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THEOREM 1.2. *Let* $I_1, \ldots, I_l$ *form a basis for the invariants of* G. *We may choose from the* I_j'*s* n *elements which are algebraically independent over* k. *Thus* $l \geqslant n$.

Proof. Let $k(x_1, \ldots, x_n)$ be the field of rational functions in the indeterminates $x_1, \ldots, x_n$ with coefficients in k, a similar meaning being attached to $k(I_1, \ldots, I_l)$. We show that $k(x_1, \ldots, x_n)$ is a finite extension of $k(I_1, \ldots, I_l)$. Let $x_i(x) = x_i$ and set

$$(1.7) \qquad p_i(X) = \prod_{\sigma \varepsilon G} (X - x_i(\sigma x)) = X^{|G|-1} + a_1 X^{|G|-1} + a_1 X^{|G|-1} + \ldots + a_{|G|}$$

It is readily checked that the coefficients a_j are polynomials which are invariant under G. Thus each $a_j \in k(I_1, \ldots, I_l)$. Since $p_i(x_i) = 0$, we conclude that $x_i, \ldots, x_n$ are algebraic over $k(I_1, \ldots, I_l)$. Hence $k(x_1, \ldots, x_n)$ is a finite extension of $k(I_i, \ldots, I_l)$.

Let $K = k(\alpha_1, \ldots, \alpha_s)$ be the field obtained by adjoining $\alpha_1, \ldots, \alpha_s$ to k. We may define the transcendence degree of K over k to be the maximum number of α_i's which are algebraically independent over k ([22], Vol. 1, p. 201). We denote this degree by Tr.deg. K/k. If we have three fields $k \subset K \subset L$, then it is known that

$$(1.8) \qquad \text{Tr.deg. } L/k = \text{Tr.deg. } L/K + \text{Tr.deg. } K/k \quad ([22], \text{Vol. 1, p. 202}).$$

Apply (1.8) with $L = k(x_1, \ldots, x_n)$, $K = k(I_1, \ldots, I_l)$. Then Tr.deg. $L/k = n$ and the finiteness of L over K means that Tr.deg. $L/K = 0$. Hence Tr.deg. $K/k = n$, which means that we may choose n I_j's which are algebraically independent over k.

2. MOLIEN'S FORMULA

For each integer $m \geqslant 0$, the homogeneous invariants of degree m form a finite dimensional vector space over k of dimension δ_m. We derive an interesting and useful formula for the δ_m's.

THEOREM 1.3. (Molien's Formula [16]). *Let* $\omega_1(\sigma), \ldots, \omega_n(\sigma)$ *be the eigenvalues of* σ. *Then*

$$(1.9) \qquad \sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{|G|} \sum_{\sigma \varepsilon G} \frac{1}{(1 - \omega_1(\sigma) t) \ldots (1 - \omega_n(\sigma) t)}$$

REMARK. (1.9) is to be interpreted as an identity between two formal power series. I.e. if the right side is expanded as a formal power series, then its coefficients are identical with the δ_m's.

We require the following

LEMMA 1.2. Let W be the subspace fixed by G.

Then $\dim W = \frac{1}{|G|} \sum_{\sigma \varepsilon G} \mathrm{Tr}\,(\sigma)$.

Proof. Let $\{v_1, \dots, v_r\}$ be a basis for W and augment this to a basis $\{v_1, \dots, v_n\}$ for V. For $\sigma_1 \in G$ and $v \in V$, we have

$$\sigma_1 \Big(\sum_{\sigma \varepsilon G} \sigma v \Big) = \sum_{\sigma \varepsilon G} (\sigma_1 \sigma) v = \sum_{\sigma \varepsilon G} \sigma v ,$$

so that $\sum_{\sigma \varepsilon G} \sigma v \in W$. It follows that

$$\frac{1}{|G|} \sum_{\sigma \varepsilon G} \sigma v_i = v_i, \ 1 \leqslant i \leqslant r ,$$

and

$$\frac{1}{|G|} \sum_{\sigma \varepsilon G} \sigma v_i = \sum_{j=1}^{r} a_{ij} v_j, \ r + 1 \leqslant i \leqslant n ,$$

the a_{ij}'s $\in k$. Hence

$$\frac{1}{|G|} \sum_{\sigma \varepsilon G} Tr\ \sigma = TR \left(\frac{1}{|G|} \sum_{\sigma \varepsilon G} \sigma \right) = r = \dim W .$$

Proof of Theorem 1.3. Let $\tilde{k}$ = algebraic closure of k. For any $\sigma \in G$, we can find a matrix τ with entries in $\tilde{k}$ so that $\tau\, \sigma\, \tau^{-1} = d$, d being diagonal and the diagonal entries being the eigenvalues of σ. Let R_m, $\tilde{R}_m$ denote respectively the space of homogeneous polynomials with coefficients from $k, \tilde{k}$. Let $(\mathrm{Tr}\,\sigma)_m$ = trace of σ as a transformation on R_m = trace of σ as a tranformation on $\tilde{R}_m$. Let $(\mathrm{Tr}\,d)_m$ = trace of d as a transformation on $\tilde{R}_m$. We have $d\,(P\,(x)) = P\,(d^{-1}x)$ for any polynomial $P(x)$. In particular, for any monomial x^a, we have $d\,(x^a) = \omega^a\,(\sigma^{-1})$, where $\omega\,(\sigma) = (\omega_1\,(\sigma), \dots, \omega_n\,(\sigma))$. The monomials x^a form a basis for R_m and $\tilde{R}_m$. We conclude that

$$(1.10) \qquad (Tr\,\sigma)_m = (Tr\,d\,)_m = \sum_{|a|=m} \omega^a\,(\sigma^{-1}) .$$

(1.10) and Lemma 1.2 yield

$$(1.11) \qquad \delta_m = \frac{1}{|G|} \sum_{\sigma \varepsilon G} (Tr\ \sigma)_m = \frac{1}{|G|} \sum_{\sigma \varepsilon G} \sum_{|a|=m} \omega^a(\sigma)\,.$$

Multiply both sides of (1.11) by t^m and sum over m from 0 to ∞. We get

$$\sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{\sigma \varepsilon G} \sum_{|a|=m} \omega^a(\sigma)\, t^m$$

$$= \frac{1}{|G|} \sum_{\sigma \varepsilon G} \left\{ \sum_{m=0}^{\infty} \omega_1^m(\sigma)\, t^m \ldots \sum_{m=0}^{\infty} \omega_n^m(\sigma)\, t^m \right\}$$

$$= \frac{1}{|G|} \sum_{\sigma \varepsilon G} \frac{1}{(1-\omega_1(\sigma)\,t) \ldots (1-\omega_n(\sigma)\,t)}$$

CHAPTER II

INVARIANT THEORETIC CHARACTERIZATION OF FINITE REFLECTION GROUPS

1. Chevalley's Theorem

We showed in chapter I that we can always find a finite number of homogeneous invariants forming a basis for the invariants of G and that this set must contain at least n elements, where n = dim V. We show that this lower bound is attained only for the finite reflection groups. We first define these groups.

Definition 2.1. Let σ be a linear transformation acting on the n-dimensional vector space V. σ is a reflection $\Leftrightarrow$ σ fixes an $n-1$ dimensional hyperplane π and σ is of finite order > 1. π is called the reflecting hyperplane (r.h.) of σ.

Remark. Choose $v \notin \pi$. and let $\sigma v = \zeta v + p$, $p \in \pi$. If $\zeta = 1$, then $\sigma^m v = v + mp$, contradicting that σ is of finite order. Hence $\zeta \neq 1$. Let $v' = v + (\zeta - 1)^{-1} p$ and choose $p_1, \ldots, p_{n-1}$ as a basis for π. Then $\sigma p_i = p_i$, $1 \leqslant i \leqslant n-1$, $\sigma v' = \zeta v'$. ζ is a root of 1 in k which is distinct from 1, as σ is of finite order > 1. Thus σ is a reflection iff relative to some basis, the matrix for σ is diagonal, $n-1$ of the diagonal entries equalling 1 and the remaining one equalling a root of 1 in k distinct from 1.