n) \delta_{ik}, \quad 1 \leq k \leq n, \end{aligned}$$

δ_{ik} denoting the Kronecker symbol. (2.14) may be rewritten in matrix notation as

$$(2.15) \quad \left(\frac{\partial G_i}{\partial y_j} \right) \cdot \left(\frac{\partial f_i}{\partial x_j} \right) = D$$

where the entries of D are

$$- \delta_{ij} \frac{\partial G_i}{\partial x_j}.$$

$\det D \neq 0$, as $x_i - \text{degree of } \frac{\partial G_i}{\partial x_i} < x_i - \text{degree of } G_i$, $1 \leq i \leq n$.

It follows from (2.15) that $\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \neq 0$.

THEOREM 2.4. (Shephard and Todd [19]). *Let G be a finite group acting on the n -dimensional space V . Suppose there exists a basis of n homogeneous polynomials for the invariants of G . Then G is a finite reflection group.*

Proof. Let H be the subgroup of G generated by the reflections in G . By assumption G has n basic homogeneous invariants which, by Theorem 1.2, are algebraically independent. Since H is a finite reflection group, we conclude from Chevalley's Theorem that H has n basic homogeneous invariants J_1, \dots, J_n which are algebraically independent. Each I_i is invariant under H so that $I_i = I_i(J_1, \dots, J_n)$, the latter quantity denoting a polynomial in the J_i 's. We may assume that $I_i(J_1, \dots, J_n)$ is a linear combination of monomials $J_1^{a_1} \dots J_n^{a_n}$ whose x -degree = $\deg I_i$. We have

$$(2.16) \quad \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)} = \frac{\partial (I_1, \dots, I_n)}{\partial (J_1, \dots, J_n)} \cdot \frac{\partial (J_1, \dots, J_n)}{\partial (x_1, \dots, x_n)}$$

By Theorem 2.3,

$$\frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)} \neq 0$$

and (2.16) then shows that

$$\frac{\partial (I_1, \dots, I_n)}{\partial (J_1, \dots, J_n)} \neq 0.$$

It follows that there is a rearrangement k_1, \dots, k_n of $1, \dots, n$ so that

$$\frac{\partial I_{k_1}}{\partial J_1} \dots \frac{\partial I_{k_n}}{\partial J_n} \neq 0.$$

Hence $I_{k_i}(J_1, \dots, J_n)$ is of positive degree in J_i and $\deg I_{k_i} \geq \deg J_i$, $1 \leq i \leq n$. Applying Theorem 2.2 both to G and H , we obtain

$$(2.17) \quad \prod_{i=1}^n \deg J_i = |H|, \quad \prod_{i=1}^n \deg I_i = |G|$$

$$(2.18) \quad \sum_{i=1}^n (\deg J_i - 1) = \sum_{i=1}^n (\deg I_i - 1) = r$$

where r = number of reflections in G = number of reflections in H .

Since $\deg I_{k_i} \geq \deg J_i$, $1 \leq i \leq n$, we conclude from (2.18) that $\deg I_{k_i} = \deg J_i$, $1 \leq i \leq n$. Hence $\prod_{i=1}^n \deg I_i = \prod_{i=1}^n \deg J_i$, and we conclude from (2.17) that $|G| = |H|$. Thus $G = H$ and G is a finite reflection group.