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(1.10) and Lemma 1.2 yield

(1.11)
$$\delta_m = \frac{1}{|G|} \sum_{\sigma \in G} (Tr \sigma)_m = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma).$$

Multiply both sides of (1.11) by t^m and sum over m from 0 to ∞ . We get

$$\sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{\sigma \in G} \sum_{\substack{|a|=m \\ |a|=m}} \omega^a(\sigma) t^m$$
$$= \frac{1}{|G|} \sum_{\sigma \in G} \left\{ \sum_{m=0}^{\infty} \omega_1^m(\sigma) t^m \dots \sum_{m=0}^{\infty} \omega_n^m(\sigma) t^m \right\}$$
$$= \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1 - \omega_1(\sigma) t) \dots (1 - \omega_n(\sigma) t)}$$

CHAPTER II

INVARIANT THEORETIC CHARACTERIZATION OF FINITE REFLECTION GROUPS

1. CHEVALLEY'S THEOREM

We showed in chapter I that we can always find a finite number of homogeneous invariants forming a basis for the invariants of G and that this set must contain at least n elements, where $n = \dim V$. We show that this lower bound is attained only for the finite reflection groups. We first define these groups.

DEFINITION 2.1. Let σ be a linear transformation acting on the *n*-dimensional vector space V. σ is a reflection $\Leftrightarrow \sigma$ fixes an n - 1 dimensional hyperplane π and σ is of finite order > 1. π is called the reflecting hyperplane (r.h.) of σ .

REMARK. Choose $v \notin \pi$. and let $\sigma v = \zeta v + p$, $p \in \pi$. If $\zeta = 1$, then $\sigma^m v = v + mp$, contradicting that σ is of finite order. Hence $\zeta \neq 1$. Let $v' = v + (\zeta - 1)^{-1} p$ and choose $p_1, ..., p_{n-1}$ as a basis for π . Then $\sigma p_i = p_i, 1 \leq i \leq n-1, \sigma v' = \zeta v'$. ζ is a root of 1 in k which is distinct from 1, as σ is of finite order > 1. Thus σ is a reflection iff relative to some basis, the matrix for σ is diagonal, n - 1 of the diagonal entries equalling 1 and the remaining one equalling a root of 1 in k distinct from 1. DEFINITION 2.2. G is a finite reflection group acting on $V \Leftrightarrow G$ is a finite group generated by reflections on V.

As an example of a finite reflection group, let $G = S_n$. It is well known that S_n is generated by transpositions. The transposition of the variables $x_i, x_j \ (i \neq j)$ fixes the hyperplane $x_i - x_j = 0$, so that it is a reflection.

We have the following result

THEOREM 2.1 (Chevalley [4]). Let G be a finite reflection group acting on the n-dimensional vector space V. The invariants of G have a basis consisting of n homogeneous elements which are algebraically independent over k.

Let k [x] denote the ring of polynomials in $x_i, ..., x_n$ with coefficients in k. We prove the following.

LEMMA 2.1. Let $I_1, ..., I_m$ be invariant polynomials of $G, I_1 \notin (I_2, ..., I_m)$ = the ideal in k [x] generated by $I_2, ..., I_m$. Suppose that $P_1 I_1 + ...$ + $P_m I_m = 0$, the P_i 's being polynomials with P_1 homogeneous. Then $P_1 \in \mathcal{I}$, where \mathcal{I} is the ideal in k [x] generated by the homogeneous invariants of positive degree.

Proof of Lemma 2.1. The proof proceeds by induction on deg P_1 . Suppose deg $P_1 = 0$, so that $P_1 = c \in k$. If $c \neq 0$, then $I_1 \in (I_2, ..., I_m)$, contrary to assumption. Hence $c = 0 \Rightarrow P_1 \in \mathscr{I}$. Let deg $P_1 = n > 0$. Let σ be a reflection in G and L = 0 the equation of its r.h. (L is a linear homogeneous polynomial). We have $P_1(x) I_1(x) + ... + P_m(x) I_m(x) = 0$, $P_1(\sigma x) I_1(x) + ... + P_m(\sigma x) I_m(x) = 0$. Hence $[P_1(\sigma x) - P_1(x)] I_1(x) + ... + [P_m(\sigma x) - P_m(x)] I_m(x)$. For L(x) = 0, $\sigma(x) = x$, so that $P_i(\sigma x) - P_i(x) = 0$ whenever L(x) = 0, $1 \leq i \leq m$. Since L(x) is irreducible it follows that

$$\frac{P_i\left(\sigma x\right) - P_i\left(x\right)}{L(x)}$$

is a polynomial, $1 \leq i \leq m$. We have

$$\begin{bmatrix} \frac{P_1(\sigma x) - P_1(x)}{L(x)} \end{bmatrix} I_1(x) + \dots + \begin{bmatrix} \frac{P_m(\sigma x) - P_m(x)}{L(x)} \end{bmatrix} I_m(x) = 0.$$
$$\deg \begin{bmatrix} \frac{P_1(\sigma x) - P_1(x)}{L(x)} \end{bmatrix} < \deg P_1(x) ,$$

so that by the induction hypothesis

$$\frac{P_1(\sigma x) - P_1(x)}{L(x)} \equiv 0 \pmod{\mathscr{I}}$$

Hence $P_1(\sigma x) \equiv P_1(x) \pmod{\mathscr{I}}$. Since the σ 's generate G, this congruence holds for $\sigma \in G$. We conclude that

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$$P_1(x) \equiv \frac{1}{\mid G \mid} \sum_{\sigma \in G} P_1(\sigma x) \pmod{\mathscr{I}}.$$

The polynomial $\frac{1}{|G|} \sum_{\sigma \in G} P_1(\sigma x)$ is invariant and homogeneous of degree $n \ge 1$. Hence it $\in \mathcal{I}$, so that $P_1 \in \mathcal{I}$.

Proof of Theorem 2.1. We choose $I_1, ..., I_r$ to be homogeneous invariants of positive degree forming a minimal basis for \mathscr{I} . Hilbert's proof of Theorem 1.1 shows that $I_1, ..., I_r$ form a basis for the invariants of G. We show that $I_1, ..., I_r$ are algebraically independent, so that r = n.

Suppose, to the contrary, that $I_1, ..., I_r$ are algebraically dependent. Choose $H(y_1, ..., y_r)$ to be a polynomial of minimal positive degree so that $H(I_1(x), ..., I_r(x)) = 0$. Let x-degree of any monomial $y_1^{a_1} ... y_r^{a_r}$ be $d_1 a_1 + ... + d_r a_r$, where $d_i = \deg I_i$. We may assume that all x-degrees of the monomials appearing in H are the same. Let

$$H_i(x) = \frac{\partial H}{\partial y_i} \left(I_1(x), \dots, I_r(x) \right), \ 1 \leq i \leq r.$$

The H_i 's are invariant homogeneous polynomials, as all monomials in H have equal x-degree. Since $H(y_1, ..., y_n)$ is of positive degree, some $\frac{\partial H}{\partial y_i} \neq 0$, It follows that the corresponding $H_i(x) \neq 0$, as H was chosen to be of minimal degree; i.e. not all H_i 's = 0. We relabel indices so that $H_1, ..., H_s, 1 \leq s \leq r$, are ideally independent (i.e. none of the H_i 's is in the ideal generated by the others) and $H_{s+j} \in (H_1, ..., H_s)$. $1 \leq j \leq r - s$. Thus $H_{s+j} = \sum_{i=1}^{s} V_{ji} H_i$, $1 \leq j \leq r - s$, where each V_{ji} is a homogeneous polynomial of degree $d_i - d_{s+j} (V_{ji}$ is interpreted to be 0 if this degree is negative). Differentiating the relation $H(I_1(x), ..., I_r(x)) = 0$ with respect to x_k , we obtain

(2.1)
$$\sum_{i=1}^{r} H_{i} \frac{\partial I_{i}}{\partial x_{k}} = \sum_{i=1}^{s} H_{i} \frac{\partial I_{i}}{\partial x_{k}} + \sum_{l=1}^{r-s} H_{s+l} \frac{\partial I_{s+l}}{\partial x_{k}}$$
$$= \sum_{i=1}^{s} H_{i} \left[\frac{\partial I_{i}}{\partial x_{k}} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_{k}} \right] = 0.$$

Since

$$\frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k}$$

is homogeneous of degree $d_i - 1$, we conclude from Lemma 2.1 that

(2.2)
$$\frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k} = \sum_{j=1}^r B_j I_j, \ 1 \leq i \leq s ,$$

where the B_j 's are homogeneous and each term in (2.2) is homogeneous of degree $d_i - 1$. This forces $B_i = 0$. Multiply both sides of (2.2) by x_k and sum over k. We conclude, by Euler's identity for homogeneous polynomials,

(2.3)
$$d_i I_i + \sum_{l=1}^{r-s} V_{li} d_{s+l} I_{s+l} = \sum_{j=1}^r A_j I_j$$

the A_i 's being homogeneous with $A_i = 0$.

(2.3) shows that $I_i \in (I_1, ..., I_{i-1}, I_{i+1}, ..., I_r)$, contradicting the minimality of the basis $I_1, ..., I_r$. Hence $I_1, ..., I_r$ are algebraically independent and r = n.

2. The Theorem of Shephard and Todd

We obtain in this section a converse to Chevalley's Theorem, thereby obtaining an invariant theoretical characterization of finite reflection groups. We first prove several preliminary results.

LEMMA 2.2. Let *H* be a finite group of linear transformations acting on the *n*-dimensional space *V* and fixing the n - 1 dimensional hyperplane π . The elements of *H* have a common eigenvector $v \in V - \pi$. Let $\sigma(v) = \zeta(\sigma) v, \sigma \in H$. $\zeta(\sigma)$ is an isomorphism from *H* into the multiplicative group of the roots of unity in *k*. It follows that *H* is a cyclic group.

REMARK. The above lemma is a consequence of Maschke's Theorem proven in section 2.3. We provide another proof below.

Proof. Let $\sigma_1 \in H$, $\sigma_1 \neq e$ (the identity of H). By the remark following Definition 2.1, there exists $v \in V - \pi$ such that $\sigma_1(v) = \zeta_1 v$, ζ_1 being a root of unity $\neq 1$. For $\sigma \in H$, let $\sigma(v) = \zeta(\sigma)v + p(\sigma)$, $\zeta(\sigma) \in k$ and $p(\sigma) \in \pi$. Let $\sigma^* = \sigma_1^{-1} \sigma^{-1} \sigma_1 \sigma$. Then $\sigma^*(v) = v + (1 - \zeta_1) p(\sigma)$. Since σ^* is of finite order, $(1 - \zeta_1) p(\sigma) = 0 \Rightarrow p(\sigma) = 0$. Hence $\sigma(v) = \zeta(\sigma) v$. $\zeta(\sigma)$ is clearly an isomorphism from H into U, the multiplicative group of