

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 24 (1978)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** INVARIANTS OF FINITE REFLECTION GROUPS  
**Autor:** Flatto, Leopold  
**Kapitel:** 2. Molien's Formula  
**DOI:** <https://doi.org/10.5169/seals-49704>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 27.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

**THEOREM 1.2.** *Let  $I_1, \dots, I_l$  form a basis for the invariants of  $G$ . We may choose from the  $I_j$ 's  $n$  elements which are algebraically independent over  $k$ . Thus  $l \geq n$ .*

*Proof.* Let  $k(x_1, \dots, x_n)$  be the field of rational functions in the indeterminates  $x_1, \dots, x_n$  with coefficients in  $k$ , a similar meaning being attached to  $k(I_1, \dots, I_l)$ . We show that  $k(x_1, \dots, x_n)$  is a finite extension of  $k(I_1, \dots, I_l)$ . Let  $x_i(x) = x_i$  and set

$$(1.7) \quad p_i(X) = \prod_{\sigma \in G} (X - x_i(\sigma x)) = X^{|G|-1} + a_1 X^{|G|-2} + a_2 X^{|G|-3} + \dots + a_{|G|}$$

It is readily checked that the coefficients  $a_j$  are polynomials which are invariant under  $G$ . Thus each  $a_j \in k(I_1, \dots, I_l)$ . Since  $p_i(x_i) = 0$ , we conclude that  $x_i, \dots, x_n$  are algebraic over  $k(I_1, \dots, I_l)$ . Hence  $k(x_1, \dots, x_n)$  is a finite extension of  $k(I_1, \dots, I_l)$ .

Let  $K = k(\alpha_1, \dots, \alpha_s)$  be the field obtained by adjoining  $\alpha_1, \dots, \alpha_s$  to  $k$ . We may define the transcendence degree of  $K$  over  $k$  to be the maximum number of  $\alpha_i$ 's which are algebraically independent over  $k$  ([22], Vol. 1, p. 201). We denote this degree by  $\text{Tr.deg. } K/k$ . If we have three fields  $k \subset K \subset L$ , then it is known that

$$(1.8) \quad \text{Tr.deg. } L/k = \text{Tr.deg. } L/K + \text{Tr.deg. } K/k \text{ ([22], Vol. 1, p. 202).}$$

Apply (1.8) with  $L = k(x_1, \dots, x_n)$ ,  $K = k(I_1, \dots, I_l)$ . Then  $\text{Tr.deg. } L/k = n$  and the finiteness of  $L$  over  $K$  means that  $\text{Tr.deg. } L/K = 0$ . Hence  $\text{Tr.deg. } K/k = n$ , which means that we may choose  $n$   $I_j$ 's which are algebraically independent over  $k$ .

## 2. MOLLIEN'S FORMULA

For each integer  $m \geq 0$ , the homogeneous invariants of degree  $m$  form a finite dimensional vector space over  $k$  of dimension  $\delta_m$ . We derive an interesting and useful formula for the  $\delta_m$ 's.

**THEOREM 1.3.** (Molien's Formula [16]). *Let  $\omega_1(\sigma), \dots, \omega_n(\sigma)$  be the eigenvalues of  $\sigma$ . Then*

$$(1.9) \quad \sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1 - \omega_1(\sigma)t) \dots (1 - \omega_n(\sigma)t)}$$

REMARK. (1.9) is to be interpreted as an identity between two formal power series. I.e. if the right side is expanded as a formal power series, then its coefficients are identical with the  $\delta_m$ 's.

We require the following

LEMMA 1.2. Let  $W$  be the subspace fixed by  $G$ .

$$\text{Then } \dim W = \frac{1}{|G|} \sum_{\sigma \in G} \text{Tr}(\sigma).$$

*Proof.* Let  $\{v_1, \dots, v_r\}$  be a basis for  $W$  and augment this to a basis  $\{v_1, \dots, v_n\}$  for  $V$ . For  $\sigma_1 \in G$  and  $v \in V$ , we have

$$\sigma_1 \left( \sum_{\sigma \in G} \sigma v \right) = \sum_{\sigma \in G} (\sigma_1 \sigma) v = \sum_{\sigma \in G} \sigma v,$$

so that  $\sum_{\sigma \in G} \sigma v \in W$ . It follows that

$$\frac{1}{|G|} \sum_{\sigma \in G} \sigma v_i = v_i, \quad 1 \leq i \leq r,$$

and

$$\frac{1}{|G|} \sum_{\sigma \in G} \sigma v_i = \sum_{j=1}^r a_{ij} v_j, \quad r+1 \leq i \leq n,$$

the  $a_{ij}$ 's  $\in k$ . Hence

$$\frac{1}{|G|} \sum_{\sigma \in G} \text{Tr} \sigma = \text{TR} \left( \frac{1}{|G|} \sum_{\sigma \in G} \sigma \right) = r = \dim W.$$

*Proof of Theorem 1.3.* Let  $\tilde{k}$  = algebraic closure of  $k$ . For any  $\sigma \in G$ , we can find a matrix  $\tau$  with entries in  $\tilde{k}$  so that  $\tau \sigma \tau^{-1} = d$ ,  $d$  being diagonal and the diagonal entries being the eigenvalues of  $\sigma$ . Let  $R_m, \tilde{R}_m$  denote respectively the space of homogeneous polynomials with coefficients from  $k, \tilde{k}$ . Let  $(\text{Tr} \sigma)_m$  = trace of  $\sigma$  as a transformation on  $R_m$  = trace of  $\sigma$  as a transformation on  $\tilde{R}_m$ . Let  $(\text{Tr} d)_m$  = trace of  $d$  as a transformation on  $\tilde{R}_m$ . We have  $d(P(x)) = P(d^{-1}x)$  for any polynomial  $P(x)$ . In particular, for any monomial  $x^a$ , we have  $d(x^a) = \omega^a(\sigma^{-1})$ , where  $\omega(\sigma) = (\omega_1(\sigma), \dots, \omega_n(\sigma))$ . The monomials  $x^a$  form a basis for  $R_m$  and  $\tilde{R}_m$ . We conclude that

$$(1.10) \quad (\text{Tr} \sigma)_m = (\text{Tr} d)_m = \sum_{|a|=m} \omega^a(\sigma^{-1}).$$

(1.10) and Lemma 1.2 yield

$$(1.11) \quad \delta_m = \frac{1}{|G|} \sum_{\sigma \in G} (\text{Tr } \sigma)_m = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma).$$

Multiply both sides of (1.11) by  $t^m$  and sum over  $m$  from 0 to  $\infty$ . We get

$$\begin{aligned} \sum_{m=0}^{\infty} \delta_m t^m &= \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma) t^m \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \left\{ \sum_{m=0}^{\infty} \omega_1^m(\sigma) t^m \dots \sum_{m=0}^{\infty} \omega_n^m(\sigma) t^m \right\} \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1 - \omega_1(\sigma) t) \dots (1 - \omega_n(\sigma) t)} \end{aligned}$$

## CHAPTER II

### INVARIANT THEORETIC CHARACTERIZATION OF FINITE REFLECTION GROUPS

#### 1. CHEVALLEY'S THEOREM

We showed in chapter I that we can always find a finite number of homogeneous invariants forming a basis for the invariants of  $G$  and that this set must contain at least  $n$  elements, where  $n = \dim V$ . We show that this lower bound is attained only for the finite reflection groups. We first define these groups.

**DEFINITION 2.1.** Let  $\sigma$  be a linear transformation acting on the  $n$ -dimensional vector space  $V$ .  $\sigma$  is a reflection  $\Leftrightarrow \sigma$  fixes an  $n - 1$  dimensional hyperplane  $\pi$  and  $\sigma$  is of finite order  $> 1$ .  $\pi$  is called the reflecting hyperplane (r.h.) of  $\sigma$ .

**REMARK.** Choose  $v \notin \pi$ . and let  $\sigma v = \zeta v + p$ ,  $p \in \pi$ . If  $\zeta = 1$ , then  $\sigma^m v = v + mp$ , contradicting that  $\sigma$  is of finite order. Hence  $\zeta \neq 1$ . Let  $v' = v + (\zeta - 1)^{-1} p$  and choose  $p_1, \dots, p_{n-1}$  as a basis for  $\pi$ . Then  $\sigma p_i = p_i$ ,  $1 \leq i \leq n - 1$ ,  $\sigma v' = \zeta v'$ .  $\zeta$  is a root of 1 in  $k$  which is distinct from 1, as  $\sigma$  is of finite order  $> 1$ . Thus  $\sigma$  is a reflection iff relative to some basis, the matrix for  $\sigma$  is diagonal,  $n - 1$  of the diagonal entries equalling 1 and the remaining one equalling a root of 1 in  $k$  distinct from 1.