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**THEOREM 1.2.** Let  $I_1, \dots, I_l$  form a basis for the invariants of  $G$ . We may choose from the  $I_j$ 's  $n$  elements which are algebraically independent over  $k$ . Thus  $l \geq n$ .

*Proof.* Let  $k(x_1, \dots, x_n)$  be the field of rational functions in the indeterminates  $x_1, \dots, x_n$  with coefficients in  $k$ , a similar meaning being attached to  $k(I_1, \dots, I_l)$ . We show that  $k(x_1, \dots, x_n)$  is a finite extension of  $k(I_1, \dots, I_l)$ . Let  $x_i(x) = x_i$  and set

$$(1.7) \quad p_i(X) = \prod_{\sigma \in G} (X - x_i(\sigma x)) = X^{|G|-1} + a_1 X^{|G|-2} + \dots + a_{|G|}$$

It is readily checked that the coefficients  $a_j$  are polynomials which are invariant under  $G$ . Thus each  $a_j \in k(I_1, \dots, I_l)$ . Since  $p_i(x_i) = 0$ , we conclude that  $x_1, \dots, x_n$  are algebraic over  $k(I_1, \dots, I_l)$ . Hence  $k(x_1, \dots, x_n)$  is a finite extension of  $k(I_1, \dots, I_l)$ .

Let  $K = k(\alpha_1, \dots, \alpha_s)$  be the field obtained by adjoining  $\alpha_1, \dots, \alpha_s$  to  $k$ . We may define the transcendence degree of  $K$  over  $k$  to be the maximum number of  $\alpha_i$ 's which are algebraically independent over  $k$  ([22], Vol. 1, p. 201). We denote this degree by  $\text{Tr.deg. } K/k$ . If we have three fields  $k \subset K \subset L$ , then it is known that

$$(1.8) \quad \text{Tr.deg. } L/k = \text{Tr.deg. } L/K + \text{Tr.deg. } K/k \text{ ([22], Vol. 1, p. 202).}$$

Apply (1.8) with  $L = k(x_1, \dots, x_n)$ ,  $K = k(I_1, \dots, I_l)$ . Then  $\text{Tr.deg. } L/k = n$  and the finiteness of  $L$  over  $K$  means that  $\text{Tr.deg. } L/K = 0$ . Hence  $\text{Tr.deg. } K/k = n$ , which means that we may choose  $n I_j$ 's which are algebraically independent over  $k$ .

## 2. MOLIEN'S FORMULA

For each integer  $m \geq 0$ , the homogeneous invariants of degree  $m$  form a finite dimensional vector space over  $k$  of dimension  $\delta_m$ . We derive an interesting and useful formula for the  $\delta_m$ 's.

**THEOREM 1.3.** (Molien's Formula [16]). Let  $\omega_1(\sigma), \dots, \omega_n(\sigma)$  be the eigenvalues of  $\sigma$ . Then

$$(1.9) \quad \sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1 - \omega_1(\sigma) t) \dots (1 - \omega_n(\sigma) t)}$$

REMARK. (1.9) is to be interpreted as an identity between two formal power series. I.e. if the right side is expanded as a formal power series, then its coefficients are identical with the  $\delta_m$ 's.

We require the following

LEMMA 1.2. Let  $W$  be the subspace fixed by  $G$ .

$$\text{Then } \dim W = \frac{1}{|G|} \sum_{\sigma \in G} \text{Tr}(\sigma).$$

*Proof.* Let  $\{v_1, \dots, v_r\}$  be a basis for  $W$  and augment this to a basis  $\{v_1, \dots, v_n\}$  for  $V$ . For  $\sigma_1 \in G$  and  $v \in V$ , we have

$$\sigma_1 \left( \sum_{\sigma \in G} \sigma v \right) = \sum_{\sigma \in G} (\sigma_1 \sigma) v = \sum_{\sigma \in G} \sigma v,$$

so that  $\sum_{\sigma \in G} \sigma v \in W$ . It follows that

$$\frac{1}{|G|} \sum_{\sigma \in G} \sigma v_i = v_i, \quad 1 \leq i \leq r,$$

and

$$\frac{1}{|G|} \sum_{\sigma \in G} \sigma v_i = \sum_{j=1}^r a_{ij} v_j, \quad r+1 \leq i \leq n,$$

the  $a_{ij}$ 's  $\in k$ . Hence

$$\frac{1}{|G|} \sum_{\sigma \in G} \text{Tr} \sigma = \text{TR} \left( \frac{1}{|G|} \sum_{\sigma \in G} \sigma \right) = r = \dim W.$$

*Proof of Theorem 1.3.* Let  $\tilde{k}$  = algebraic closure of  $k$ . For any  $\sigma \in G$ , we can find a matrix  $\tau$  with entries in  $\tilde{k}$  so that  $\tau \sigma \tau^{-1} = d$ ,  $d$  being diagonal and the diagonal entries being the eigenvalues of  $\sigma$ . Let  $R_m$ ,  $\tilde{R}_m$  denote respectively the space of homogeneous polynomials with coefficients from  $k$ ,  $\tilde{k}$ . Let  $(\text{Tr } \sigma)_m$  = trace of  $\sigma$  as a transformation on  $R_m$  = trace of  $\sigma$  as a transformation on  $\tilde{R}_m$ . Let  $(\text{Tr } d)_m$  = trace of  $d$  as a transformation on  $\tilde{R}_m$ . We have  $d(P(x)) = P(d^{-1}x)$  for any polynomial  $P(x)$ . In particular, for any monomial  $x^a$ , we have  $d(x^a) = \omega^a(\sigma^{-1})$ , where  $\omega(\sigma) = (\omega_1(\sigma), \dots, \omega_n(\sigma))$ . The monomials  $x^a$  form a basis for  $R_m$  and  $\tilde{R}_m$ . We conclude that

$$(1.10) \quad (\text{Tr } \sigma)_m = (\text{Tr } d)_m = \sum_{|a|=m} \omega^a(\sigma^{-1}).$$

(1.10) and Lemma 1.2 yield

$$(1.11) \quad \delta_m = \frac{1}{|G|} \sum_{\sigma \in G} (Tr \sigma)_m = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma).$$

Multiply both sides of (1.11) by  $t^m$  and sum over  $m$  from 0 to  $\infty$ . We get

$$\begin{aligned} \sum_{m=0}^{\infty} \delta_m t^m &= \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{\sigma \in G} \sum_{|a|=m} \omega^a(\sigma) t^m \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \left\{ \sum_{m=0}^{\infty} \omega_1^m(\sigma) t^m \dots \sum_{m=0}^{\infty} \omega_n^m(\sigma) t^m \right\} \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{(1 - \omega_1(\sigma)t) \dots (1 - \omega_n(\sigma)t)} \end{aligned}$$

## CHAPTER II

### INVARIANT THEORETIC CHARACTERIZATION OF FINITE REFLECTION GROUPS

#### 1. CHEVALLEY'S THEOREM

We showed in chapter I that we can always find a finite number of homogeneous invariants forming a basis for the invariants of  $G$  and that this set must contain at least  $n$  elements, where  $n = \dim V$ . We show that this lower bound is attained only for the finite reflection groups. We first define these groups.

**DEFINITION 2.1.** Let  $\sigma$  be a linear transformation acting on the  $n$ -dimensional vector space  $V$ .  $\sigma$  is a reflection  $\Leftrightarrow \sigma$  fixes an  $n - 1$  dimensional hyperplane  $\pi$  and  $\sigma$  is of finite order  $> 1$ .  $\pi$  is called the reflecting hyperplane (r.h.) of  $\sigma$ .

**REMARK.** Choose  $v \notin \pi$ . and let  $\sigma v = \zeta v + p$ ,  $p \in \pi$ . If  $\zeta = 1$ , then  $\sigma^m v = v + mp$ , contradicting that  $\sigma$  is of finite order. Hence  $\zeta \neq 1$ . Let  $v' = v + (\zeta - 1)^{-1} p$  and choose  $p_1, \dots, p_{n-1}$  as a basis for  $\pi$ . Then  $\sigma p_i = p_i$ ,  $1 \leq i \leq n - 1$ ,  $\sigma v' = \zeta v'$ .  $\zeta$  is a root of 1 in  $k$  which is distinct from 1, as  $\sigma$  is of finite order  $> 1$ . Thus  $\sigma$  is a reflection iff relative to some basis, the matrix for  $\sigma$  is diagonal,  $n - 1$  of the diagonal entries equalling 1 and the remaining one equalling a root of 1 in  $k$  distinct from 1.