**Zeitschrift:** L'Enseignement Mathématique
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique
**Band:** 24 (1978)
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** INVARIANTS OF FINITE REFLECTION GROUPS
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**Kapitel:** 2. Molien's Formula
**DOI:** https://doi.org/10.5169/seals-49704

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THEOREM 1.2. *Let* $I_1, \ldots, I_l$ *form a basis for the invariants of* $G$. *We may choose from the* $I_j$'s $n$ *elements which are algebraically independent over* $k$. *Thus* $l \geqslant n$.

*Proof.* Let $k(x_1, \ldots, x_n)$ be the field of rational functions in the indeterminates $x_1, \ldots, x_n$ with coefficients in $k$, a similar meaning being attached to $k(I_1, \ldots, I_l)$. We show that $k(x_1, \ldots, x_n)$ is a finite extension of $k(I_1, \ldots, I_l)$. Let $x_i(x) = x_i$ and set

$$(1.7) \qquad p_i(X) = \prod_{\sigma \varepsilon G} (X - x_i(\sigma x)) = X^{|G|-1} + a_1 X^{|G|-1} + a_1 X^{|G|-1} + \ldots + a_{|G|}$$

It is readily checked that the coefficients $a_j$ are polynomials which are invariant under $G$. Thus each $a_j \in k(I_1, \ldots, I_l)$. Since $p_i(x_i) = 0$, we conclude that $x_i, \ldots, x_n$ are algebraic over $k(I_1, \ldots, I_l)$. Hence $k(x_1, \ldots, x_n)$ is a finite extension of $k(I_i, \ldots, I_l)$.

Let $K = k(\alpha_1, \ldots, \alpha_s)$ be the field obtained by adjoining $\alpha_1, \ldots, \alpha_s$ to $k$. We may define the transcendence degree of $K$ over $k$ to be the maximum number of $\alpha_i$'s which are algebraically independent over $k$ ([22], Vol. 1, p. 201). We denote this degree by Tr.deg. $K/k$. If we have three fields $k \subset K \subset L$, then it is known that

$$(1.8) \qquad \text{Tr.deg. } L/k = \text{Tr.deg. } L/K + \text{Tr.deg. } K/k \text{ ([22], Vol. 1, p. 202).}$$

Apply (1.8) with $L = k(x_1, \ldots, x_n)$, $K = k(I_1, \ldots, I_l)$. Then Tr.deg. $L/k = n$ and the finiteness of $L$ over $K$ means that Tr.deg. $L/K = 0$. Hence Tr.deg. $K/k = n$, which means that we may choose $n$ $I_j$'s which are algebraically independent over $k$.

## 2. MOLIEN'S FORMULA

For each integer $m \geqslant 0$, the homogeneous invariants of degree $m$ form a finite dimensional vector space over $k$ of dimension $\delta_m$. We derive an interesting and useful formula for the $\delta_m$'s.

THEOREM 1.3. (Molien's Formula [16]). *Let* $\omega_1(\sigma), \ldots, \omega_n(\sigma)$ *be the eigenvalues of* $\sigma$. *Then*

$$(1.9) \qquad \sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{|G|} \sum_{\sigma \varepsilon G} \frac{1}{(1-\omega_1(\sigma)t)\ldots(1-\omega_n(\sigma)t)}$$

REMARK. (1.9) is to be interpreted as an identity between two formal power series. I.e. if the right side is expanded as a formal power series, then its coefficients are identical with the $\delta_m$'s.

We require the following

LEMMA 1.2. Let $W$ be the subspace fixed by $G$.

Then $\dim W = \dfrac{1}{|G|} \sum_{\sigma \varepsilon G} \operatorname{Tr}(\sigma)$.

*Proof.* Let $\{v_1, \ldots, v_r\}$ be a basis for $W$ and augment this to a basis $\{v_1, \ldots, v_n\}$ for $V$. For $\sigma_1 \in G$ and $v \in V$, we have

$$\sigma_1 \Big( \sum_{\sigma \varepsilon G} \sigma v \Big) = \sum_{\sigma \varepsilon G} (\sigma_1 \sigma) v = \sum_{\sigma \varepsilon G} \sigma v ,$$

so that $\sum_{\sigma \varepsilon G} \sigma v \in W$. It follows that

$$\frac{1}{|G|} \sum_{\sigma \varepsilon G} \sigma v_i = v_i, \; 1 \leqslant i \leqslant r ,$$

and

$$\frac{1}{|G|} \sum_{\sigma \varepsilon G} \sigma v_i = \sum_{j=1}^{r} a_{ij} v_j, \; r+1 \leqslant i \leqslant n ,$$

the $a_{ij}$'s $\in k$. Hence

$$\frac{1}{|G|} \sum_{\sigma \varepsilon G} Tr\, \sigma = TR \left( \frac{1}{|G|} \sum_{\sigma \varepsilon G} \sigma \right) = r = \dim W .$$

*Proof of Theorem 1.3.* Let $\tilde{k}$ = algebraic closure of $k$. For any $\sigma \in G$, we can find a matrix $\tau$ with entries in $\tilde{k}$ so that $\tau \sigma \tau^{-1} = d$, $d$ being diagonal and the diagonal entries being the eigenvalues of $\sigma$. Let $R_m$, $\tilde{R}_m$ denote respectively the space of homogeneous polynomials with coefficients from $k$, $\tilde{k}$. Let $(\operatorname{Tr} \sigma)_m$ = trace of $\sigma$ as a transformation on $R_m$ = trace of $\sigma$ as a tranformation on $\tilde{R}_m$. Let $(\operatorname{Tr} d)_m$ = trace of $d$ as a transformation on $\tilde{R}_m$. We have $d(P(x)) = P(d^{-1}x)$ for any polynomial $P(x)$. In particular, for any monomial $x^a$, we have $d(x^a) = \omega^a(\sigma^{-1})$, where $\omega(\sigma) = (\omega_1(\sigma), \ldots, \omega_n(\sigma))$. The monomials $x^a$ form a basis for $R_m$ and $\tilde{R}_m$. We conclude that

$$(1.10) \qquad (Tr\, \sigma)_m = (Tr\, d)_m = \sum_{|a| = m} \omega^a(\sigma^{-1}) .$$

(1.10) and Lemma 1.2 yield

$$(1.11) \qquad \delta_m = \frac{1}{|G|} \sum_{\sigma \varepsilon G} (Tr\, \sigma)_m = \frac{1}{|G|} \sum_{\sigma \varepsilon G} \sum_{|a|=m} \omega^a(\sigma) .$$

Multiply both sides of (1.11) by $t^m$ and sum over $m$ from 0 to $\infty$. We get

$$\sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{\sigma \varepsilon G} \sum_{|a|=m} \omega^a(\sigma) t^m$$

$$= \frac{1}{|G|} \sum_{\sigma \varepsilon G} \{ \sum_{m=0}^{\infty} \omega_1^m(\sigma) t^m \ldots \sum_{m=0}^{\infty} \omega_n^m(\sigma) t^m \}$$

$$= \frac{1}{|G|} \sum_{\sigma \varepsilon G} \frac{1}{(1-\omega_1(\sigma) t) \ldots (1-\omega_n(\sigma) t)}$$

# CHAPTER II

# INVARIANT THEORETIC CHARACTERIZATION OF FINITE REFLECTION GROUPS

## 1. Chevalley's Theorem

We showed in chapter I that we can always find a finite number of homogeneous invariants forming a basis for the invariants of $G$ and that this set must contain at least $n$ elements, where $n = \dim V$. We show that this lower bound is attained only for the finite reflection groups. We first define these groups.

Definition 2.1. Let $\sigma$ be a linear transformation acting on the $n$-dimensional vector space $V$. $\sigma$ is a reflection $\Leftrightarrow$ $\sigma$ fixes an $n-1$ dimensional hyperplane $\pi$ and $\sigma$ is of finite order $> 1$. $\pi$ is called the reflecting hyperplane (r.h.) of $\sigma$.

Remark. Choose $v \notin \pi$. and let $\sigma v = \zeta v + p$, $p \in \pi$. If $\zeta = 1$, then $\sigma^m v = v + mp$, contradicting that $\sigma$ is of finite order. Hence $\zeta \neq 1$. Let $v' = v + (\zeta - 1)^{-1} p$ and choose $p_1, \ldots, p_{n-1}$ as a basis for $\pi$. Then $\sigma p_i = p_i$, $1 \leqslant i \leqslant n-1$, $\sigma v' = \zeta v'$. $\zeta$ is a root of 1 in $k$ which is distinct from 1, as $\sigma$ is of finite order $> 1$. Thus $\sigma$ is a reflection iff relative to some basis, the matrix for $\sigma$ is diagonal, $n-1$ of the diagonal entries equalling 1 and the remaining one equalling a root of 1 in $k$ distinct from 1.