

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 24 (1978)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** INVARIANTS OF FINITE REFLECTION GROUPS  
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**Kapitel:** 1. Chevalley's Theorem  
**DOI:** <https://doi.org/10.5169/seals-49704>

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(1.10) and Lemma 1.2 yield

$$(1.11) \quad \delta_m = \frac{1}{|G|} \sum_{\sigma \in G} (\text{Tr } \sigma)_m = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma).$$

Multiply both sides of (1.11) by  $t^m$  and sum over  $m$  from 0 to  $\infty$ . We get

$$\begin{aligned} \sum_{m=0}^{\infty} \delta_m t^m &= \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma) t^m \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \left\{ \sum_{m=0}^{\infty} \omega_1^m(\sigma) t^m \dots \sum_{m=0}^{\infty} \omega_n^m(\sigma) t^m \right\} \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1 - \omega_1(\sigma) t) \dots (1 - \omega_n(\sigma) t)} \end{aligned}$$

## CHAPTER II

### INVARIANT THEORETIC CHARACTERIZATION OF FINITE REFLECTION GROUPS

#### 1. CHEVALLEY'S THEOREM

We showed in chapter I that we can always find a finite number of homogeneous invariants forming a basis for the invariants of  $G$  and that this set must contain at least  $n$  elements, where  $n = \dim V$ . We show that this lower bound is attained only for the finite reflection groups. We first define these groups.

**DEFINITION 2.1.** Let  $\sigma$  be a linear transformation acting on the  $n$ -dimensional vector space  $V$ .  $\sigma$  is a reflection  $\Leftrightarrow \sigma$  fixes an  $n - 1$  dimensional hyperplane  $\pi$  and  $\sigma$  is of finite order  $> 1$ .  $\pi$  is called the reflecting hyperplane (r.h.) of  $\sigma$ .

**REMARK.** Choose  $v \notin \pi$ . and let  $\sigma v = \zeta v + p$ ,  $p \in \pi$ . If  $\zeta = 1$ , then  $\sigma^m v = v + mp$ , contradicting that  $\sigma$  is of finite order. Hence  $\zeta \neq 1$ . Let  $v' = v + (\zeta - 1)^{-1} p$  and choose  $p_1, \dots, p_{n-1}$  as a basis for  $\pi$ . Then  $\sigma p_i = p_i$ ,  $1 \leq i \leq n - 1$ ,  $\sigma v' = \zeta v'$ .  $\zeta$  is a root of 1 in  $k$  which is distinct from 1, as  $\sigma$  is of finite order  $> 1$ . Thus  $\sigma$  is a reflection iff relative to some basis, the matrix for  $\sigma$  is diagonal,  $n - 1$  of the diagonal entries equalling 1 and the remaining one equalling a root of 1 in  $k$  distinct from 1.

DEFINITION 2.2.  $G$  is a finite reflection group acting on  $V \Leftrightarrow G$  is a finite group generated by reflections on  $V$ .

As an example of a finite reflection group, let  $G = S_n$ . It is well known that  $S_n$  is generated by transpositions. The transposition of the variables  $x_i, x_j (i \neq j)$  fixes the hyperplane  $x_i - x_j = 0$ , so that it is a reflection.

We have the following result

THEOREM 2.1 (Chevalley [4]). *Let  $G$  be a finite reflection group acting on the  $n$ -dimensional vector space  $V$ . The invariants of  $G$  have a basis consisting of  $n$  homogeneous elements which are algebraically independent over  $k$ .*

Let  $k[x]$  denote the ring of polynomials in  $x_1, \dots, x_n$  with coefficients in  $k$ . We prove the following.

LEMMA 2.1. Let  $I_1, \dots, I_m$  be invariant polynomials of  $G$ ,  $I_1 \notin (I_2, \dots, I_m)$  = the ideal in  $k[x]$  generated by  $I_2, \dots, I_m$ . Suppose that  $P_1 I_1 + \dots + P_m I_m = 0$ , the  $P_i$ 's being polynomials with  $P_1$  homogeneous. Then  $P_1 \in \mathcal{I}$ , where  $\mathcal{I}$  is the ideal in  $k[x]$  generated by the homogeneous invariants of positive degree.

*Proof of Lemma 2.1.* The proof proceeds by induction on  $\deg P_1$ . Suppose  $\deg P_1 = 0$ , so that  $P_1 = c \in k$ . If  $c \neq 0$ , then  $I_1 \in (I_2, \dots, I_m)$ , contrary to assumption. Hence  $c = 0 \Rightarrow P_1 \in \mathcal{I}$ . Let  $\deg P_1 = n > 0$ . Let  $\sigma$  be a reflection in  $G$  and  $L = 0$  the equation of its r.h. ( $L$  is a linear homogeneous polynomial). We have  $P_1(x) I_1(x) + \dots + P_m(x) I_m(x) = 0$ ,  $P_1(\sigma x) I_1(x) + \dots + P_m(\sigma x) I_m(x) = 0$ . Hence  $[P_1(\sigma x) - P_1(x)] I_1(x) + \dots + [P_m(\sigma x) - P_m(x)] I_m(x) = 0$ . For  $L(x) = 0$ ,  $\sigma(x) = x$ , so that  $P_i(\sigma x) - P_i(x) = 0$  whenever  $L(x) = 0$ ,  $1 \leq i \leq m$ . Since  $L(x)$  is irreducible it follows that

$$\frac{P_i(\sigma x) - P_i(x)}{L(x)}$$

is a polynomial,  $1 \leq i \leq m$ . We have

$$\left[ \frac{P_1(\sigma x) - P_1(x)}{L(x)} \right] I_1(x) + \dots + \left[ \frac{P_m(\sigma x) - P_m(x)}{L(x)} \right] I_m(x) = 0.$$

$$\deg \left[ \frac{P_1(\sigma x) - P_1(x)}{L(x)} \right] < \deg P_1(x),$$

so that by the induction hypothesis

$$\frac{P_1(\sigma x) - P_1(x)}{L(x)} \equiv 0 \pmod{\mathcal{I}}.$$

Hence  $P_1(\sigma x) \equiv P_1(x) \pmod{\mathcal{I}}$ . Since the  $\sigma$ 's generate  $G$ , this congruence holds for  $\sigma \in G$ . We conclude that

$$P_1(x) \equiv \frac{1}{|G|} \sum_{\sigma \in G} P_1(\sigma x) \pmod{\mathcal{I}}.$$

The polynomial  $\frac{1}{|G|} \sum_{\sigma \in G} P_1(\sigma x)$  is invariant and homogeneous of degree  $n \geq 1$ . Hence it  $\in \mathcal{I}$ , so that  $P_1 \in \mathcal{I}$ .

*Proof of Theorem 2.1.* We choose  $I_1, \dots, I_r$  to be homogeneous invariants of positive degree forming a minimal basis for  $\mathcal{I}$ . Hilbert's proof of Theorem 1.1 shows that  $I_1, \dots, I_r$  form a basis for the invariants of  $G$ . We show that  $I_1, \dots, I_r$  are algebraically independent, so that  $r = n$ .

Suppose, to the contrary, that  $I_1, \dots, I_r$  are algebraically dependent. Choose  $H(y_1, \dots, y_r)$  to be a polynomial of minimal positive degree so that  $H(I_1(x), \dots, I_r(x)) = 0$ . Let  $x$ -degree of any monomial  $y_1^{a_1} \dots y_r^{a_r}$  be  $d_1 a_1 + \dots + d_r a_r$ , where  $d_i = \deg I_i$ . We may assume that all  $x$ -degrees of the monomials appearing in  $H$  are the same. Let

$$H_i(x) = \frac{\partial H}{\partial y_i}(I_1(x), \dots, I_r(x)), \quad 1 \leq i \leq r.$$

The  $H_i$ 's are invariant homogeneous polynomials, as all monomials in  $H$  have equal  $x$ -degree. Since  $H(y_1, \dots, y_r)$  is of positive degree, some  $\frac{\partial H}{\partial y_i} \neq 0$ . It follows that the corresponding  $H_i(x) \neq 0$ , as  $H$  was chosen

to be of minimal degree; i.e. not all  $H_i$ 's = 0. We relabel indices so that  $H_1, \dots, H_s, 1 \leq s \leq r$ , are ideally independent (i.e. none of the  $H_i$ 's is in the ideal generated by the others) and  $H_{s+j} \in (H_1, \dots, H_s), 1 \leq j \leq r - s$ .

Thus  $H_{s+j} = \sum_{i=1}^s V_{ji} H_i, 1 \leq j \leq r - s$ , where each  $V_{ji}$  is a homogeneous polynomial of degree  $d_i - d_{s+j}$  ( $V_{ji}$  is interpreted to be 0 if this degree is negative). Differentiating the relation  $H(I_1(x), \dots, I_r(x)) = 0$  with respect to  $x_k$ , we obtain

$$(2.1) \quad \begin{aligned} \sum_{i=1}^r H_i \frac{\partial I_i}{\partial x_k} &= \sum_{i=1}^s H_i \frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} H_{s+l} \frac{\partial I_{s+l}}{\partial x_k} \\ &= \sum_{i=1}^s H_i \left[ \frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k} \right] = 0. \end{aligned}$$

Since

$$\frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k}$$

is homogeneous of degree  $d_i - 1$ , we conclude from Lemma 2.1 that

$$(2.2) \quad \frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k} = \sum_{j=1}^r B_j I_j, \quad 1 \leq i \leq s,$$

where the  $B_j$ 's are homogeneous and each term in (2.2) is homogeneous of degree  $d_i - 1$ . This forces  $B_i = 0$ . Multiply both sides of (2.2) by  $x_k$  and sum over  $k$ . We conclude, by Euler's identity for homogeneous polynomials,

$$(2.3) \quad d_i I_i + \sum_{l=1}^{r-s} V_{li} d_{s+l} I_{s+l} = \sum_{j=1}^r A_j I_j,$$

the  $A_j$ 's being homogeneous with  $A_i = 0$ .

(2.3) shows that  $I_i \in (I_1, \dots, I_{i-1}, I_{i+1}, \dots, I_r)$ , contradicting the minimality of the basis  $I_1, \dots, I_r$ . Hence  $I_1, \dots, I_r$  are algebraically independent and  $r = n$ .

## 2. THE THEOREM OF SHEPHARD AND TODD

We obtain in this section a converse to Chevalley's Theorem, thereby obtaining an invariant theoretical characterization of finite reflection groups. We first prove several preliminary results.

LEMMA 2.2. Let  $H$  be a finite group of linear transformations acting on the  $n$ -dimensional space  $V$  and fixing the  $n - 1$  dimensional hyperplane  $\pi$ . The elements of  $H$  have a common eigenvector  $v \in V - \pi$ . Let  $\sigma(v) = \zeta(\sigma)v$ ,  $\sigma \in H$ .  $\zeta(\sigma)$  is an isomorphism from  $H$  into the multiplicative group of the roots of unity in  $k$ . It follows that  $H$  is a cyclic group.

REMARK. The above lemma is a consequence of Maschke's Theorem proven in section 2.3. We provide another proof below.

*Proof.* Let  $\sigma_1 \in H$ ,  $\sigma_1 \neq e$  (the identity of  $H$ ). By the remark following Definition 2.1, there exists  $v \in V - \pi$  such that  $\sigma_1(v) = \zeta_1 v$ ,  $\zeta_1$  being a root of unity  $\neq 1$ . For  $\sigma \in H$ , let  $\sigma(v) = \zeta(\sigma)v + p(\sigma)$ ,  $\zeta(\sigma) \in k$  and  $p(\sigma) \in \pi$ . Let  $\sigma^* = \sigma_1^{-1} \sigma^{-1} \sigma_1 \sigma$ . Then  $\sigma^*(v) = v + (1 - \zeta_1)p(\sigma)$ . Since  $\sigma^*$  is of finite order,  $(1 - \zeta_1)p(\sigma) = 0 \Rightarrow p(\sigma) = 0$ . Hence  $\sigma(v) = \zeta(\sigma)v$ .  $\zeta(\sigma)$  is clearly an isomorphism from  $H$  into  $U$ , the multiplicative group of