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(1.10) and Lemma 1.2 yield

$$(1.11) \quad \delta_m = \frac{1}{|G|} \sum_{\sigma \in G} (\text{Tr } \sigma)_m = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma).$$

Multiply both sides of (1.11) by  $t^m$  and sum over  $m$  from 0 to  $\infty$ . We get

$$\begin{aligned} \sum_{m=0}^{\infty} \delta_m t^m &= \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma) t^m \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \left\{ \sum_{m=0}^{\infty} \omega_1^m(\sigma) t^m \dots \sum_{m=0}^{\infty} \omega_n^m(\sigma) t^m \right\} \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1 - \omega_1(\sigma) t) \dots (1 - \omega_n(\sigma) t)} \end{aligned}$$

## CHAPTER II

### INVARIANT THEORETIC CHARACTERIZATION OF FINITE REFLECTION GROUPS

#### 1. CHEVALLEY'S THEOREM

We showed in chapter I that we can always find a finite number of homogeneous invariants forming a basis for the invariants of  $G$  and that this set must contain at least  $n$  elements, where  $n = \dim V$ . We show that this lower bound is attained only for the finite reflection groups. We first define these groups.

**DEFINITION 2.1.** Let  $\sigma$  be a linear transformation acting on the  $n$ -dimensional vector space  $V$ .  $\sigma$  is a reflection  $\Leftrightarrow \sigma$  fixes an  $n - 1$  dimensional hyperplane  $\pi$  and  $\sigma$  is of finite order  $> 1$ .  $\pi$  is called the reflecting hyperplane (r.h.) of  $\sigma$ .

**REMARK.** Choose  $v \notin \pi$ . and let  $\sigma v = \zeta v + p$ ,  $p \in \pi$ . If  $\zeta = 1$ , then  $\sigma^m v = v + mp$ , contradicting that  $\sigma$  is of finite order. Hence  $\zeta \neq 1$ . Let  $v' = v + (\zeta - 1)^{-1} p$  and choose  $p_1, \dots, p_{n-1}$  as a basis for  $\pi$ . Then  $\sigma p_i = p_i$ ,  $1 \leq i \leq n - 1$ ,  $\sigma v' = \zeta v'$ .  $\zeta$  is a root of 1 in  $k$  which is distinct from 1, as  $\sigma$  is of finite order  $> 1$ . Thus  $\sigma$  is a reflection iff relative to some basis, the matrix for  $\sigma$  is diagonal,  $n - 1$  of the diagonal entries equalling 1 and the remaining one equalling a root of 1 in  $k$  distinct from 1.

DEFINITION 2.2.  $G$  is a finite reflection group acting on  $V \Leftrightarrow G$  is a finite group generated by reflections on  $V$ .

As an example of a finite reflection group, let  $G = S_n$ . It is well known that  $S_n$  is generated by transpositions. The transposition of the variables  $x_i, x_j$  ( $i \neq j$ ) fixes the hyperplane  $x_i - x_j = 0$ , so that it is a reflection.

We have the following result

THEOREM 2.1 (Chevalley [4]). *Let  $G$  be a finite reflection group acting on the  $n$ -dimensional vector space  $V$ . The invariants of  $G$  have a basis consisting of  $n$  homogeneous elements which are algebraically independent over  $k$ .*

Let  $k[x]$  denote the ring of polynomials in  $x_1, \dots, x_n$  with coefficients in  $k$ . We prove the following.

LEMMA 2.1. Let  $I_1, \dots, I_m$  be invariant polynomials of  $G$ ,  $I_1 \notin (I_2, \dots, I_m)$  = the ideal in  $k[x]$  generated by  $I_2, \dots, I_m$ . Suppose that  $P_1 I_1 + \dots + P_m I_m = 0$ , the  $P_i$ 's being polynomials with  $P_1$  homogeneous. Then  $P_1 \in \mathcal{J}$ , where  $\mathcal{J}$  is the ideal in  $k[x]$  generated by the homogeneous invariants of positive degree.

*Proof of Lemma 2.1.* The proof proceeds by induction on  $\deg P_1$ . Suppose  $\deg P_1 = 0$ , so that  $P_1 = c \in k$ . If  $c \neq 0$ , then  $I_1 \in (I_2, \dots, I_m)$ , contrary to assumption. Hence  $c = 0 \Rightarrow P_1 \in \mathcal{J}$ . Let  $\deg P_1 = n > 0$ . Let  $\sigma$  be a reflection in  $G$  and  $L = 0$  the equation of its r.h. ( $L$  is a linear homogeneous polynomial). We have  $P_1(x) I_1(x) + \dots + P_m(x) I_m(x) = 0$ ,  $P_1(\sigma x) I_1(x) + \dots + P_m(\sigma x) I_m(x) = 0$ . Hence  $[P_1(\sigma x) - P_1(x)] I_1(x) + \dots + [P_m(\sigma x) - P_m(x)] I_m(x) = 0$ . For  $L(x) = 0$ ,  $\sigma(x) = x$ , so that  $P_i(\sigma x) - P_i(x) = 0$  whenever  $L(x) = 0$ ,  $1 \leq i \leq m$ . Since  $L(x)$  is irreducible it follows that

$$\frac{P_i(\sigma x) - P_i(x)}{L(x)}$$

is a polynomial,  $1 \leq i \leq m$ . We have

$$\left[ \frac{P_1(\sigma x) - P_1(x)}{L(x)} \right] I_1(x) + \dots + \left[ \frac{P_m(\sigma x) - P_m(x)}{L(x)} \right] I_m(x) = 0.$$

$$\deg \left[ \frac{P_1(\sigma x) - P_1(x)}{L(x)} \right] < \deg P_1(x),$$

so that by the induction hypothesis

$$\frac{P_1(\sigma x) - P_1(x)}{L(x)} \equiv 0 \pmod{\mathcal{J}}.$$

Hence  $P_1(\sigma x) \equiv P_1(x) \pmod{\mathcal{J}}$ . Since the  $\sigma$ 's generate  $G$ , this congruence holds for  $\sigma \in G$ . We conclude that

$$P_1(x) \equiv \frac{1}{|G|} \sum_{\sigma \in G} P_1(\sigma x) \pmod{\mathcal{J}}.$$

The polynomial  $\frac{1}{|G|} \sum_{\sigma \in G} P_1(\sigma x)$  is invariant and homogeneous of degree  $n \geq 1$ . Hence it  $\in \mathcal{J}$ , so that  $P_1 \in \mathcal{J}$ .

*Proof of Theorem 2.1.* We choose  $I_1, \dots, I_r$  to be homogeneous invariants of positive degree forming a minimal basis for  $\mathcal{J}$ . Hilbert's proof of Theorem 1.1 shows that  $I_1, \dots, I_r$  form a basis for the invariants of  $G$ . We show that  $I_1, \dots, I_r$  are algebraically independent, so that  $r = n$ .

Suppose, to the contrary, that  $I_1, \dots, I_r$  are algebraically dependent. Choose  $H(y_1, \dots, y_r)$  to be a polynomial of minimal positive degree so that  $H(I_1(x), \dots, I_r(x)) = 0$ . Let  $x$ -degree of any monomial  $y_1^{a_1} \dots y_r^{a_r}$  be  $d_1 a_1 + \dots + d_r a_r$ , where  $d_i = \deg I_i$ . We may assume that all  $x$ -degrees of the monomials appearing in  $H$  are the same. Let

$$H_i(x) = \frac{\partial H}{\partial y_i}(I_1(x), \dots, I_r(x)), \quad 1 \leq i \leq r.$$

The  $H_i$ 's are invariant homogeneous polynomials, as all monomials in  $H$  have equal  $x$ -degree. Since  $H(y_1, \dots, y_r)$  is of positive degree, some  $\frac{\partial H}{\partial y_i} \neq 0$ . It follows that the corresponding  $H_i(x) \neq 0$ , as  $H$  was chosen

to be of minimal degree; i.e. not all  $H_i$ 's = 0. We relabel indices so that  $H_1, \dots, H_s, 1 \leq s \leq r$ , are ideally independent (i.e. none of the  $H_i$ 's is in the ideal generated by the others) and  $H_{s+j} \in (H_1, \dots, H_s), 1 \leq j \leq r - s$ .

Thus  $H_{s+j} = \sum_{i=1}^s V_{ji} H_i, 1 \leq j \leq r - s$ , where each  $V_{ji}$  is a homogeneous polynomial of degree  $d_i - d_{s+j}$  ( $V_{ji}$  is interpreted to be 0 if this degree is negative). Differentiating the relation  $H(I_1(x), \dots, I_r(x)) = 0$  with respect to  $x_k$ , we obtain

$$\begin{aligned} (2.1) \quad \sum_{i=1}^r H_i \frac{\partial I_i}{\partial x_k} &= \sum_{i=1}^s H_i \frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} H_{s+l} \frac{\partial I_{s+l}}{\partial x_k} \\ &= \sum_{i=1}^s H_i \left[ \frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k} \right] = 0. \end{aligned}$$



Since

$$\frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k}$$

is homogeneous of degree  $d_i - 1$ , we conclude from Lemma 2.1 that

$$(2.2) \quad \frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k} = \sum_{j=1}^r B_j I_j, \quad 1 \leq i \leq s,$$

where the  $B_j$ 's are homogeneous and each term in (2.2) is homogeneous of degree  $d_i - 1$ . This forces  $B_i = 0$ . Multiply both sides of (2.2) by  $x_k$  and sum over  $k$ . We conclude, by Euler's identity for homogeneous polynomials,

$$(2.3) \quad d_i I_i + \sum_{l=1}^{r-s} V_{li} d_{s+l} I_{s+l} = \sum_{j=1}^r A_j I_j,$$

the  $A_j$ 's being homogeneous with  $A_i = 0$ .

(2.3) shows that  $I_i \in (I_1, \dots, I_{i-1}, I_{i+1}, \dots, I_r)$ , contradicting the minimality of the basis  $I_1, \dots, I_r$ . Hence  $I_1, \dots, I_r$  are algebraically independent and  $r = n$ .

## 2. THE THEOREM OF SHEPHARD AND TODD

We obtain in this section a converse to Chevalley's Theorem, thereby obtaining an invariant theoretical characterization of finite reflection groups. We first prove several preliminary results.

LEMMA 2.2. Let  $H$  be a finite group of linear transformations acting on the  $n$ -dimensional space  $V$  and fixing the  $n - 1$  dimensional hyperplane  $\pi$ . The elements of  $H$  have a common eigenvector  $v \in V - \pi$ . Let  $\sigma(v) = \zeta(\sigma)v$ ,  $\sigma \in H$ .  $\zeta(\sigma)$  is an isomorphism from  $H$  into the multiplicative group of the roots of unity in  $k$ . It follows that  $H$  is a cyclic group.

REMARK. The above lemma is a consequence of Maschke's Theorem proven in section 2.3. We provide another proof below.

*Proof.* Let  $\sigma_1 \in H$ ,  $\sigma_1 \neq e$  (the identity of  $H$ ). By the remark following Definition 2.1, there exists  $v \in V - \pi$  such that  $\sigma_1(v) = \zeta_1 v$ ,  $\zeta_1$  being a root of unity  $\neq 1$ . For  $\sigma \in H$ , let  $\sigma(v) = \zeta(\sigma)v + p(\sigma)$ ,  $\zeta(\sigma) \in k$  and  $p(\sigma) \in \pi$ . Let  $\sigma^* = \sigma_1^{-1} \sigma \sigma_1$ . Then  $\sigma^*(v) = v + (1 - \zeta_1)p(\sigma)$ . Since  $\sigma^*$  is of finite order,  $(1 - \zeta_1)p(\sigma) = 0 \Rightarrow p(\sigma) = 0$ . Hence  $\sigma(v) = \zeta(\sigma)v$ .  $\zeta(\sigma)$  is clearly an isomorphism from  $H$  into  $U$ , the multiplicative group of

the roots of unity in  $k$ .  $U$  is known to be cyclic ([22], Vol. 1, p. 112). It follows that  $\zeta(H)$ , a subgroup of  $U$ , is cyclic and so  $H$  is cyclic.

**THEOREM 2.2.** *Let  $G$  be a finite group acting on the  $n$ -dimensional space  $V$ . Let  $I_1, \dots, I_n$  be homogeneous polynomials forming a basis for the invariants of  $G$ . Let  $d_1, \dots, d_n$  be the respective degrees of  $I_1, \dots, I_n$ . Then*

$$(2.4) \quad \prod_{i=1}^n d_i = |G|, \quad \sum_{i=1}^n (d_i - 1) = r$$

where  $r = \text{number of reflections in } G$ .

*Proof.* By Theorem 1.2,  $I_1, \dots, I_n$  are algebraically independent. Let  $I(x)$  be a homogeneous invariant of degree  $m$ . Then  $I$  is a linear combination of the monomials  $I_1^{a_1} \dots I_n^{a_n}$  where  $a_1 d_1 + \dots + a_n d_n = m$ . Furthermore, these monomials are linearly independent over  $k$ , as  $I_1, \dots, I_n$  are algebraically independent over  $k$ . It follows that the dimension  $\delta_m$  of homogeneous invariants of degree  $m = \text{number of non-negative integer solutions to } a_1 d_1 + \dots + a_n d_n = m$ . Hence

$$(2.5) \quad \sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{(1-t^{d_1}) \dots (1-t^{d_n})}.$$

(1.9) and (2.5) yield

$$(2.6) \quad \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1-\omega_1(\sigma)t) \dots (1-\omega_n(\sigma)t)} = \frac{1}{(1-t^{d_1}) \dots (1-t^{d_n})}$$

Expand both sides of (2.6) in powers of  $(1-t)$ . Let  $\mathcal{R} = \text{set of reflections in } G$  and  $\zeta(\sigma) = \text{eigenvalue of the reflection } \sigma \text{ which } \neq 1$ . We have

$$(2.7) \quad \begin{aligned} & \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1-\omega_1(\sigma)t) \dots (1-\omega_n(\sigma)t)} \\ &= \frac{1}{|G|} \frac{1}{(1-t)^n} + \frac{1}{|G|} \sum_{\sigma \in \mathcal{R}} \frac{1}{1-\zeta(\sigma)} \frac{1}{(1-t)^{n-1}} + \dots \end{aligned}$$

$$(2.8) \quad \begin{aligned} & \frac{1}{(1-t^{d_1}) \dots (1-t^{d_n})} = \prod_{i=1}^n \frac{1}{d_i(1-t) - \binom{d_i}{2}(1-t)^2 + \dots \pm (1-t)^{d_i}} \\ &= \frac{1}{\prod_{i=1}^n d_i} \frac{1}{(1-t)^n} + \frac{\frac{1}{2} \sum_{i=1}^n (d_i - 1)}{\prod_{i=1}^n d_i} \frac{1}{(1-t)^{n-1}} + \dots \end{aligned}$$

Equating coefficients of (2.7), (2.8), we get

$$(2.9) \quad \prod_{i=1}^n d_i = |G|, \quad \sum_{i=1}^n (d_i - 1) = 2 \sum_{\sigma \in \mathcal{R}} \frac{1}{1 - \zeta(\sigma)}.$$

We evaluate the sum

$$\sum_{\sigma \in \mathcal{R}} \frac{1}{1 - \zeta(\sigma)} :$$

Let  $\pi$  be any r.h. Let  $H_\pi = \{\sigma \mid \sigma \in G \text{ and } \sigma \text{ fixes } \pi\}$ . Thus  $H_\pi$  is the subgroup of  $G$  consisting of the identity and those reflections in  $G$  with r.h.  $\pi$ . Applying Lemma 2.2 to  $H_\pi$ , we conclude that there exists  $v \notin \pi$  such that  $\sigma(v) = \zeta(\sigma)v$  for  $\sigma \in H_\pi$ . Let  $H'_\pi = H_\pi - \{e\}$ . Since  $\zeta(\sigma^{-1}) = (\zeta(\sigma))^{-1}$ , we obtain

$$(2.10) \quad \begin{aligned} \sum_{\sigma \in H'_\pi} \frac{1}{1 - \zeta(\sigma)} &= \sum_{\sigma \in H'_\pi} \frac{1}{1 - \zeta(\sigma^{-1})} \\ &= \sum_{\sigma \in H'_\pi} \left(1 - \frac{1}{1 - \zeta(\sigma)}\right) = |H'_\pi| - \sum_{\sigma \in H'_\pi} \frac{1}{1 - \zeta(\sigma)}. \end{aligned}$$

Hence

$$(2.11) \quad \sum_{\sigma \in H'_\pi} \frac{1}{1 - \zeta(\sigma)} = \frac{|H'_\pi|}{2}.$$

Summing both sides of (2.11) over all r.h.  $\pi$ , we get

$$(2.12) \quad \sum_{\sigma \in \mathcal{R}} \frac{1}{1 - \zeta(\sigma)} = \frac{r}{2}.$$

(2.9), (2.12) yield Theorem 2.2.

**THEOREM 2.3.** *Let  $f_1, \dots, f_n$  be polynomials in the variables  $x_1, \dots, x_n$ .  $f_1, \dots, f_n$  are algebraically independent over  $k \Leftrightarrow$*

$$\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \neq 0.$$

*Proof.* Suppose that  $f_1, \dots, f_n$  are algebraically independent. Then  $G(f_1, \dots, f_n) = 0$  for some polynomial  $G = G(y_1, \dots, y_n)$ . Assume that  $G(y_1, \dots, y_n)$  is of minimal positive degree. Differentiating this relation with respect to  $x_j$ , we get

$$(2.13) \quad \sum_{i=1}^n \frac{\partial G}{\partial y_i} (f_1, \dots, f_n) \frac{\partial f_i}{\partial x_j} = 0, \quad 1 \leq j \leq n.$$

(2.13) is a system of linear equations (with coefficients in  $k(x_1, \dots, x_n)$ ) in the unknowns  $H_i(x) = \frac{\partial G}{\partial y_i} (f_1, \dots, f_n)$ ,  $1 \leq i \leq n$ .  $\frac{\partial G}{\partial y_i} \neq 0$  for some  $i$ , as  $G$  is not constant, and  $\deg \frac{\partial G}{\partial y_i} < \deg G$ . It follows that the corresponding  $H_i(x) \neq 0$ . Thus the linear system (2.13) has a non-zero solution, so that its determinant

$$\frac{\partial (f_1, \dots, f_n)}{\partial (x_1, \dots, x_n)} \neq 0.$$

Conversely, let  $f_1, \dots, f_n$  be algebraically independent. For each  $i$ ,  $x_i, f_1, \dots, f_n$  are algebraically dependent. Hence there exists a polynomial  $G_i(x_i, y_1, \dots, y_n)$  of minimal positive degree in  $x_i$  such that  $G_i(x_i, f_1, \dots, f_n) = 0$ . Differentiating these relations with respect to  $x_k$ , we get

$$(2.14) \quad \sum_{j=1}^n \frac{\partial G_i}{\partial y_j} (x_i, f_1, \dots, f_n) \frac{\partial f_j}{\partial x_k} + \frac{\partial G_i}{\partial x_k} (x_i, f_1, \dots, f_n) \delta_{ik}, \quad 1 \leq k \leq n,$$

$\delta_{ik}$  denoting the Kronecker symbol. (2.14) may be rewritten in matrix notation as

$$(2.15) \quad \left( \frac{\partial G_i}{\partial y_j} \right) \cdot \left( \frac{\partial f_i}{\partial x_j} \right) = D$$

where the entries of  $D$  are

$$- \delta_{ij} \frac{\partial G_i}{\partial x_j}.$$

$\det D \neq 0$ , as  $x_i - \text{degree of } \frac{\partial G_i}{\partial x_i} < x_i - \text{degree of } G_i$ ,  $1 \leq i \leq n$ .

It follows from (2.15) that  $\frac{\partial (f_1, \dots, f_n)}{\partial (x_1, \dots, x_n)} \neq 0$ .

**THEOREM 2.4.** (Shephard and Todd [19]). *Let  $G$  be a finite group acting on the  $n$ -dimensional space  $V$ . Suppose there exists a basis of  $n$  homogeneous polynomials for the invariants of  $G$ . Then  $G$  is a finite reflection group.*

*Proof.* Let  $H$  be the subgroup of  $G$  generated by the reflections in  $G$ . By assumption  $G$  has  $n$  basic homogeneous invariants which, by Theorem 1.2, are algebraically independent. Since  $H$  is a finite reflection group, we conclude from Chevalley's Theorem that  $H$  has  $n$  basic homogeneous invariants  $J_1, \dots, J_n$  which are algebraically independent. Each  $I_i$  is invariant under  $H$  so that  $I_i = I_i(J_1, \dots, J_n)$ , the latter quantity denoting a polynomial in the  $J_i$ 's. We may assume that  $I_i(J_1, \dots, J_n)$  is a linear combination of monomials  $J_1^{a_1} \dots J_n^{a_n}$  whose  $x$ -degree =  $\deg I_i$ . We have

$$(2.16) \quad \frac{\partial(I_1, \dots, I_n)}{\partial(x_1, \dots, x_n)} = \frac{\partial(I_1, \dots, I_n)}{\partial(J_1, \dots, J_n)} \cdot \frac{\partial(J_1, \dots, J_n)}{\partial(x_1, \dots, x_n)}$$

By Theorem 2.3,

$$\frac{\partial(I_1, \dots, I_n)}{\partial(x_1, \dots, x_n)} \neq 0$$

and (2.16) then shows that

$$\frac{\partial(I_1, \dots, I_n)}{\partial(J_1, \dots, J_n)} \neq 0.$$

It follows that there is a rearrangement  $k_1, \dots, k_n$  of  $1, \dots, n$  so that

$$\frac{\partial I_{k_1}}{\partial J_1} \dots \frac{\partial I_{k_n}}{\partial J_n} \neq 0.$$

Hence  $I_{k_i}(J_1, \dots, J_n)$  is of positive degree in  $J_i$  and  $\deg I_{k_i} \geq \deg J_i$ ,  $1 \leq i \leq n$ . Applying Theorem 2.2 both to  $G$  and  $H$ , we obtain

$$(2.17) \quad \prod_{i=1}^n \deg J_i = |H|, \quad \prod_{i=1}^n \deg I_i = |G|$$

$$(2.18) \quad \sum_{i=1}^n (\deg J_i - 1) = \sum_{i=1}^n (\deg I_i - 1) = r$$

where  $r$  = number of reflections in  $G$  = number of reflections in  $H$ .

Since  $\deg I_{k_i} \geq \deg J_i$ ,  $1 \leq i \leq n$ , we conclude from (2.18) that  $\deg I_{k_i} = \deg J_i$ ,  $1 \leq i \leq n$ . Hence  $\prod_{i=1}^n \deg I_i = \prod_{i=1}^n \deg J_i$ , and we conclude from (2.17) that  $|G| = |H|$ . Thus  $G = H$  and  $G$  is a finite reflection group.

### 3. A FORMULA FOR $\frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)}$

We obtain a formula which shall be used in Chapter III.

**THEOREM 2.5.** *Let  $G$  be a finite reflection group acting on the  $n$ -dimensional space  $V$ . Let  $I_1, \dots, I_n$  be a basic set of homogeneous invariants for  $G$ . Let  $x$  be a coordinate system for  $V$  and  $L_i(x) = 0$ ,  $1 \leq i \leq r$ , the r.h.'s for  $G$ , each  $L_i$  being linear and homogeneous. Then*

$$(2.19) \quad \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)} = c \prod_{i=1}^r L_i(x)$$

$c$  being a constant  $\neq 0$ .

*Proof.* Let  $J$  the left hand side of (2.19). We observe that  $J$  is a non-zero homogeneous polynomial of degree  $\sum_{i=1}^n (d_i - 1)$ . By Theorem 2.2,

$\sum_{i=1}^n (d_i - 1) = r$ , so that  $\deg J = r$ . If  $k$  is the real field  $R$ , we have the following simple proof of (2.19).  $I_i = I_i(x_1, \dots, x_n)$ ,  $1 \leq i \leq n$ , is a mapping from  $x$ -space to  $I$ -space. This mapping is not 1 - 1 in any neighborhood of a point  $x$  lying in the r.h.  $L_i(x) = 0$ , as any point and its reflection get mapped into the same point  $I$ . It follows from the Implicit Function Theorem that  $J(x) = 0$  whenever  $L_i(x) = 0$ . Thus  $L_i | J$ ,  $1 \leq i \leq r$ , and so  $\prod_{i=1}^r L_i | J$ . Since  $J$ ,  $\prod_{i=1}^r L_i$  have the same degree  $r$ , we have

$$J = c \prod_{i=1}^r L_i, \quad c \neq 0.$$

For an arbitrary field  $k$ , the theorem is proven as follows. Let  $\pi$  be an r.h. with equation  $L(x) = 0$  and  $H$  the subgroup of  $h$  elements in  $G$  fixing  $\pi$ . Thus there are  $h - 1$  reflections in  $G$  with r.h.  $\pi$ . We show that  $L^{h-1} | J$ . By Lemma 2.2,  $H$  is a cyclic group generated by an element  $\sigma$ . Furthermore there exists  $v \notin \pi$  and a primitive  $h$ -th root of 1 such that  $\sigma(v) = \zeta v$ . Choose a coordinate system  $y = (y_1, \dots, y_n)$  in  $V$  so that  $\pi$  has the equation  $y_n = 0$  and  $v = (0, \dots, 0, 1)$   $\sigma$  then becomes the transformation  $(y_1, \dots, y_{n-1}, y_n) \rightarrow (y_1, \dots, y_{n-1}, \zeta y_n)$ . Let  $x = \tau y$  and  $J_i(y) = I_i(\tau y)$ ,  $1 \leq i \leq n$ . We have

$$(2.20) \quad J_i(y_1, \dots, y_{n-1}, \zeta y_n) = J_i(y_1, \dots, y_{n-1}, y_n), \quad 1 \leq i \leq n$$

Let  $J_i = \sum A_m y_n^m$ , the  $A_m$ 's being polynomials in  $y_1, \dots, y_{n-1}$ . (2.20) implies that  $A_m = 0$  whenever  $h \nmid m$ , so that  $A_m = 0$ ,  $0 \leq m \leq h-1$ . Since

$$\frac{\partial J_i}{\partial y_m} = \sum_m A_m y_n^{m-1},$$

we conclude

$$y_n^{h-1} \left| \frac{\partial J_i}{\partial y_n}, 1 \leq i \leq n. \right.$$

Hence

$$(2.21) \quad y_n^{h-1} \left| \frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)}, \right.$$

Since

$$\frac{\partial (J_1, \dots, J_n)}{\partial (y_1, \dots, y_n)} = J(x) \cdot \det \tau,$$

(2.21) is equivalent to  $L^{h-1}(x) \mid J(x)$ . It follows that if  $L_i(x) = 0$ ,  $1 \leq i \leq r$ , are the r.h.'s for  $G$ , then  $\prod_{i=1}^r L_i \mid J$ . But  $J, \prod_{i=1}^r L_i$  have the same degree  $r$ , so that  $J = c \prod_{i=1}^r L_i$   $c \neq 0$ .

#### 4. DECOMPOSITION OF FINITE REFLECTION GROUPS

We shall decompose every finite reflection group into a direct product of irreducible ones and show that it suffices to study the invariant theory of the irreducible groups.

**DEFINITION 2.3.** Let the group  $G$  act on  $V$ .  $G$  is said to be reducible iff there exists a proper subspace  $W$  invariant under  $G$ ; i.e.  $\sigma w \in W$  for  $\sigma \in G, w \in W$ .  $G$  is said to be completely reducible iff  $V = V_1 \oplus V_2$ ,  $V_1$  and  $V_2$  being proper invariant subspaces.  $G$  is said to be irreducible iff it is not reducible.

**THEOREM 2.6.** (Maschke [22], Vol. 2, p. 179). *Let  $G$  be a finite group acting on the vector space  $V$ . If  $G$  is reducible, then it is completely reducible.*

*Proof.* Let  $V_1$  be a proper invariant subspace of  $V$ . Let  $V_2$  be a complementary subspace. Thus for  $v \in V$ , we have a unique decomposition

$v = v_1 + v_2, v_i \in V_i (i=1, 2)$ . Let  $\eta v = v_2$  and set  $\tau = \frac{1}{|G|} \sum_{\sigma \in G} \sigma \eta \sigma^{-1}$ .

$\tau$  satisfies the following:

- i)  $\tau \sigma = \sigma \tau, \sigma \in G$ . For  $\sigma \tau = \frac{1}{|G|} \sum_{\sigma_1 \in G} \sigma \sigma_1 \eta (\sigma \sigma_1)^{-1} \sigma = \tau \sigma$
- ii)  $\tau v_1 = 0, v_1 \in V_1$ . For  $\sigma^{-1} v_1 \in V_1, \sigma \in G$ , so that  $\eta \sigma^{-1} v_1 = 0 \Rightarrow \tau v_1 = 0$
- iii)  $(1-\tau) v \in V_1, v \in V, 1$  denoting the identity of  $G$ . For  $(1-\eta) v \in V_1$ , so that  $(1-\eta) \sigma^{-1} v \in V_1 \Rightarrow \sigma (1-\eta) \sigma^{-1} v \in V_1, \sigma \in G$ . It follows that  $(1-\tau) v = \frac{1}{|G|} \sum_{\sigma \in G} \sigma (1-\eta) \sigma^{-1} v \in V_1$ .

Let  $V'_2 = \tau V$ .  $V'_2$  is invariant under  $G$  as  $\sigma(\tau v) = \tau(\sigma v)$ . For any  $v, v = \tau v + (1-\tau) v$ . It follows from iii) that  $V = V_1 + V'_2$ . ii), iii) imply  $\tau(1-\tau) = 0 \Leftrightarrow \tau = \tau^2$ . Hence  $\tau v'_2 = v'_2$  for  $v'_2 \in V'_2$ . Let  $v_1 + v'_2 = 0$ , where  $v_1 \in V_1, v'_2 \in V'_2$ . Applying  $\tau$  to both sides, we get  $v'_2 = 0$  and so  $v_1 = 0$ . Hence  $V = V_1 \oplus V'_2$ .

Repeated application of Maschke's Theorem yields the

**COROLLARY.** *Let  $G$  be a finite group acting on the finite-dimensional vector space  $V$ . Then  $V = V_1 \oplus \dots \oplus V_s$ , the  $V_i$ 's being invariant subspaces of  $V$  and  $G$  acting irreducibly on each  $V_i$ .*

For finite reflection groups, we have

**THEOREM 2.7.** *Let  $G$  be a finite reflection group acting on  $V$ . There exists a decomposition  $V = V_1 \oplus \dots \oplus V_s$  into invariant subspaces such that :*

- 1) *Let  $G_i = G|_{V_i}$  = group of restrictions of elements of  $G$  to  $V_i$ . Then  $G$  is isomorphic to  $G_1 \times \dots \times G_s$*
- 2) *Each  $G_i, 1 \leq i \leq s$ , is a reflection group acting irreducibly on  $V_i$ .*

*Proof.* By the corollary to Theorem 2.6, there exists a decomposition  $V = V_1 \oplus \dots \oplus V_s$ , the  $V_i$ 's being invariant subspaces and  $G_i$  irreducible for  $1 \leq i \leq s$ . We label the  $V_i$ 's so that  $V_1, \dots, V_r$  are 1-dimensional and  $G|_{V_i} = \text{identity}$ .

By the remark following Definition 2.1, for each reflection  $\sigma$  there exists an eigenvector  $v \in V - \pi, \pi$  being the r.h. for  $\sigma$ . Call  $v$  a root of  $G$ . We have

$$(2.22) \quad \dim(V_i + \pi) + \dim(V_i \cap \pi) = \dim V_i + \dim \pi.$$



If  $V_i \not\subset \pi$ , then  $V_i + \pi = V$  and we conclude from (2.22) that  $\dim V_i = \dim (V_i \cap \pi) + 1$ . I.e.  $V_i \cap \pi$  is a hyperplane in  $V_i$  and  $\sigma|_{V_i}$  a reflection on  $V_i$ . Choose  $u \in V_i - \pi$  so that  $u$  is an eigenvector of  $\sigma$ .  $u$  is a multiple of the root  $v$ , so that  $v \in V_i$ . Thus  $\sigma|_{V_i}$  is a reflection of  $V_i$  if  $v \in V_i$ , and the identity if  $v \notin V_i$ . Furthermore, each root  $v$  is in some  $V_i$ ,  $r + 1 \leq i \leq s$ , otherwise the corresponding reflection  $\sigma$  would have been the identity.

Let  $\tilde{G}_i =$  subgroup generated by those reflections whose roots are in  $V_i$ ,  $1 \leq i \leq s$ . It is readily checked that  $G = \tilde{G}_1 \times \dots \times \tilde{G}_s$ ,  $G_i = \tilde{G}_i|_{V_i}$ . If  $\sigma \in \tilde{G}_i$  and  $\sigma|_{V_i} =$  identity then  $\sigma =$  identity. The mapping  $\sigma \rightarrow \sigma|_{V_i}$  is thus an isomorphism from  $\tilde{G}_i$  onto  $G_i$ .

**THEOREM 2.8.** *Let  $G$  be a finite reflection group acting on  $V$  and decompose  $V$  as in Theorem 2.7. Every polynomial invariant under  $G$  is a polynomial in the invariant polynomials of  $G_1, \dots, G_s$ .*

*Proof.* For each  $v \in V$ , write  $v = v_1 + \dots + v_s$ ,  $v_i \in V_i$ . By Theorem 2.7, for each  $\sigma \in G$ , we may write  $\sigma v = \sigma_1 v_1 + \dots + \sigma_s v_s$ ,  $\sigma_i \in G_i$ . For any polynomial function  $p(v)$  on  $V$ , we have  $p(v) = \sum_{i=1}^N p_{i1}(v_1) \dots p_{is}(v_s)$  where  $p_{ij}(v_j)$  is a polynomial function on  $V_j$ . If  $p(v)$  is invariant under  $G$ , then

$$(2.23) \quad p(v) = \frac{1}{|G|} \sum_{\sigma \in G} p(\sigma v) = \sum_{i=1}^N I_{i1}(v_1) \dots I_{is}(v_s)$$

where

$$(2.24) \quad I_{ij}(v_j) = \frac{1}{|G_j|} \sum_{\sigma_j \in G_j} p_{ij}(\sigma_j v_j)$$

is an invariant of  $G_j$ .

## CHAPTER III

### THE DEGREES OF THE BASIC INVARIANTS

We determine the degrees of the basic homogeneous invariants in case  $G$  is a finite reflection group. We present two different methods. The first one (Theorem 3.8), restricts itself to the case where  $k$  is the real field and has the advantage of providing an effective method for computing the