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(1.10) and Lemma 1.2 yield

$$(1.11) \quad \delta_m = \frac{1}{|G|} \sum_{\sigma \in G} (\text{Tr } \sigma)_m = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma).$$

Multiply both sides of (1.11) by t^m and sum over m from 0 to ∞ . We get

$$\begin{aligned} \sum_{m=0}^{\infty} \delta_m t^m &= \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma) t^m \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \left\{ \sum_{m=0}^{\infty} \omega_1^m(\sigma) t^m \dots \sum_{m=0}^{\infty} \omega_n^m(\sigma) t^m \right\} \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1 - \omega_1(\sigma) t) \dots (1 - \omega_n(\sigma) t)} \end{aligned}$$

CHAPTER II

INVARIANT THEORETIC CHARACTERIZATION OF FINITE REFLECTION GROUPS

1. CHEVALLEY'S THEOREM

We showed in chapter I that we can always find a finite number of homogeneous invariants forming a basis for the invariants of G and that this set must contain at least n elements, where $n = \dim V$. We show that this lower bound is attained only for the finite reflection groups. We first define these groups.

DEFINITION 2.1. Let σ be a linear transformation acting on the n -dimensional vector space V . σ is a reflection $\Leftrightarrow \sigma$ fixes an $n - 1$ dimensional hyperplane π and σ is of finite order > 1 . π is called the reflecting hyperplane (r.h.) of σ .

REMARK. Choose $v \notin \pi$. and let $\sigma v = \zeta v + p$, $p \in \pi$. If $\zeta = 1$, then $\sigma^m v = v + mp$, contradicting that σ is of finite order. Hence $\zeta \neq 1$. Let $v' = v + (\zeta - 1)^{-1} p$ and choose p_1, \dots, p_{n-1} as a basis for π . Then $\sigma p_i = p_i$, $1 \leq i \leq n - 1$, $\sigma v' = \zeta v'$. ζ is a root of 1 in k which is distinct from 1, as σ is of finite order > 1 . Thus σ is a reflection iff relative to some basis, the matrix for σ is diagonal, $n - 1$ of the diagonal entries equalling 1 and the remaining one equalling a root of 1 in k distinct from 1.

DEFINITION 2.2. G is a finite reflection group acting on $V \Leftrightarrow G$ is a finite group generated by reflections on V .

As an example of a finite reflection group, let $G = S_n$. It is well known that S_n is generated by transpositions. The transposition of the variables $x_i, x_j (i \neq j)$ fixes the hyperplane $x_i - x_j = 0$, so that it is a reflection.

We have the following result

THEOREM 2.1 (Chevalley [4]). *Let G be a finite reflection group acting on the n -dimensional vector space V . The invariants of G have a basis consisting of n homogeneous elements which are algebraically independent over k .*

Let $k[x]$ denote the ring of polynomials in x_1, \dots, x_n with coefficients in k . We prove the following.

LEMMA 2.1. Let I_1, \dots, I_m be invariant polynomials of G , $I_1 \notin (I_2, \dots, I_m)$ = the ideal in $k[x]$ generated by I_2, \dots, I_m . Suppose that $P_1 I_1 + \dots + P_m I_m = 0$, the P_i 's being polynomials with P_1 homogeneous. Then $P_1 \in \mathcal{I}$, where \mathcal{I} is the ideal in $k[x]$ generated by the homogeneous invariants of positive degree.

Proof of Lemma 2.1. The proof proceeds by induction on $\deg P_1$. Suppose $\deg P_1 = 0$, so that $P_1 = c \in k$. If $c \neq 0$, then $I_1 \in (I_2, \dots, I_m)$, contrary to assumption. Hence $c = 0 \Rightarrow P_1 \in \mathcal{I}$. Let $\deg P_1 = n > 0$. Let σ be a reflection in G and $L = 0$ the equation of its r.h. (L is a linear homogeneous polynomial). We have $P_1(x) I_1(x) + \dots + P_m(x) I_m(x) = 0$, $P_1(\sigma x) I_1(x) + \dots + P_m(\sigma x) I_m(x) = 0$. Hence $[P_1(\sigma x) - P_1(x)] I_1(x) + \dots + [P_m(\sigma x) - P_m(x)] I_m(x) = 0$. For $L(x) = 0$, $\sigma(x) = x$, so that $P_i(\sigma x) - P_i(x) = 0$ whenever $L(x) = 0$, $1 \leq i \leq m$. Since $L(x)$ is irreducible it follows that

$$\frac{P_i(\sigma x) - P_i(x)}{L(x)}$$

is a polynomial, $1 \leq i \leq m$. We have

$$\left[\frac{P_1(\sigma x) - P_1(x)}{L(x)} \right] I_1(x) + \dots + \left[\frac{P_m(\sigma x) - P_m(x)}{L(x)} \right] I_m(x) = 0.$$

$$\deg \left[\frac{P_1(\sigma x) - P_1(x)}{L(x)} \right] < \deg P_1(x),$$

so that by the induction hypothesis

$$\frac{P_1(\sigma x) - P_1(x)}{L(x)} \equiv 0 \pmod{\mathcal{I}}.$$

Hence $P_1(\sigma x) \equiv P_1(x) \pmod{\mathcal{I}}$. Since the σ 's generate G , this congruence holds for $\sigma \in G$. We conclude that

$$P_1(x) \equiv \frac{1}{|G|} \sum_{\sigma \in G} P_1(\sigma x) \pmod{\mathcal{I}}.$$

The polynomial $\frac{1}{|G|} \sum_{\sigma \in G} P_1(\sigma x)$ is invariant and homogeneous of degree $n \geq 1$. Hence it $\in \mathcal{I}$, so that $P_1 \in \mathcal{I}$.

Proof of Theorem 2.1. We choose I_1, \dots, I_r to be homogeneous invariants of positive degree forming a minimal basis for \mathcal{I} . Hilbert's proof of Theorem 1.1 shows that I_1, \dots, I_r form a basis for the invariants of G . We show that I_1, \dots, I_r are algebraically independent, so that $r = n$.

Suppose, to the contrary, that I_1, \dots, I_r are algebraically dependent. Choose $H(y_1, \dots, y_r)$ to be a polynomial of minimal positive degree so that $H(I_1(x), \dots, I_r(x)) = 0$. Let x -degree of any monomial $y_1^{a_1} \dots y_r^{a_r}$ be $d_1 a_1 + \dots + d_r a_r$, where $d_i = \deg I_i$. We may assume that all x -degrees of the monomials appearing in H are the same. Let

$$H_i(x) = \frac{\partial H}{\partial y_i}(I_1(x), \dots, I_r(x)), \quad 1 \leq i \leq r.$$

The H_i 's are invariant homogeneous polynomials, as all monomials in H have equal x -degree. Since $H(y_1, \dots, y_r)$ is of positive degree, some $\frac{\partial H}{\partial y_i} \neq 0$. It follows that the corresponding $H_i(x) \neq 0$, as H was chosen

to be of minimal degree; i.e. not all H_i 's = 0. We relabel indices so that $H_1, \dots, H_s, 1 \leq s \leq r$, are ideally independent (i.e. none of the H_i 's is in the ideal generated by the others) and $H_{s+j} \in (H_1, \dots, H_s), 1 \leq j \leq r - s$.

Thus $H_{s+j} = \sum_{i=1}^s V_{ji} H_i, 1 \leq j \leq r - s$, where each V_{ji} is a homogeneous polynomial of degree $d_i - d_{s+j}$ (V_{ji} is interpreted to be 0 if this degree is negative). Differentiating the relation $H(I_1(x), \dots, I_r(x)) = 0$ with respect to x_k , we obtain

$$\begin{aligned} (2.1) \quad \sum_{i=1}^r H_i \frac{\partial I_i}{\partial x_k} &= \sum_{i=1}^s H_i \frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} H_{s+l} \frac{\partial I_{s+l}}{\partial x_k} \\ &= \sum_{i=1}^s H_i \left[\frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k} \right] = 0. \end{aligned}$$

Since

$$\frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k}$$

is homogeneous of degree $d_i - 1$, we conclude from Lemma 2.1 that

$$(2.2) \quad \frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k} = \sum_{j=1}^r B_j I_j, \quad 1 \leq i \leq s,$$

where the B_j 's are homogeneous and each term in (2.2) is homogeneous of degree $d_i - 1$. This forces $B_i = 0$. Multiply both sides of (2.2) by x_k and sum over k . We conclude, by Euler's identity for homogeneous polynomials,

$$(2.3) \quad d_i I_i + \sum_{l=1}^{r-s} V_{li} d_{s+l} I_{s+l} = \sum_{j=1}^r A_j I_j,$$

the A_j 's being homogeneous with $A_i = 0$.

(2.3) shows that $I_i \in (I_1, \dots, I_{i-1}, I_{i+1}, \dots, I_r)$, contradicting the minimality of the basis I_1, \dots, I_r . Hence I_1, \dots, I_r are algebraically independent and $r = n$.

2. THE THEOREM OF SHEPHARD AND TODD

We obtain in this section a converse to Chevalley's Theorem, thereby obtaining an invariant theoretical characterization of finite reflection groups. We first prove several preliminary results.

LEMMA 2.2. Let H be a finite group of linear transformations acting on the n -dimensional space V and fixing the $n - 1$ dimensional hyperplane π . The elements of H have a common eigenvector $v \in V - \pi$. Let $\sigma(v) = \zeta(\sigma)v$, $\sigma \in H$. $\zeta(\sigma)$ is an isomorphism from H into the multiplicative group of the roots of unity in k . It follows that H is a cyclic group.

REMARK. The above lemma is a consequence of Maschke's Theorem proven in section 2.3. We provide another proof below.

Proof. Let $\sigma_1 \in H$, $\sigma_1 \neq e$ (the identity of H). By the remark following Definition 2.1, there exists $v \in V - \pi$ such that $\sigma_1(v) = \zeta_1 v$, ζ_1 being a root of unity $\neq 1$. For $\sigma \in H$, let $\sigma(v) = \zeta(\sigma)v + p(\sigma)$, $\zeta(\sigma) \in k$ and $p(\sigma) \in \pi$. Let $\sigma^* = \sigma_1^{-1} \sigma^{-1} \sigma_1 \sigma$. Then $\sigma^*(v) = v + (1 - \zeta_1)p(\sigma)$. Since σ^* is of finite order, $(1 - \zeta_1)p(\sigma) = 0 \Rightarrow p(\sigma) = 0$. Hence $\sigma(v) = \zeta(\sigma)v$. $\zeta(\sigma)$ is clearly an isomorphism from H into U , the multiplicative group of

the roots of unity in k . U is known to be cyclic ([22], Vol. 1, p. 112). It follows that $\zeta(H)$, a subgroup of U , is cyclic and so H is cyclic.

THEOREM 2.2. *Let G be a finite group acting on the n -dimensional space V . Let I_1, \dots, I_n be homogeneous polynomials forming a basis for the invariants of G . Let d_1, \dots, d_n be the respective degrees of I_1, \dots, I_n . Then*

$$(2.4) \quad \prod_{i=1}^n d_i = |G|, \quad \sum_{i=1}^n (d_i - 1) = r$$

where $r =$ number of reflections in G .

Proof. By Theorem 1.2, I_1, \dots, I_n are algebraically independent. Let $I(x)$ be a homogeneous invariant of degree m . Then I is a linear combination of the monomials $I_1^{a_1} \dots I_n^{a_n}$ where $a_1 d_1 + \dots + a_n d_n = m$. Furthermore, these monomials are linearly independent over k , as I_1, \dots, I_n are algebraically independent over k . It follows that the dimension δ_m of homogeneous invariants of degree $m =$ number of non-negative integer solutions to $a_1 d_1 + \dots + a_n d_n = m$. Hence

$$(2.5) \quad \sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{(1-t^{d_1}) \dots (1-t^{d_n})}.$$

(1.9) and (2.5) yield

$$(2.6) \quad \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1-\omega_1(\sigma)t) \dots (1-\omega_n(\sigma)t)} = \frac{1}{(1-t^{d_1}) \dots (1-t^{d_n})}$$

Expand both sides of (2.6) in powers of $(1-t)$. Let $\mathcal{R} =$ set of reflections in G and $\zeta(\sigma) =$ eigenvalue of the reflection σ which $\neq 1$. We have

$$(2.7) \quad \begin{aligned} & \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1-\omega_1(\sigma)t) \dots (1-\omega_n(\sigma)t)} \\ &= \frac{1}{|G|} \frac{1}{(1-t)^n} + \frac{1}{|G|} \sum_{\sigma \in \mathcal{R}} \frac{1}{1-\zeta(\sigma)} \frac{1}{(1-t)^{n-1}} + \dots \end{aligned}$$

$$(2.8) \quad \begin{aligned} \frac{1}{(1-t^{d_1}) \dots (1-t^{d_n})} &= \prod_{i=1}^n \frac{1}{d_i(1-t) - \binom{d_i}{2}(1-t)^2 + \dots \pm (1-t)^{d_i}} \\ &= \frac{1}{\prod_{i=1}^n d_i} \frac{1}{(1-t)^n} + \frac{1/2 \sum_{i=1}^n (d_i - 1)}{\prod_{i=1}^n d_i} \frac{1}{(1-t)^{n-1}} + \dots \end{aligned}$$

Equating coefficients of (2.7), (2.8), we get

$$(2.9) \quad \prod_{i=1}^n d_i = |G|, \quad \sum_{i=1}^n (d_i - 1) = 2 \sum_{\sigma \in \mathcal{R}} \frac{1}{1 - \zeta(\sigma)}.$$

We evaluate the sum

$$\sum_{\sigma \in \mathcal{R}} \frac{1}{1 - \zeta(\sigma)} :$$

Let π be any r.h. Let $H_\pi = \{\sigma \mid \sigma \in G \text{ and } \sigma \text{ fixes } \pi\}$. Thus H_π is the subgroup of G consisting of the identity and those reflections in G with r.h. π . Applying Lemma 2.2 to H_π , we conclude that there exists $v \notin \pi$ such that $\sigma(v) = \zeta(\sigma)v$ for $\sigma \in H_\pi$. Let $H'_\pi = H_\pi - \{e\}$. Since $\zeta(\sigma^{-1}) = (\zeta(\sigma))^{-1}$, we obtain

$$(2.10) \quad \begin{aligned} \sum_{\sigma \in H'_\pi} \frac{1}{1 - \zeta(\sigma)} &= \sum_{\sigma \in H'_\pi} \frac{1}{1 - \zeta(\sigma^{-1})} \\ &= \sum_{\sigma \in H'_\pi} \left(1 - \frac{1}{1 - \zeta(\sigma)}\right) = |H'_\pi| - \sum_{\sigma \in H'_\pi} \frac{1}{1 - \zeta(\sigma)}. \end{aligned}$$

Hence

$$(2.11) \quad \sum_{\sigma \in H'_\pi} \frac{1}{1 - \zeta(\sigma)} = \frac{|H'_\pi|}{2}.$$

Summing both sides of (2.11) over all r.h. π , we get

$$(2.12) \quad \sum_{\sigma \in \mathcal{R}} \frac{1}{1 - \zeta(\sigma)} = \frac{r}{2}.$$

(2.9), (2.12) yield Theorem 2.2.

THEOREM 2.3. *Let f_1, \dots, f_n be polynomials in the variables x_1, \dots, x_n . f_1, \dots, f_n are algebraically independent over $k \Leftrightarrow$*

$$\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \neq 0.$$

Proof. Suppose that f_1, \dots, f_n are algebraically independent. Then $G(f_1, \dots, f_n) = 0$ for some polynomial $G = G(y_1, \dots, y_n)$. Assume that $G(y_1, \dots, y_n)$ is of minimal positive degree. Differentiating this relation with respect to x_j , we get

$$(2.13) \quad \sum_{i=1}^n \frac{\partial G}{\partial y_i} (f_1, \dots, f_n) \frac{\partial f_i}{\partial x_j} = 0, \quad 1 \leq j \leq n.$$

(2.13) is a system of linear equations (with coefficients in $k(x_1, \dots, x_n)$) in the unknowns $H_i(x) = \frac{\partial G}{\partial y_i} (f_1, \dots, f_n)$, $1 \leq i \leq n$. $\frac{\partial G}{\partial y_i} \neq 0$ for some i , as G is not constant, and $\deg \frac{\partial G}{\partial y_i} < \deg G$. It follows that the corresponding $H_i(x) \neq 0$. Thus the linear system (2.13) has a non-zero solution, so that its determinant

$$\frac{\partial (f_1, \dots, f_n)}{\partial (x_1, \dots, x_n)} \neq 0.$$

Conversely, let f_1, \dots, f_n be algebraically independent. For each i , x_i, f_1, \dots, f_n are algebraically dependent. Hence there exists a polynomial $G_i(x_i, y_1, \dots, y_n)$ of minimal positive degree in x_i such that $G_i(x_i, f_1, \dots, f_n) = 0$. Differentiating these relations with respect to x_k , we get

$$(2.14) \quad \sum_{j=1}^n \frac{\partial G_i}{\partial y_j} (x_i, f_1, \dots, f_n) \frac{\partial f_j}{\partial x_k} + \frac{\partial G_i}{\partial x_k} (x_i, f_1, \dots, f_n) \delta_{ik}, \quad 1 \leq k \leq n,$$

δ_{ik} denoting the Kronecker symbol. (2.14) may be rewritten in matrix notation as

$$(2.15) \quad \begin{pmatrix} \frac{\partial G_i}{\partial y_j} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f_i}{\partial x_j} \end{pmatrix} = D$$

where the entries of D are

$$- \delta_{ij} \frac{\partial G_i}{\partial x_j}.$$

$\det D \neq 0$, as x_i - degree of $\frac{\partial G_i}{\partial x_i} < x_i$ - degree of G_i , $1 \leq i \leq n$.

It follows from (2.15) that $\frac{\partial (f_1, \dots, f_n)}{\partial (x_1, \dots, x_n)} \neq 0$.

THEOREM 2.4. (Shephard and Todd [19]). *Let G be a finite group acting on the n -dimensional space V . Suppose there exists a basis of n homogeneous polynomials for the invariants of G . Then G is a finite reflection group.*

Proof. Let H be the subgroup of G generated by the reflections in G . By assumption G has n basic homogeneous invariants which, by Theorem 1.2, are algebraically independent. Since H is a finite reflection group, we conclude from Chevalley's Theorem that H has n basic homogeneous invariants J_1, \dots, J_n which are algebraically independent. Each I_i is invariant under H so that $I_i = I_i(J_1, \dots, J_n)$, the latter quantity denoting a polynomial in the J_i 's. We may assume that $I_i(J_1, \dots, J_n)$ is a linear combination of monomials $J_1^{a_1} \dots J_n^{a_n}$ whose x -degree = $\deg I_i$. We have

$$(2.16) \quad \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)} = \frac{\partial (I_1, \dots, I_n)}{\partial (J_1, \dots, J_n)} \cdot \frac{\partial (J_1, \dots, J_n)}{\partial (x_1, \dots, x_n)}$$

By Theorem 2.3,

$$\frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)} \neq 0$$

and (2.16) then shows that

$$\frac{\partial (I_1, \dots, I_n)}{\partial (J_1, \dots, J_n)} \neq 0.$$

It follows that there is a rearrangement k_1, \dots, k_n of $1, \dots, n$ so that

$$\frac{\partial I_{k_1}}{\partial J_1} \dots \frac{\partial I_{k_n}}{\partial J_n} \neq 0.$$

Hence $I_{k_i}(J_1, \dots, J_n)$ is of positive degree in J_i and $\deg I_{k_i} \geq \deg J_i$, $1 \leq i \leq n$. Applying Theorem 2.2 both to G and H , we obtain

$$(2.17) \quad \prod_{i=1}^n \deg J_i = |H|, \quad \prod_{i=1}^n \deg I_i = |G|$$

$$(2.18) \quad \sum_{i=1}^n (\deg J_i - 1) = \sum_{i=1}^n (\deg I_i - 1) = r$$

where r = number of reflections in G = number of reflections in H .

Since $\deg I_{k_i} \geq \deg J_i$, $1 \leq i \leq n$, we conclude from (2.18) that $\deg I_{k_i} = \deg J_i$, $1 \leq i \leq n$. Hence $\prod_{i=1}^n \deg I_i = \prod_{i=1}^n \deg J_i$, and we conclude from (2.17) that $|G| = |H|$. Thus $G = H$ and G is a finite reflection group.

3. A FORMULA FOR $\frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)}$

We obtain a formula which shall be used in Chapter III.

THEOREM 2.5. Let G be a finite reflection group acting on the n -dimensional space V . Let I_1, \dots, I_n be a basic set of homogeneous invariants for G . Let x be a coordinate system for V and $L_i(x) = 0$, $1 \leq i \leq r$, the r.h.'s for G , each L_i being linear and homogeneous. Then

$$(2.19) \quad \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)} = c \prod_{i=1}^r L_i(x)$$

c being a constant $\neq 0$.

Proof. Let J the left hand side of (2.19). We observe that J is a non-zero homogeneous polynomial of degree $\sum_{i=1}^n (d_i - 1)$. By Theorem 2.2,

$\sum_{i=1}^n (d_i - 1) = r$, so that $\deg J = r$. If k is the real field R , we have the following simple proof of (2.19). $I_i = I_i(x_1, \dots, x_n)$, $1 \leq i \leq n$, is a mapping from x -space to I -space. This mapping is not 1 - 1 in any neighborhood of a point x lying in the r.h. $L_i(x) = 0$, as any point and its reflection get mapped into the same point I . It follows from the Implicit Function Theorem that $J(x) = 0$ whenever $L_i(x) = 0$. Thus $L_i \mid J$, $1 \leq i \leq r$,

and so $\prod_{i=1}^r L_i \mid J$. Since $J, \prod_{i=1}^r L_i$ have the same degree r , we have

$$J = c \prod_{i=1}^r L_i, \quad c \neq 0.$$

For an arbitrary field k , the theorem is proven as follows. Let π be an r.h. with equation $L(x) = 0$ and H the subgroup of h elements in G fixing π . Thus there are $h - 1$ reflections in G with r.h. π . We show that $L^{h-1} \mid J$. By Lemma 2.2, H is a cyclic group generated by an element σ . Furthermore there exists $v \notin \pi$ and a primitive h -th root of 1 such that $\sigma(v) = \zeta v$. Choose a coordinate system $y = (y_1, \dots, y_n)$ in V so that π has the equation $y_n = 0$ and $v = (0, \dots, 0, 1)$ σ then becomes the transformation $(y_1, \dots, y_{n-1}, y_n) \rightarrow (y_1, \dots, y_{n-1}, \zeta y_n)$. Let $x = \tau y$ and $J_i(y) = I_i(\tau y)$, $1 \leq i \leq n$. We have

$$(2.20) \quad J_i(y_1, \dots, y_{n-1}, \zeta y_n) = J_i(y_1, \dots, y_{n-1}, y_n), \quad 1 \leq i \leq n$$

Let $J_i = \sum A_m y_n^m$, the A_m 's being polynomials in y_1, \dots, y_{n-1} . (2.20) implies that $A_m = 0$ whenever $h \nmid m$, so that $A_m = 0, 0 \leq m \leq h - 1$. Since

$$\frac{\partial J_i}{\partial y_m} = \sum_m A_m y_n^{m-1},$$

we conclude

$$y_n^{h-1} \left| \frac{\partial J_i}{\partial y_n}, 1 \leq i \leq n. \right.$$

Hence

$$(2.21) \quad y_n^{h-1} \left| \frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)}, \right.$$

Since

$$\frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)} = J(x) \cdot \det \tau,$$

(2.21) is equivalent to $L^{h-1}(x) \mid J(x)$. It follows that if $L_i(x) = 0, 1 \leq i \leq r$, are the r.h.'s for G , then $\prod_{i=1}^r L_i \mid J$. But $J, \prod_{i=1}^r L_i$ have the same degree r , so that $J = c \prod_{i=1}^r L_i, c \neq 0$.

4. DECOMPOSITION OF FINITE REFLECTION GROUPS

We shall decompose every finite reflection group into a direct product of irreducible ones and show that it suffices to study the invariant theory of the irreducible groups.

DEFINITION 2.3. Let the group G act on V . G is said to be reducible iff there exists a proper subspace W invariant under G ; i.e. $\sigma w \in W$ for $\sigma \in G, w \in W$. G is said to be completely reducible iff $V = V_1 \oplus V_2, V_1$ and V_2 being proper invariant subspaces. G is said to be irreducible iff it is not reducible.

THEOREM 2.6. (Maschke [22], Vol. 2, p. 179). *Let G be a finite group acting on the vector space V . If G is reducible, then it is completely reducible.*

Proof. Let V_1 be a proper invariant subspace of V . Let V_2 be a complementary subspace. Thus for $v \in V$, we have a unique decomposition

$v = v_1 + v_2, v_i \in V_i (i=1, 2)$. Let $\eta v = v_2$ and set $\tau = \frac{1}{|G|} \sum_{\sigma \in G} \sigma \eta \sigma^{-1}$.

τ satisfies the following:

- i) $\tau \sigma = \sigma \tau, \sigma \in G$. For $\sigma \tau = \frac{1}{|G|} \sum_{\sigma_1 \in G} \sigma \sigma_1 \eta (\sigma \sigma_1)^{-1} \sigma = \tau \sigma$
- ii) $\tau v_1 = 0, v_1 \in V_1$. For $\sigma^{-1} v_1 \in V_1, \sigma \in G$, so that $\eta \sigma^{-1} v_1 = 0 \Rightarrow \tau v_1 = 0$
- iii) $(1-\tau)v \in V_1, v \in V, 1$ denoting the identity of G . For $(1-\eta)v \in V_1$, so that $(1-\eta)\sigma^{-1}v \in V_1 \Rightarrow \sigma(1-\eta)\sigma^{-1}v \in V_1, \sigma \in G$. It follows that $(1-\tau)v = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(1-\eta)\sigma^{-1}v \in V_1$.

Let $V'_2 = \tau V$. V'_2 is invariant under G as $\sigma(\tau v) = \tau(\sigma v)$. For any $v, v = \tau v + (1-\tau)v$. It follows from iii) that $V = V_1 + V'_2$. ii), iii) imply $\tau(1-\tau) = 0 \Leftrightarrow \tau = \tau^2$. Hence $\tau v'_2 = v'_2$ for $v'_2 \in V'_2$. Let $v_1 + v'_2 = 0$, where $v_1 \in V_1, v'_2 \in V'_2$. Applying τ to both sides, we get $v'_2 = 0$ and so $v_1 = 0$. Hence $V = V_1 \oplus V'_2$.

Repeated application of Maschke's Theorem yields the

COROLLARY. *Let G be a finite group acting on the finite-dimensional vector space V . Then $V = V_1 \oplus \dots \oplus V_s$, the V_i 's being invariant subspaces of V and G acting irreducibly on each V_i .*

For finite reflection groups, we have

THEOREM 2.7. *Let G be a finite reflection group acting on V . There exists a decomposition $V = V_1 \oplus \dots \oplus V_s$ into invariant subspaces such that:*

- 1) *Let $G_i = G|_{V_i}$ = group of restrictions of elements of G to V_i . Then G is isomorphic to $G_1 \times \dots \times G_s$*
- 2) *Each $G_i, 1 \leq i \leq s$, is a reflection group acting irreducibly on V_i .*

Proof. By the corollary to Theorem 2.6, there exists a decomposition $V = V_1 \oplus \dots \oplus V_s$, the V_i 's being invariant subspaces and G_i irreducible for $1 \leq i \leq s$. We label the V_i 's so that V_1, \dots, V_r are 1-dimensional and $G|_{V_i} = \text{identity}$.

By the remark following Definition 2.1, for each reflection σ there exists an eigenvector $v \in V - \pi, \pi$ being the r.h. for σ . Call v a root of G . We have

$$(2.22) \quad \dim(V_i + \pi) + \dim(V_i \cap \pi) = \dim V_i + \dim \pi.$$

If $V_i \not\subset \pi$, then $V_i + \pi = V$ and we conclude from (2.22) that $\dim V_i = \dim (V_i \cap \pi) + 1$. I.e. $V_i \cap \pi$ is a hyperplane in V_i and $\sigma|_{V_i}$ a reflection on V_i . Choose $u \in V_i - \pi$ so that u is an eigenvector of σ . u is a multiple of the root v , so that $v \in V_i$. Thus $\sigma|_{V_i}$ is a reflection of V_i if $v \in V_i$, and the identity if $v \notin V_i$. Furthermore, each root v is in some V_i , $r + 1 \leq i \leq s$, otherwise the corresponding reflection σ would have been the identity.

Let $\tilde{G}_i =$ subgroup generated by those reflections whose roots are in V_i , $1 \leq i \leq s$. It is readily checked that $G = \tilde{G}_1 \times \dots \times \tilde{G}_s$, $G_i = \tilde{G}_i|_{V_i}$. If $\sigma \in \tilde{G}_i$ and $\sigma|_{V_i} =$ identity then $\sigma =$ identity. The mapping $\sigma \rightarrow \sigma|_{V_i}$ is thus an isomorphism from \tilde{G}_i onto G_i .

THEOREM 2.8. *Let G be a finite reflection group acting on V and decompose V as in Theorem 2.7. Every polynomial invariant under G is a polynomial in the invariant polynomials of G_1, \dots, G_s .*

Proof. For each $v \in V$, write $v = v_1 + \dots + v_s$, $v_i \in V_i$. By Theorem 2.7, for each $\sigma \in G$, we may write $\sigma v = \sigma_1 v_1 + \dots + \sigma_s v_s$, $\sigma_i \in G_i$. For any polynomial function $p(v)$ on V , we have $p(v) = \sum_{i=1}^N p_{i1}(v_1) \dots p_{is}(v_s)$ where $p_{ij}(v_j)$ is a polynomial function on V_j . If $p(v)$ is invariant under G , then

$$(2.23) \quad p(v) = \frac{1}{|G|} \sum_{\sigma \in G} p(\sigma v) = \sum_{i=1}^N I_{i1}(v_1) \dots I_{is}(v_s)$$

where

$$(2.24) \quad I_{ij}(v_j) = \frac{1}{|G_j|} \sum_{\sigma_j \in G_j} p_{ij}(\sigma_j v_j)$$

is an invariant of G_j .

CHAPTER III

THE DEGREES OF THE BASIC INVARIANTS

We determine the degrees of the basic homogeneous invariants in case G is a finite reflection group. We present two different methods. The first one (Theorem 3.8), restricts itself to the case where k is the real field and has the advantage of providing an effective method for computing the