**Zeitschrift:** L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

**Band:** 24 (1978)

**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INVARIANTS OF FINITE REFLECTION GROUPS

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Kapitel: CHAPTER II INVARIANT THEORETIC CHARACTERIZATION OF

FINITE REFLECTION GROUPS

**DOI:** https://doi.org/10.5169/seals-49704

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(1.10) and Lemma 1.2 yield

(1.11) 
$$\delta_m = \frac{1}{|G|} \sum_{\sigma \in G} (Tr \, \sigma)_m = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma).$$

Multiply both sides of (1.11) by  $t^m$  and sum over m from 0 to  $\infty$ . We get

$$\sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{\sigma \in G} \sum_{|a|=m}^{\infty} \omega^a(\sigma) t^m$$

$$= \frac{1}{|G|} \sum_{\sigma \in G} \left\{ \sum_{m=0}^{\infty} \omega_1^m(\sigma) t^m \dots \sum_{m=0}^{\infty} \omega_n^m(\sigma) t^m \right\}$$

$$= \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1 - \omega_1(\sigma) t) \dots (1 - \omega_n(\sigma) t)}$$

## CHAPTER II

# INVARIANT THEORETIC CHARACTERIZATION OF FINITE REFLECTION GROUPS

### 1. CHEVALLEY'S THEOREM

We showed in chapter I that we can always find a finite number of homogeneous invariants forming a basis for the invariants of G and that this set must contain at least n elements, where  $n = \dim V$ . We show that this lower bound is attained only for the finite reflection groups. We first define these groups.

DEFINITION 2.1. Let  $\sigma$  be a linear transformation acting on the *n*-dimensional vector space V.  $\sigma$  is a reflection  $\Leftrightarrow \sigma$  fixes an n-1 dimensional hyperplane  $\pi$  and  $\sigma$  is of finite order > 1.  $\pi$  is called the reflecting hyperplane (r.h.) of  $\sigma$ .

REMARK. Choose  $v \notin \pi$  and let  $\sigma v = \zeta v + p$ ,  $p \in \pi$ . If  $\zeta = 1$ , then  $\sigma^m v = v + mp$ , contradicting that  $\sigma$  is of finite order. Hence  $\zeta \neq 1$ . Let  $v' = v + (\zeta - 1)^{-1} p$  and choose  $p_1, ..., p_{n-1}$  as a basis for  $\pi$ . Then  $\sigma p_i = p_i$ ,  $1 \le i \le n-1$ ,  $\sigma v' = \zeta v'$ .  $\zeta$  is a root of 1 in k which is distinct from 1, as  $\sigma$  is of finite order > 1. Thus  $\sigma$  is a reflection iff relative to some basis, the matrix for  $\sigma$  is diagonal, n-1 of the diagonal entries equalling 1 and the remaining one equalling a root of 1 in k distinct from 1.

DEFINITION 2.2. G is a finite reflection group acting on  $V \Leftrightarrow G$  is a finite group generated by reflections on V.

As an example of a finite reflection group, let  $G = S_n$ . It is well known that  $S_n$  is generated by transpositions. The transposition of the variables  $x_i$ ,  $x_j$  ( $i \neq j$ ) fixes the hyperplane  $x_i - x_j = 0$ , so that it is a reflection.

We have the following result

Theorem 2.1 (Chevalley [4]). Let G be a finite reflection group acting on the n-dimensional vector space V. The invariants of G have a basis consisting of n homogeneous elements which are algebraically independent over k.

Let k[x] denote the ring of polynomials in  $x_i, ..., x_n$  with coefficients in k. We prove the following.

LEMMA 2.1. Let  $I_1, ..., I_m$  be invariant polynomials of  $G, I_1 \notin (I_2, ..., I_m)$  = the ideal in k[x] generated by  $I_2, ..., I_m$ . Suppose that  $P_1 I_1 + ... + P_m I_m = 0$ , the  $P_i$ 's being polynomials with  $P_1$  homogeneous. Then  $P_1 \in \mathcal{I}$ , where  $\mathcal{I}$  is the ideal in k[x] generated by the homogeneous invariants of positive degree.

Proof of Lemma 2.1. The proof proceeds by induction on  $\deg P_1$ . Suppose  $\deg P_1=0$ , so that  $P_1=c\in k$ . If  $c\neq 0$ , then  $I_1\in (I_2,...,I_m)$ , contrary to assumption. Hence  $c=0\Rightarrow P_1\in \mathscr{I}$ . Let  $\deg P_1=n>0$ . Let  $\sigma$  be a reflection in G and L=0 the equation of its r.h. (L is a linear homogeneous polynomial). We have  $P_1(x)I_1(x)+...+P_m(x)I_m(x)=0$ ,  $P_1(\sigma x)I_1(x)+...+P_m(\sigma x)I_m(x)=0$ . Hence  $\begin{bmatrix} P_1(\sigma x)-P_1(x)\end{bmatrix}I_1(x)+...+\begin{bmatrix} P_m(\sigma x)-P_m(x)\end{bmatrix}I_m(x)$ . For L(x)=0,  $\sigma(x)=x$ , so that  $P_i(\sigma x)-P_i(x)=0$  whenever L(x)=0,  $1\leqslant i\leqslant m$ . Since L(x) is irreducible it follows that

$$\frac{P_i\left(\sigma x\right) - P_i\left(x\right)}{L(x)}$$

is a polynomial,  $1 \le i \le m$ . We have

$$\label{eq:local_equation} \begin{split} \left[\frac{P_1\left(\sigma x\right) \,-\, P_1\left(x\right)}{L\left(x\right)}\right] \,I_1\left(x\right) \,+\, \ldots \,+\, \left[\frac{P_m\left(\sigma x\right) \,-\, P_m\left(x\right)}{L\left(x\right)}\right] \,I_m\left(x\right) \,=\, 0 \;. \\ \deg \, \left[\frac{P_1\left(\sigma x\right) \,-\, P_1\left(x\right)}{L\left(x\right)}\right] \,\,<\, \deg \, P_1\left(x\right) \;, \end{split}$$

so that by the induction hypothesis

$$\frac{P_1(\sigma x) - P_1(x)}{L(x)} \equiv 0 \pmod{\mathscr{I}}.$$

Hence  $P_1(\sigma x) \equiv P_1(x) \pmod{\mathscr{I}}$ . Since the  $\sigma$ 's generate G, this congruence holds for  $\sigma \in G$ . We conclude that

$$P_1(x) \equiv \frac{1}{|G|} \sum_{\sigma \in G} P_1(\sigma x) \pmod{\mathscr{I}}.$$

The polynomial  $\frac{1}{|G|} \sum_{\sigma \in G} P_1(\sigma x)$  is invariant and homogeneous of degree  $n \ge 1$ . Hence it  $\in \mathcal{I}$ , so that  $P_1 \in \mathcal{I}$ .

*Proof of Theorem 2.1.* We choose  $I_1, ..., I_r$  to be homogeneous invariants of positive degree forming a minimal basis for  $\mathcal{I}$ . Hilbert's proof of Theorem 1.1 shows that  $I_1, ..., I_r$  form a basis for the invariants of G. We show that  $I_1, ..., I_r$  are algebraically independent, so that r = n.

Suppose, to the contrary, that  $I_1, ..., I_r$  are algebraically dependent. Choose  $H(y_1, ..., y_r)$  to be a polynomial of minimal positive degree so that  $H(I_1(x), ..., I_r(x)) = 0$ . Let x-degree of any monomial  $y_1^{a_1} ... y_r^{a_r}$  be  $d_1 a_1 + ... + d_r a_r$ , where  $d_i = \deg I_i$ . We may assume that all x-degrees of the monomials appearing in H are the same. Let

$$H_{i}\left(x\right) = \frac{\partial H}{\partial y_{i}}\left(I_{1}\left(x\right), ..., I_{r}\left(x\right)\right), \ 1 \leqslant i \leqslant r.$$

The  $H_i$ 's are invariant homogeneous polynomials, as all monomials in H have equal x-degree. Since  $H(y_1, ..., y_n)$  is of positive degree, some  $\frac{\partial H}{\partial y_i} \neq 0$ , It follows that the corresponding  $H_i(x) \neq 0$ , as H was chosen to be of minimal degree; i.e. not all  $H_i$ 's = 0. We relabel indices so that  $H_1, ..., H_s, 1 \leq s \leq r$ , are ideally independent (i.e. none of the  $H_i$ 's is in the ideal generated by the others) and  $H_{s+j} \in (H_1, ..., H_s)$ .  $1 \leq j \leq r - s$ . Thus  $H_{s+j} = \sum_{i=1}^{s} V_{ji} H_i$ ,  $1 \leq j \leq r - s$ , where each  $V_{ji}$  is a homogeneous polynomial of degree  $d_i - d_{s+j} (V_{ji}$  is interpreted to be 0 if this degree is negative). Differentiating the relation  $H(I_1(x), ..., I_r(x)) = 0$  with respect to  $x_k$ , we obtain

(2.1) 
$$\sum_{i=1}^{r} H_{i} \frac{\partial I_{i}}{\partial x_{k}} = \sum_{i=1}^{s} H_{i} \frac{\partial I_{i}}{\partial x_{k}} + \sum_{l=1}^{r-s} H_{s+l} \frac{\partial I_{s+l}}{\partial x_{k}}$$
$$= \sum_{i=1}^{s} H_{i} \left[ \frac{\partial I_{i}}{\partial x_{k}} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_{k}} \right] = 0.$$

Since

$$\frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k}$$

is homogeneous of degree  $d_i - 1$ , we conclude from Lemma 2.1 that

(2.2) 
$$\frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k} = \sum_{j=1}^r B_j I_j, \ 1 \leqslant i \leqslant s,$$

where the  $B_j$ 's are homogeneous and each term in (2.2) is homogeneous of degree  $d_i - 1$ . This forces  $B_i = 0$ . Multiply both sides of (2.2) by  $x_k$  and sum over k. We conclude, by Euler's identity for homogeneous polynomials,

(2.3) 
$$d_i I_i + \sum_{l=1}^{r-s} V_{li} d_{s+l} I_{s+l} = \sum_{j=1}^{r} A_j I_j ,$$

the  $A_i$ 's being homogeneous with  $A_i = 0$ .

(2.3) shows that  $I_i \in (I_1, ..., I_{i-1}, I_{i+1}, ..., I_r)$ , contradicting the minimality of the basis  $I_1, ..., I_r$ . Hence  $I_1, ..., I_r$  are algebraically independent and r = n.

### 2. The Theorem of Shephard and Todd

We obtain in this section a converse to Chevalley's Theorem, thereby obtaining an invariant theoretical characterization of finite reflection groups. We first prove several preliminary results.

LEMMA 2.2. Let H be a finite group of linear transformations acting on the n-dimensional space V and fixing the n-1 dimensional hyperplane  $\pi$ . The elements of H have a common eigenvector  $v \in V - \pi$ . Let  $\sigma(v) = \zeta(\sigma)v$ ,  $\sigma \in H$ .  $\zeta(\sigma)$  is an isomorphism from H into the multiplicative group of the roots of unity in k. It follows that H is a cyclic group.

REMARK. The above lemma is a consequence of Maschke's Theorem proven in section 2.3. We provide another proof below.

*Proof.* Let  $\sigma_1 \in H$ ,  $\sigma_1 \neq e$  (the identity of H). By the remark following Definition 2.1, there exists  $v \in V - \pi$  such that  $\sigma_1(v) = \zeta_1 v$ ,  $\zeta_1$  being a root of unity  $\neq 1$ . For  $\sigma \in H$ , let  $\sigma(v) = \zeta(\sigma)v + p(\sigma)$ ,  $\zeta(\sigma) \in k$  and  $p(\sigma) \in \pi$ . Let  $\sigma^* = \sigma_1^{-1} \sigma^{-1} \sigma_1 \sigma$ . Then  $\sigma^*(v) = v + (1 - \zeta_1) p(\sigma)$ . Since  $\sigma^*$  is of finite order,  $(1 - \zeta_1) p(\sigma) = 0 \Rightarrow p(\sigma) = 0$ . Hence  $\sigma(v) = \zeta(\sigma) v$ .  $\zeta(\sigma)$  is clearly an isomorphism from H into U, the multiplicative group of

the roots of unity in k. U is known to be cyclic ([22], Vol. 1, p. 112). It follows that  $\zeta(H)$ , a subgroup of U, is cyclic and so H is cyclic.

Theorem 2.2. Let G be a finite group acting on the n-dimensional space V. Let  $I_1, ..., I_n$  be homogeneous polynomials forming a basis for the invariants of G. Let  $d_1, ..., d_n$  be the respective degrees of  $I_1, ..., I_n$ . Then

(2.4) 
$$\prod_{i=1}^{n} d_{i} = |G|, \quad \sum_{i=1}^{n} (d_{i}-1) = r$$

where r = number of reflections in G.

*Proof.* By Theorem 1.2,  $I_1, ..., I_n$  are algebraically independent. Let I(x) be a homogeneous invariant of degree m. Then I is a linear combination of the monomials  $I_1^{a_1} ... I_n^{a_n}$  where  $a_1 d_1 + ... a_n d_n = m$ . Furthermore, these monomials are linearly independent over k, as  $I_1, ..., I_n$  are algebraically independent over k. It follows that the dimension  $\delta_m$  of homogeneous invariants of degree m = number of non-negative integer solutions to  $a_1 d_1 + ... + a_n d_n = m$ . Hence

(2.5) 
$$\sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{(1-t^{d_1}) \dots (1-t^{d_n})}.$$

(1.9) and (2.5) yield

$$(2.6) \frac{1}{|G|} \sum_{\sigma e G} \frac{1}{(1 - \omega_1(\sigma) t) \dots (1 - \omega_n(\sigma) t)} = \frac{1}{(1 - t^{d_1}) \dots (1 - t^{d_n})}$$

Expand both sides of (2.6) in powers of (1-t). Let  $\mathcal{R} = \text{set}$  of reflections in G and  $\zeta(\sigma) = \text{eigenvalue}$  of the reflection  $\sigma$  which  $\neq 1$ . We have

(2.7) 
$$\frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1 - \omega_{1})(\sigma)t) \dots (1 - \omega_{n}(\sigma)t)}$$

$$= \frac{1}{|G|} \frac{1}{(1 - t)^{n}} + \frac{1}{|G|} \sum_{\sigma \in \mathcal{R}} \frac{1}{1 - \zeta(\sigma)} \frac{1}{(1 - t)^{n-1}} + \dots$$
(2.8) 
$$\frac{1}{(1 - t^{d_{1}}) \dots (1 - t^{d_{n}})} = \prod_{i=1}^{n} \frac{1}{d_{i}(1 - t) - {d_{i} \choose 2}(1 - t)^{2} + \dots \pm (1 - t)^{d_{i}}}$$

$$= \frac{1}{\prod_{i=1}^{n} d_{i}(1 - t)^{n}} + \frac{\frac{1}{2} \sum_{i=1}^{n} (d_{i} - 1)}{\prod_{i=1}^{n} d_{i}} \frac{1}{(1 - t)^{n-1}} + \dots$$

Equating coefficients of (2.7), (2.8), we get

(2.9) 
$$\prod_{i=1}^{n} d_{i} = |G|, \sum_{i=1}^{n} (d_{i}-1) = 2 \sum_{\sigma \in \mathcal{R}} \frac{1}{1-\zeta(\sigma)}.$$

We evaluate the sum

$$\sum_{\sigma \in \mathcal{R}} \frac{1}{1 - \zeta(\sigma)} :$$

Let  $\pi$  be any r.h. Let  $H_{\pi} = \{ \sigma \mid \sigma \in G \text{ and } \sigma \text{ fixes } \pi \}$ . Thus  $H_{\pi}$  is the subgroup of G consisting of the identity and those reflections in G with r.h.  $\pi$ . Applying Lemma 2.2 to  $H_{\pi}$ , we conclude that there exists  $v \notin \pi$  such that  $\sigma(v) = \zeta(\sigma)v$  for  $\sigma \in H_{\pi}$ . Let  $H'_{\pi} = H_{\pi} - \{e\}$ . Since  $\zeta(\sigma^{-1}) = (\zeta(\sigma))^{-1}$ , we obtain

(2.10) 
$$\sum_{\sigma \varepsilon H_{\pi}^{'}} \frac{1}{1 - \zeta(\sigma)} = \sum_{\sigma \varepsilon H_{\pi}^{'}} \frac{1}{1 - \zeta(\sigma^{-1})}$$

$$= \sum_{\sigma \varepsilon H_{\pi}^{'}} \left(1 - \frac{1}{1 - \zeta(\sigma)}\right) = |H_{\pi}^{'}| - \sum_{\sigma \varepsilon H_{\pi}^{'}} \frac{1}{1 - \zeta(\sigma)}.$$

Hence

(2.11) 
$$\sum_{\sigma \in H_{\pi}'} \frac{1}{1 - \zeta(\sigma)} = \frac{|H_{\pi}'|}{2}.$$

Summing both sides of (2.11) over all r.h.  $\pi$ , we get

(2.12) 
$$\sum_{\sigma \in \mathcal{R}} \frac{1}{1 - \zeta(\sigma)} = \frac{r}{2} .$$

(2.9), (2.12) yield Theorem 2.2.

THEOREM 2.3. Let  $f_1, ..., f_n$  be polynomials in the variables  $x_1, ..., x_n$ .  $f_1, ..., f_n$  are algebraically independent over  $k \Leftrightarrow$ 

$$\frac{\partial (f_1, ..., f_n)}{\partial (x_1, ..., x_n)} \neq 0.$$

*Proof.* Suppose that  $f_1, ..., f_n$  are algebraically independent. Then  $G(f_1, ..., f_n) = 0$  for some polynomial  $G = G(y_1, ..., y_n)$ . Assume that  $G(y_1, ..., y_n)$  is of minimal positive degree. Differentiating this relation with respect to  $x_i$ , we get

(2.13) 
$$\sum_{i=1}^{n} \frac{\partial G}{\partial y_i}(f_1, ..., f_n) \frac{\partial f_i}{\partial x_j} = 0, \ 1 \leqslant j \leqslant n.$$

(2.13) is a system of linear equations (with coefficients in  $k(x_1, ..., x_n)$ ) in the unknowns  $H_i(x) = \frac{\partial G}{\partial y_i}(f_1, ..., f_n)$ ,  $1 \le i \le n$ .  $\frac{\partial G}{\partial y_i} \ne 0$  for some i, as G is not constant, and deg  $\frac{\partial G}{\partial y_i} < \deg G$ . It follows that the corresponding  $H_i(x) \ne 0$ . Thus the linear system (2.13) has a non-zero solution, so that its determinant

$$\frac{\partial (f_1, ..., f_n)}{\partial (x_1, ..., x_n)} \neq 0.$$

Conversely, let  $f_1, ..., f_n$  be algebraically independent. For each i,  $x_i, f_1, ..., f_n$  are algebraically dependent. Hence there exists a polynomial  $G_i$   $(x_i, y_1, ..., y_n)$  of minimal positive degree in  $x_i$  such that  $G_i(x_i, f_1, ..., f_n) = 0$ . Differentiating these relations with respect to  $x_k$ , we get

(2.14) 
$$\sum_{j=1}^{n} \frac{\partial G_{i}}{\partial y_{j}}(x_{i}, f_{1}, ..., f_{n}) \frac{\partial f_{j}}{\partial x_{k}} + \frac{\partial G_{i}}{\partial x_{k}}(x_{i}, f_{1}, ..., f_{n}) \delta_{ik}, 1 \leq k \leq n,$$

 $\delta_{ik}$  denoting the Kronecker symbol. (2.14) may be rewritten in matrix notation as

$$\left(\frac{\partial G_i}{\partial y_j}\right) \cdot \left(\frac{\partial f_i}{\partial x_j}\right) = D$$

where the entries of D are

$$-\delta_{ij} \frac{\partial G_i}{\partial x_j}.$$

det  $D \neq 0$ , as  $x_i$  – degree of  $\frac{\partial G_i}{\partial x_i} < x_i$  – degree of  $G_i$ ,  $1 \leqslant i \leqslant n$ .

It follows from (2.15) that  $\frac{\partial (f_1, ..., f_n)}{\partial (x_1, ..., x_n)} \neq 0$ .

Theorem 2.4. (Shephard and Todd [19]). Let G be a finite group acting on the n-dimensional space V. Suppose there exists a basis of n homogeneous polynomials for the invariants of G. Then G is a finite reflection group.

*Proof.* Let H be the subgroup of G generated by the reflections in G. By assumption G has n basic homogeneous invariants which, by Theorem 1.2, are algebraically independent. Since H is a finite reflection group, we conclude from Chevalley's Theorem that H has n basic homogeneous invariants  $J_1, ..., J_n$  which are algebraically independent. Each  $I_i$  is invariant under H so that  $I_i = I_i (J_1, ..., J_n)$ , the latter quantity denoting a polynomial in the  $J_i$ 's. We may assume that  $I_i (J_1, ..., J_n)$  is a linear combination of monomials  $J_1^{a_1} ... J_n^{a_n}$  whose x-degree  $= \deg I_i$ . We have

(2.16) 
$$\frac{\partial (I_1, ..., I_n)}{\partial (x_1, ..., x_n)} = \frac{\partial (I_1, ..., I_n)}{\partial (J_1, ..., J_n)} \cdot \frac{\partial (J_1, ..., J_n)}{\partial (x_1, ..., x_n)}$$

By Theorem 2.3,

$$\frac{\partial (I_1, ..., I_n)}{\partial (x_1, ..., x_n)} \neq 0$$

and (2.16) then shows that

$$\frac{\partial (I_1, ..., I_n)}{\partial (J_1, ..., J_n)} \neq 0.$$

It follows that there is a rearrangement  $k_1, ..., k_n$  of 1, ..., n so that

$$\frac{\partial I_{k_1}}{\partial J_1} \dots \frac{\partial I_{k_n}}{\partial J_n} \neq 0.$$

Hence  $I_{k_i}(J_1, ..., J_n)$  is of positive degree in  $J_i$  and  $\deg I_{k_i} \geqslant \deg J_i$ ,  $1 \leqslant i \leqslant n$ . Applying Theorem 2.2 both to G and H, we obtain

(2.17) 
$$\prod_{i=1}^{n} \deg J_{i} = |H|, \prod_{i=1}^{n} \deg I_{i} = |G|$$

(2.18) 
$$\sum_{i=1}^{n} (\deg J_i - 1) = \sum_{i=1}^{n} (\deg I_i - 1) = r$$

where r = number of reflections in G = number of reflections in H.

Since  $\deg I_{k_i} \geqslant \deg J_i$ ,  $1 \leqslant i \leqslant n$ , we conclude from (2.18) that  $\deg I_{k_i} = \deg J_i$ ,  $1 \leqslant i \leqslant n$ . Hence  $\prod_{i=1}^n \deg I_i = \prod_{i=1}^n \deg J_i$ , and we conclude from (2.17) that |G| = |H|. Thus G = H and G is a finite reflection group.

3. A FORMULA FOR 
$$\frac{\partial (I_1, ..., I_n)}{\partial (x_1, ..., x_n)}$$

We obtain a formula which shall be used in Chapter III.

THEOREM 2.5. Let G be a finite reflection group acting on the n-dimensional space V. Let  $I_1, ..., I_n$  be a basic set of homogeneous invariants for G. Let x be a coordinate system for V and  $L_i(x) = 0, 1 \le i \le r$ , the r.h.'s for G, each  $L_i$  being linear and homogeneous. Then

(2.19) 
$$\frac{\partial (I_1, ..., I_n)}{\partial (x_1, ..., x_n)} = c \prod_{i=1}^r L_i(x)$$

c being a constant  $\neq 0$ .

Proof. Let J the left hand side of (2.19). We observe that J is a non-zero homogeneous polynomial of degree  $\sum_{i=1}^{n} (d_i - 1)$ . By Theorem 2.2,  $\sum_{i=1}^{n} (d_i - 1) = r$ , so that deg J = r. If k is the real field R, we have the following simple proof of (2.19).  $I_i = I_i(x_1, ..., x_n)$ ,  $1 \le i \le n$ , is a mapping from x-space to I-space. This mapping is not 1 - 1 in any neighborhood of a point x lying in the r.h.  $L_i(x) = 0$ , as any point and its reflection get mapped into the same point I. It follows from the Implicit Function Theorem that J(x) = 0, whenever  $L_i(x) = 0$ . Thus  $L_i \mid J$ ,  $1 \le i \le r$ , and so  $\prod_{i=1}^{r} L_i \mid J$ . Since J,  $\prod_{i=1}^{r} L_i$  have the same degree r, we have  $J = c \prod_{i=1}^{r} L_i$ ,  $c \ne 0$ .

For an arbitrary field k, the theorem is proven as follows. Let  $\pi$  be an r.h. with equation L(x)=0 and H the subgroup of h elements in G fixing  $\pi$ . Thus there are h-1 reflections in G with r.h.  $\pi$ . We show that  $L^{h-1} \mid J$ . By Lemma 2.2, H is a cyclic group generated by an element  $\sigma$ . Furthermore there exists  $v \notin \pi$  and a primitive h-th root of 1 such that  $\sigma(v)=\zeta v$ . Choose a coordinate system  $y=(y_1,...,y_n)$  in V so that  $\pi$  has the equation  $y_n=0$  and v=(0,...,0,1)  $\sigma$  then becomes the transformation  $(y_1,...,y_{n-1},y_n) \to (y_1,...,y_{n-1},\zeta y_n)$ . Let  $x=\tau y$  and  $J_i(y)=I_i(\tau y), 1 \leqslant i \leqslant n$ . We have

$$(2.20) J_i(y_1, ..., y_{n-1}, \zeta y_n) = J_i(y_1, ..., y_{n-1}, y_n), \ 1 \le i \le n$$

Let  $J_i = \sum A_m y_n^m$ , the  $A_m$ 's being polynomials in  $y_1, ..., y_{n-1}$ . (2.20) implies that  $A_m = 0$  whenever  $h \nmid m$ , so that  $A_m = 0$ ,  $0 \le m \le h - 1$ . Since

$$\frac{\partial J_i}{\partial y_m} = \Sigma_m A_m y_n^{m-1},$$

we conclude

$$y_n^{h-1} \left| \frac{\partial J_i}{\partial y_n}, 1 \leqslant i \leqslant n \right|.$$

Hence

(2.21) 
$$y_n^{h-1} \left| \frac{\partial (J_1, ..., J_n)}{\partial (y_1, ..., y_n)} \right|,$$

Since

$$\frac{\partial (J_1, ..., J_n)}{\partial (y_1, ..., y_n)} = J(x) \cdot \det \tau,$$

(2.21) is equivalent to  $L^{h-1}(x) \mid J(x)$ . It follows that if  $L_i(x) = 0$ ,  $1 \le i \le r$ , are the r.h.'s for G, then  $\prod_{i=1}^r L_i \mid J$ . But J,  $\prod_{i=1}^r L_i$  have the same degree r, so that  $J = c \prod_{i=1}^r L_i c \ne 0$ .

# 4. Decomposition of Finite Reflection Groups

We shall decompose every finite reflection group into a direct product of irreducible ones and show that it suffices to study the invariant theory of the irreducible groups.

DEFINITION 2.3. Let the group G act on V. G is said to be reducible iff there exists a proper subspace W invariant under G; i.e.  $\sigma w \in W$  for  $\sigma \in G$ ,  $w \in W$ . G is said to be completely reducible iff  $V = V_1 \oplus V_2$ ,  $V_1$  and  $V_2$  being proper invariant subspaces. G is said to be irreducible iff it is not reducible.

Theorem 2.6. (Maschke [22], Vol. 2, p. 179). Let G be a finite group acting on the vector space V. If G is reducible, then it is completely reducible.

*Proof.* Let  $V_1$  be a proper invariant subspace of V. Let  $V_2$  be a complementary subspace. Thus for  $v \in V$ , we have a unique decomposition

 $v = v_1 + v_2, v_i \in V_i$  (i = 1, 2). Let  $\eta v = v_2$  and set  $\tau = \frac{1}{|G|} \sum_{\sigma \in G} \sigma \eta \sigma^{-1}$ .  $\tau$  satisfies the following:

i) 
$$\tau \sigma = \sigma \tau$$
,  $\sigma \in G$ . For  $\sigma \tau = \frac{1}{|G|} \sum_{\sigma_1 \in G} \sigma \sigma_1 \eta (\sigma \sigma_1)^{-1} \sigma = \tau \sigma$ 

- ii)  $\tau v_1 = 0$ ,  $v_1 \in V_1$ . For  $\sigma^{-1} v_1 \in V_1$ ,  $\sigma \in G$ , so that  $\eta \sigma^{-1} v_1 = 0$   $\Rightarrow \tau v_1 = 0$
- iii)  $(1-\tau)$   $v \in V_1$ ,  $v \in V$ , 1 denoting the identity of G. For  $(1-\eta)$   $v \in V_1$ , so that  $(1-\eta)$   $\sigma^{-1}$   $v \in V_1$   $\Rightarrow \sigma$   $(1-\eta)$   $\sigma^{-1}$   $v \in V_1$ ,  $\sigma \in G$ . It follows that  $(1-\tau)$   $v = \frac{1}{\mid G \mid} \sum_{\sigma \in G} \sigma (1-\eta) \sigma^{-1} v \in V_1$ .

Let  $V_2' = \tau V$ .  $V_2'$  is invariant under G as  $\sigma(\tau v) = \tau(\sigma v)$ . For any v,  $v = \tau v + (1-\tau)v$ . It follows from iii) that  $V = V_1 + V_2'$ . ii), iii) imply  $\tau(1-\tau) = 0 \Leftrightarrow \tau = \tau^2$ . Hence  $\tau v_2' = v_2'$  for  $v_2' \in V_2'$ . Let  $v_1 + v_2' = 0$ , where  $v_1 \in V_1$ ,  $v_2' \in V_2'$ . Applying  $\tau$  to both sides, we get  $v_2' = 0$  and so  $v_1 = 0$ . Hence  $V = V_1 \oplus V_2'$ .

Repeated application of Maschke's Theorem yields the

COROLLARY. Let G be a finite group acting on the finite-dimensional vector space V. Then  $V = V_1 \oplus ... \oplus V_s$ , the  $V_i$ 's being invariant subspaces of V and G acting irreducibly on each  $V_i$ .

For finite reflection groups, we have

Theorem 2.7. Let G be a finite reflection group acting on V. There exists a decomposition  $V=V_1\oplus\ldots\oplus V_s$  into invariant subspaces such that:

- 1) Let  $G_i = G|_{V_i} = \text{group of restrictions of elements of } G \text{ to } V_i$ . Then G is isomorphic to  $G_1 \times ... \times G_s$
- 2) Each  $G_i$ ,  $1 \le i \le s$ , is a reflection group acting irreducibly on  $V_i$ .

*Proof.* By the corollary to Theorem 2.6, there exists a decomposition  $V = V_1 \oplus ... \oplus V_s$ , the  $V_i$ 's being invariant subspaces and  $G_i$  irreducible for  $1 \le i \le s$ . We label the  $V_i$ 's so that  $V_1, ..., V_r$  are 1-dimensional and  $G|_{V_i} = \text{identity}$ .

By the remark following Definition 2.1, for each reflection  $\sigma$  there exists an eigenvector  $v \in V - \pi$ ,  $\pi$  being the r.h. for  $\sigma$ . Call v a root of G. We have

$$(2.22) \qquad \dim (V_i + \pi) + \dim (V_i \cap \pi) = \dim V_i + \dim \pi.$$

If  $V_i \not = \pi$ , then  $V_i + \pi = V$  and we conclude from (2.22) that dim  $V_i$  = dim  $(V_i \cap \pi) + 1$ . I.e.  $V_i \cap \pi$  is a hyperplane in  $V_i$  and  $\sigma |_{V_i}$  a reflection on  $V_i$ . Choose  $u \in V_i - \pi$  so that u is an eigenvector of  $\sigma$ . u is a multiple of the root v, so that  $v \in V_i$ . Thus  $\sigma |_{V_i}$  is a reflection of  $V_i$  if  $v \in V_i$ , and the identity if  $v \notin V_i$ . Furthermore, each root v is in some  $V_i$ ,  $v \in V_i$ , otherwise the corresponding reflection  $\sigma$  would have been the identity.

Let  $G_i$  = subgroup generated by those reflections whose roots are in  $V_i$ ,  $1 \le i \le s$ . It is readily checked that  $G = G_1 \times ... \times G_s$ ,  $G_i = G_i \mid_{V_i}$ . If  $\sigma \in G_i$  and  $\sigma \mid_{V_i} = identity$  then  $\sigma = identity$ . The mapping  $\sigma \to \sigma \mid_{V_i}$  is thus an isomorphism from  $G_i$  onto  $G_i$ .

Theorem 2.8. Let G be a finite reflection group acting on V and decompose V as in Theorem 2.7. Every polynomial invariant under G is a polynomial in the invariant polynomials of  $G_1, ..., G_s$ .

*Proof.* For each  $v \in V$ , write  $v = v_1 + ... + v_s$ ,  $v_i \in V_i$ . By Theorem 2.7, for each  $\sigma \in G$ , we may write  $\sigma v = \sigma_1 v_1 + ... + \sigma_s v_s$ ,  $\sigma_i \in G_i$ . For any polynomial function p(v) on V, we have  $p(v) = \sum_{i=1}^{N} p_{i1}(v_1) ... p_{is}(v_s)$  where  $p_{ij}(v_j)$  is a polynomial function on  $V_j$ . If p(v) is invariant under G, then

(2.23) 
$$p(v) = \frac{1}{|G|} \sum_{\sigma \in G} p(\sigma v) = \sum_{i=1}^{N} I_{i1}(v_1) \dots I_{is}(v_s)$$

where

$$(2.24) I_{ij}(v_j) = \frac{1}{|G_j|} \sum_{\sigma_j \in G_j} p_{ij}(\sigma_j v_j)$$

is an invariant of  $G_j$ .

## CHAPTER III

# THE DEGREES OF THE BASIC INVARIANTS

We determine the degrees of the basic homogeneous invariants in case G is a finite reflection group. We present two different methods. The first one (Theorem 3.8), restricts itself to the case where k is the real field and has the advantage of providing an effective method for computing the