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INVARIANTS OF FINITE REFLECTION GROUPS

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INTRODUCTION

Let G be a group of linear transformations acting on a finite dimensional vector space V over a given field k. Let S be the ring of polynomial functions on V, i.e. those functions which become polynomials for any given coordinate system on V. G is made to act on S by defining

$$(\sigma s)(v) = s(\sigma^{-1}v), \qquad \sigma \in G,\ s \in S,\ v \in V$$

The elements of S fixed by G, i.e. $\sigma s = s$ for all $\sigma \in G$, are called the invariants of G. The subject of invariant theory deals with the determination of all invariants of a given group G. For finite groups, Hilbert proved in 1890 [14] the main theorem of invariant theory stating that the algebra of invariants is finitely generated. These finite sets of generators are said to form an integrity basis for the invariants of G. Later on, Noether [17] produced an explicit set of basic invariants for finite groups. However, this number is usually much more than necessary (we elaborate on this point in chapter I) and there lacks a systematic method for producing a basis which is in some sense minimal.

As we show in this expository paper, such a systematic method exists for the class of groups known as the finite reflection groups. In this case, a very detailed and beautiful theory has been worked out in the last twenty five years, bringing together various concepts from algebra, geometry, and analysis. The subject matter is closely related to other mathematical theories, such as the topology of Lie groups and the study of the Chevalley groups. For these connections, the interested reader is referred to the books of Bourbaki and Carter [2, 3], where further references are supplied.

We give here a brief description of the subject treated in this paper. A linear transformation σ acting on the n-dimensional vector space V is said to be a reflection if it fixes an $n - 1$ dimensional hyperplane π, which is then called the reflecting hyperplane (r.h.) of σ. G is a reflection group if it is generated by reflections. For finite reflection groups G acting on an

n-dimensional vector space V over a field k of characteristic 0, we have the fundamental result of Chevalley [4], stating that there are n algebraically independent homogeneous polynomials forming an integrity basis for the invariants of G. Conversely, we will show that if G is a finite group of linear transformations acting on V which is not a reflection group, than any basic set of homogeneous invariants must contain more than n elements which are algebraically dependent. Thus we may say that the finite reflection groups are distinguished to be those with the simplest possible type of invariant theory.

Let $d_1, \ldots, d_n$ be the respective degrees of the basic homogeneous invariants $I_1, \ldots, I_n$, where $d_1 \leqslant \ldots \leqslant d_n$. It can readily be shown that the d_i's are independent of the particular basis $I_1, \ldots, I_n$. We present in chapter III two methods for computing the d_i's. The first one is due to Coxeter and Coleman [7, 8] and is restricted to the case where the underlying field k is real. Coxeter has classified all real finite irreducible reflection groups [6]. If such a group G acts on the n-dimensional Euclidean space R^n, then its r.h.'s divide R^n into $|G|$ components, called the chambers of G. Each chamber is bounded by n r.h.'s called its walls. The reflections in these walls generate G. Coxeter has found a remarkable relation between the d_i's and the eigenvalues of the product of these generators. This relation, first checked individually for each of the groups listed in [7], has subsequently been proved by Coleman [8]. Coleman's Theorem (Theorem 3.8 of chapter III) may be used effectively to compute the d_i's in the real case. We also present another method due to Solomon [18] who has obtained formula 3.27) for the d_i's. Solomon's method works for all fields of characteristic 0, but cannot be used as effectively as the method of Coxeter and Coleman in the real case.

In Chapter IV, we apply the invariant theory developed in the earlier chapters to study a certain system of partial differential equations and related mean value properties. We assume that G is a finite orthogonal reflection group acting on R^n. Let I denote the set of homogeneous invariants of positive degree. For any polynomial $p\,(x)$, let $p\,(\partial)$ be the partial differential operator obtained by replacing each variable x_i by $\partial/\partial x_i$. Steinberg [21] has described the solution space of C^∞ functions satisfying the system

1) $$p(\partial)f = 0, \; p \in I$$

on some given n-dimensional region $\mathcal{R}$. We may interpret the solutions of 1) to be an analog of the harmonic functions, as the latter are the solutions of $\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = 0$ and $\sum_{i=1}^{n} x_i^2$ is the basic invariant for the orthogonal group $O(n)$ ([23] p. 53). We use Steinberg's result to describe the solution space S_y of continuous functions on $\mathscr{R}$ satisfying the mean value property

$$2) \qquad f(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} f(x + t\sigma y), \ x \in \mathscr{R}$$

and $0 < t < \varepsilon_x$, y denoting a fixed vector $\neq 0$. Observe that 2) is again an analog of the familiar mean value property characterizing harmonic functions ([15] p. 224). Flatto and Wiener [10] have shown that the solution spaces to 1) and 2) are identical, provided the degrees d_i are distinct and y does not belong to a certain algebraic manifold $\mathscr{M}$. $\mathscr{M}$ can be described by equations, the latter yielding an explicit integrity basis for the invariants of G.

I have tried to keep the present paper self-contained, defining and explaining most of the notions and results needed in it. Occasionally, I quote some well known results of algebra, most of which can be found in [22]. In Chapter IV we require some standard results on harmonic functions, which may be found in [15]. In Chapter III, we require Coxeter's classification of the irreducible finite reflection groups acting on R^n. It would have taken us too far afield to present this matter in detail. I present a brief exposition, without proof, of the main points of this theory which are required in the present paper. For a quick and readable account of the details, the reader is referred to [1].

CHAPTER I

GENERAL THEORY

1. The Main Theorem of Invariant Theory

We present in this chapter some basic notions and results of invariant theory. We assume throughout that G is a finite group of linear transformations acting on the finite dimensional vector space V over a given field k of characteristic 0. n designates the dimension of V.

Definition 1.1. Let $P(v)$ be a polynomial function on V. $P(v)$ is invariant of $G \Leftrightarrow P(\sigma v) = P(v)$ for $\sigma \in G$, $v \in V$.

Let $x_1, \ldots, x_n$ be a coordinate system for V. Then $P(v)$ becomes a polynomial which we designate by $P(x)$. σ is represented by a matrix which we again designate by σ. For this coordinate system, the above definition takes the form $P(\sigma x) = P(x)$, $\sigma \in G$ and x arbitrary. Let $P(x) = \sum_{i=0}^{m} P_i(x)$, where $m = \deg P$ and $P_i(x)$ is homogeneous of degree i. Then $P(\sigma x) = \sum_{i=0}^{m} P_i(\sigma x)$. Since $P_i(\sigma x)$ is also homogeneous of degree i, we conclude that $P(x)$ is invariant under G iff $P_i(x)$ is invariant under G for $1 \leqslant i \leqslant m$. Hence the determination of the invariants of G reduces to the determination of its homogeneous invariants.

Definition 1.2. Let $I_1(x), \ldots, I_k(x)$ be invariants of G. $I_1(x), \ldots, I_k(x)$ form an integrity basis for the invariants of $G \Leftrightarrow$ any polynomial invariant under G is a polynomial in $I_1, \ldots, I_k$.

As a concrete illustration of the above definitions, let G be the symmetric group S_n consisting of the linear transformations $x_i' = x_{\sigma(i)}$, σ being any permutation of $1, \ldots, n$. The invariants of S_n are the symmetric polynomials in $x_1, \ldots, x_n$. It is well known ([22], Vol. I, p. 79) that the elementary symmetric polynomials $I_j(x) = \sum x_{i_1} \ldots x_{i_j}$ $(1 \leqslant i_1 < \ldots < i_j \leqslant n)$, $1 \leqslant j \leqslant n$, form an integrity basis for all symmetric polynomials.

In the sequel, we shall use the term basis to mean integrity basis. The following result, due to Hilbert, is the main theorem of invariant theory.

Theorem 1.1. *The invariants of G have a finite basis.*

We present two proofs of this theorem, due respectively to Hilbert [14] and Noether [17].

Hilbert's Proof: Let I denote the set of all homogeneous invariants of positive degree. Let $\mathscr{I}$ be the ideal generated by I. By Hilbert's Basis Theorem ([22], Vol. 2, p. 18), $\mathscr{I} = (I_1, \ldots, I_k)$ where $I_1, \ldots, I_k$ are homogeneous invariants of positive degree. Since every invariant polynomial is a sum of homogeneous invariants, it suffices to show that every P in I is a polynomial in $I_1, \ldots, I_k$. Now $P \in I \Rightarrow P \in \mathscr{I}$, so that $P(x) = \sum_{j=1}^{m} Q_j(x) I_j(x)$.

Since P and the I_j's are homogeneous, the Q_j's may be chosen homogeneous. We show that the Q_j's may be chosen invariant by the following group averaging processs. Since $P(x) = P(\sigma x)$ for all $\sigma \in G$, we have

$$(1.1) \qquad P(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} P(\sigma x) = \sum_{j=1}^{k} M_j(x) I_j(x),$$

where

$$(1.2) \qquad M_j(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} Q_j(\sigma x).$$

For $\sigma_1 \in G$

$$(1.3) \quad M_j(\sigma_1 x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} Q_j(\sigma\sigma_1 x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} Q_j(\sigma x) = M_j(x).$$

Thus $M_j(x)$ is a homogeneous invariant, $1 \leqslant j \leqslant k$. Since deg $M.$ + deg I_j = deg P and deg $I_j > 0$, we have deg $M_j <$ deg P, $1 \leqslant j \leqslant k_j$ The proof of Theorem 1.1 now follows by induction. It obviously holds for deg $P = 0$ and suppose that it holds for deg $P \leqslant m - 1$. Let deg $P = m$. M_j is a polynomial in $I_1, ..., I_k$ for $1 \leqslant j \leqslant k$. It follows from (1.1) that P is a polynomial in $I_1, ..., I_k$.

Noether's Proof: We prove first a preliminary lemma. For any n-tuple $a = (a_1, ..., a_n)$ of non-negative integers, let $|a| = a_1 + ... + a_n$.

LEMMA 1.1. Let

$$x_i = (x_{i1}, ..., x_{in}),\ x_i^a = x_{i1}^{a_1} \ldots x_{in}^{a_n},\ 1 \leqslant i \leqslant N,\ a = (a_1, ..., a_n)$$

being an arbitrary n-tuple of non-negative integers. $\sum_{i=1}^{N} x_i^a$ is a polynomial in the sums $\sum_{i=1}^{N} x_i^a$, $|a| \leqslant N$

Proof. For $n = 1$, the above states the well known fact that $\sum_{i=1}^{N} x_i^a$ is a polynomial in $\sum_{i=1}^{N} x_i, ..., \sum_{i=1}^{N} x_i^N$ ([22], Vol. 1, p. 81). Suppose that the result holds for $n - 1$, $n \geqslant 2$. The case $(a_1, ..., a_{n-1}, 0)$ is identical with $(a_1, ..., a_{n-1})$. Hence the result holds for $(a_1, ..., a_n)$, $a_n = 0$. Suppose it holds for $(a_1, ..., a_n)$, where $a_n < m$ ($n \geqslant 2$ and $m \geqslant 1$). We show that it holds for $a_n = m$ and so, by induction, for all $(a_1, ..., a_n)$. Increase a_{n-1} by 1, decrease a_n by 1, keeping the other a_i's fixed, and call the new n-tuple b. Let $s_1, ..., s_l$ be a denumeration of the sums $\sum_{i=1}^{N} x_i^a$, $|a| \leqslant N$. Then

$$(1.4) \qquad \sum_{i=1}^{N} x_i^b = F(s_1, ..., s_l)$$

where $F = F(u_1, ..., u_l)$ is a polynomial in the u_i's. Differentiate both sides of (1.4) with respect to $x_{j,n-1}$ and multiply by x_{jn}. We obtain

$$(1.5)\qquad (a_{n-1}+1)\, x_j^a = \sum_{k=1}^{l} \frac{\partial F}{\partial u_k}(s_1, \ldots, s_l)\, \frac{\partial s_k}{\partial x_{j,n-1}}\, x_{jn}$$

If $s_k = \sum_{i=1}^{N} x_i^c$, $c = (c_1, \ldots, c_n)$, then

$$\frac{\partial s_k}{\partial x_{j,n-1}}\, x_{jn} = c_{n-1}\, x_j^d, \; d = (c_1, \ldots, c_{n-2}, c_{n-1}-1, c_n+1)\,.$$

It follows by summing both sides of (1.5) over j, $1 \leqslant j \leqslant N$, that $\sum_{i=1}^{N} x_i^a$ is a polynomial in $s_1, \ldots, s_l$.

We can now provide Noether's proof. Let $P(x)$ be a homogeneous invariant of degree m. Thus $P(x) = \sum_{|a|=m} c_a x^a$, the c_a's being elements of k. We have

$$(1.6)\qquad P(x) = \frac{1}{|G|} \sum P(\sigma x) = \sum_{|a|=m} \frac{c_a}{|G|} J_a(x)$$

where $J_a(x) = \sum_{\sigma \varepsilon G} (\sigma x)^a$

By Lemma 1.1, each J_a is a polynomial in the J_a's with $|a| \leqslant |G|$. It follows from (1.6) that the J_a's, $|a| \leqslant |G|$, form a basis for the invariants of G.

Comparing the two methods of proof, Noether's has the advantage of producing an explicit basis. It is however a proof of "finite type" which can not be generalized to continuous groups. Hilbert's proof goes through directly for continuous compact groups acting on the Euclidean space R^n, as we then have the notion of Haar measure and the group averaging process can be carried out.

We observe that the basis produced by Noether's method consists of $\binom{|G|+n}{n}$ elements of degree $\leqslant |G|$. The main interest in these bounds is their universality. In individual cases, they may prove to be very poor. Consider, for instance, the case $G = S_n$. Noether's method yields a basis of $\binom{n!+n}{n} \sim (n!)^{n-1}$ (as $n \to \infty$) homogeneous invariants of degrees $\leqslant n!$, while in actuality there are n basic homogeneous invariants of degree $\leqslant n$.

We obtain the following lower bound for the number of elements in a basis.

THEOREM 1.2. *Let $I_1, \ldots, I_l$ form a basis for the invariants of G. We may choose from the I_j's n elements which are algebraically independent over k. Thus $l \geqslant n$.*

Proof. Let $k(x_1, \ldots, x_n)$ be the field of rational functions in the indeterminates $x_1, \ldots, x_n$ with coefficients in k, a similar meaning being attached to $k(I_1, \ldots, I_l)$. We show that $k(x_1, \ldots, x_n)$ is a finite extension of $k(I_1, \ldots, I_l)$. Let $x_i(x) = x_i$ and set

$$(1.7) \qquad p_i(X) = \prod_{\sigma \varepsilon G} (X - x_i(\sigma x)) = X^{|G|-1} + a_1 X^{|G|-1} + a_1 X^{|G|-1} + \ldots + a_{|G|}$$

It is readily checked that the coefficients a_j are polynomials which are invariant under G. Thus each $a_j \in k(I_1, \ldots, I_l)$. Since $p_i(x_i) = 0$, we conclude that $x_i, \ldots, x_n$ are algebraic over $k(I_1, \ldots, I_l)$. Hence $k(x_1, \ldots, x_n)$ is a finite extension of $k(I_i, \ldots, I_l)$.

Let $K = k(\alpha_1, \ldots, \alpha_s)$ be the field obtained by adjoining $\alpha_1, \ldots, \alpha_s$ to k. We may define the transcendence degree of K over k to be the maximum number of α_i's which are algebraically independent over k ([22], Vol. 1, p. 201). We denote this degree by Tr.deg. K/k. If we have three fields $k \subset K \subset L$, then it is known that

$$(1.8) \qquad \text{Tr.deg. } L/k = \text{Tr.deg. } L/K + \text{Tr.deg. } K/k$$ ([22], Vol. 1, p. 202).

Apply (1.8) with $L = k(x_1, \ldots, x_n)$, $K = k(I_1, \ldots, I_l)$. Then Tr.deg. $L/k = n$ and the finiteness of L over K means that Tr.deg. $L/K = 0$. Hence Tr.deg. $K/k = n$, which means that we may choose n I_j's which are algebraically independent over k.

2. MOLIEN'S FORMULA

For each integer $m \geqslant 0$, the homogeneous invariants of degree m form a finite dimensional vector space over k of dimension δ_m. We derive an interesting and useful formula for the δ_m's.

THEOREM 1.3. (Molien's Formula [16]). *Let $\omega_1(\sigma), \ldots, \omega_n(\sigma)$ be the eigenvalues of σ. Then*

$$(1.9) \qquad \sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{|G|} \sum_{\sigma \varepsilon G} \frac{1}{(1 - \omega_1(\sigma) t) \ldots (1 - \omega_n(\sigma) t)}$$

REMARK. (1.9) is to be interpreted as an identity between two formal power series. I.e. if the right side is expanded as a formal power series, then its coefficients are identical with the δ_m's.

We require the following

LEMMA 1.2. Let W be the subspace fixed by G.
Then $\dim W = \frac{1}{|G|} \sum_{\sigma \varepsilon G} \mathrm{Tr}\,(\sigma)$.

Proof. Let $\{v_1, ..., v_r\}$ be a basis for W and augment this to a basis $\{v_1, ..., v_n\}$ for V. For $\sigma_1 \in G$ and $v \in V$, we have

$$\sigma_1 \Big(\sum_{\sigma \varepsilon G} \sigma v \Big) = \sum_{\sigma \varepsilon G} (\sigma_1 \sigma) v = \sum_{\sigma \varepsilon G} \sigma v ,$$

so that $\sum_{\sigma \varepsilon G} \sigma v \in W$. It follows that

$$\frac{1}{|G|} \sum_{\sigma \varepsilon G} \sigma v_i = v_i, \; 1 \leqslant i \leqslant r ,$$

and

$$\frac{1}{|G|} \sum_{\sigma \varepsilon G} \sigma v_i = \sum_{j=1}^{r} a_{ij} v_j, \; r+1 \leqslant i \leqslant n ,$$

the a_{ij}'s $\in k$. Hence

$$\frac{1}{|G|} \sum_{\sigma \varepsilon G} Tr\ \sigma = TR \left(\frac{1}{|G|} \sum_{\sigma \varepsilon G} \sigma \right) = r = \dim W .$$

Proof of Theorem 1.3. Let $\tilde{k}$ = algebraic closure of k. For any $\sigma \in G$, we can find a matrix τ with entries in $\tilde{k}$ so that $\tau\, \sigma\, \tau^{-1} = d$, d being diagonal and the diagonal entries being the eigenvalues of σ. Let R_m, $\tilde{R}_m$ denote respectively the space of homogeneous polynomials with coefficients from k, $\tilde{k}$. Let $(\mathrm{Tr}\,\sigma)_m$ = trace of σ as a transformation on R_m = trace of σ as a tranformation on $\tilde{R}_m$. Let $(\mathrm{Tr}\, d)_m$ = trace of d as a transformation on $\tilde{R}_m$. We have $d\,(P\,(x)) = P\,(d^{-1} x)$ for any polynomial $P(x)$. In particular, for any monomial x^a, we have $d\,(x^a) = \omega^a\,(\sigma^{-1})$, where $\omega\,(\sigma) = (\omega_1\,(\sigma), ..., \omega_n\,(\sigma))$. The monomials x^a form a basis for R_m and $\tilde{R}_m$. We conclude that

$$(1.10) \qquad (Tr\,\sigma)_m = (Tr\, d\,)_m = \sum_{|a|=m} \omega^a\,(\sigma^{-1}) .$$

(1.10) and Lemma 1.2 yield

$$(1.11)\qquad \delta_m = \frac{1}{|G|}\sum_{\sigma \varepsilon G}(Tr\,\sigma)_m = \frac{1}{|G|}\sum_{\sigma \varepsilon G}\ \sum_{|a|=m}\omega^a(\sigma)\,.$$

Multiply both sides of (1.11) by t^m and sum over m from 0 to ∞. We get

$$\sum_{m=0}^{\infty}\delta_m t^m = \frac{1}{|G|}\sum_{m=0}^{\infty}\sum_{\sigma\varepsilon G}\ \sum_{|a|=m}\omega^a(\sigma)\,t^m$$

$$= \frac{1}{|G|}\sum_{\sigma\varepsilon G}\Big\{\sum_{m=0}^{\infty}\omega_1^m(\sigma)\,t^m \dots \sum_{m=0}^{\infty}\omega_n^m(\sigma)\,t^m\Big\}$$

$$= \frac{1}{|G|}\sum_{\sigma\varepsilon G}\frac{1}{(1-\omega_1(\sigma)\,t)\dots(1-\omega_n(\sigma)\,t)}$$

CHAPTER II

INVARIANT THEORETIC CHARACTERIZATION OF FINITE REFLECTION GROUPS

1. Chevalley's Theorem

We showed in chapter I that we can always find a finite number of homogeneous invariants forming a basis for the invariants of G and that this set must contain at least n elements, where n = dim V. We show that this lower bound is attained only for the finite reflection groups. We first define these groups.

Definition 2.1. Let σ be a linear transformation acting on the n-dimensional vector space V. σ is a reflection $\Leftrightarrow$ σ fixes an $n-1$ dimensional hyperplane π and σ is of finite order > 1. π is called the reflecting hyperplane (r.h.) of σ.

Remark. Choose $v \notin \pi$. and let $\sigma v = \zeta v + p$, $p \in \pi$. If $\zeta = 1$, then $\sigma^m v = v + mp$, contradicting that σ is of finite order. Hence $\zeta \neq 1$. Let $v' = v + (\zeta - 1)^{-1} p$ and choose $p_1, \dots, p_{n-1}$ as a basis for π. Then $\sigma p_i = p_i$, $1 \leqslant i \leqslant n-1$, $\sigma v' = \zeta v'$. ζ is a root of 1 in k which is distinct from 1, as σ is of finite order > 1. Thus σ is a reflection iff relative to some basis, the matrix for σ is diagonal, $n-1$ of the diagonal entries equalling 1 and the remaining one equalling a root of 1 in k distinct from 1.

DEFINITION 2.2. G is a finite reflection group acting on $V \Leftrightarrow G$ is a finite group generated by reflections on V.

As an example of a finite reflection group, let $G = S_n$. It is well known that S_n is generated by transpositions. The transposition of the variables x_i, x_j $(i \neq j)$ fixes the hyperplane $x_i - x_j = 0$, so that it is a reflection.

We have the following result

THEOREM 2.1 (Chevalley [4]). *Let G be a finite reflection group acting on the n-dimensional vector space V. The invariants of G have a basis consisting of n homogeneous elements which are algebraically independent over k.*

Let $k[x]$ denote the ring of polynomials in $x_i, ..., x_n$ with coefficients in k. We prove the following.

LEMMA 2.1. Let $I_1, ..., I_m$ be invariant polynomials of G, $I_1 \notin (I_2, ..., I_m)$ = the ideal in $k[x]$ generated by $I_2, ..., I_m$. Suppose that $P_1 I_1 + ... + P_m I_m = 0$, the P_i's being polynomials with P_1 homogeneous. Then $P_1 \in \mathscr{I}$, where $\mathscr{I}$ is the ideal in $k[x]$ generated by the homogeneous invariants of positive degree.

Proof of Lemma 2.1. The proof proceeds by induction on $\deg P_1$. Suppose $\deg P_1 = 0$, so that $P_1 = c \in k$. If $c \neq 0$, then $I_1 \in (I_2, ..., I_m)$, contrary to assumption. Hence $c = 0 \Rightarrow P_1 \in \mathscr{I}$. Let $\deg P_1 = n > 0$. Let σ be a reflection in G and $L = 0$ the equation of its r.h. (L is a linear homogeneous polynomial). We have $P_1(x) I_1(x) + ... + P_m(x) I_m(x) = 0$, $P_1(\sigma x) I_1(x) + ... + P_m(\sigma x) I_m(x) = 0$. Hence $[P_1(\sigma x) - P_1(x)] I_1(x) + ... + [P_m(\sigma x) - P_m(x)] I_m(x)$. For $L(x) = 0$, $\sigma(x) = x$, so that $P_i(\sigma x) - P_i(x) = 0$ whenever $L(x) = 0, 1 \leqslant i \leqslant m$. Since $L(x)$ is irreducible it follows that

$$\frac{P_i(\sigma x) - P_i(x)}{L(x)}$$

is a polynomial, $1 \leqslant i \leqslant m$. We have

$$\left[\frac{P_1(\sigma x) - P_1(x)}{L(x)}\right] I_1(x) + \dots + \left[\frac{P_m(\sigma x) - P_m(x)}{L(x)}\right] I_m(x) = 0 .$$

$$\deg \left[\frac{P_1(\sigma x) - P_1(x)}{L(x)}\right] < \deg P_1(x) ,$$

so that by the induction hypothesis

$$\frac{P_1(\sigma x) - P_1(x)}{L(x)} \equiv 0 \pmod{\mathscr{I}}.$$

Hence $P_1(\sigma x) \equiv P_1(x) \pmod{\mathscr{I}}$. Since the σ's generate G, this congruence holds for $\sigma \in G$. We conclude that

$$P_1(x) \equiv \frac{1}{|G|} \sum_{\sigma \varepsilon G} P_1(\sigma x) \pmod{\mathscr{I}}.$$

The polynomial $\frac{1}{|G|} \sum_{\sigma \varepsilon G} P_1(\sigma x)$ is invariant and homogeneous of degree $n \geqslant 1$. Hence it $\in \mathscr{I}$, so that $P_1 \in \mathscr{I}$.

Proof of Theorem 2.1. We choose $I_1, \ldots, I_r$ to be homogeneous invariants of positive degree forming a minimal basis for $\mathscr{I}$. Hilbert's proof of Theorem 1.1 shows that $I_1, \ldots, I_r$ form a basis for the invariants of G. We show that $I_1, \ldots, I_r$ are algebraically independent, so that $r = n$.

Suppose, to the contrary, that $I_1, \ldots, I_r$ are algebraically dependent. Choose $H(y_1, \ldots, y_r)$ to be a polynomial of minimal positive degree so that $H(I_1(x), \ldots, I_r(x)) = 0$. Let x-degree of any monomial $y_1^{a_1} \ldots y_r^{a_r}$ be $d_1 a_1 + \ldots + d_r a_r$, where $d_i = \deg I_i$. We may assume that all x-degrees of the monomials appearing in H are the same. Let

$$H_i(x) = \frac{\partial H}{\partial y_i}(I_1(x), \ldots, I_r(x)), \ 1 \leqslant i \leqslant r.$$

The H_i's are invariant homogeneous polynomials, as all monomials in H have equal x-degree. Since $H(y_1, \ldots, y_n)$ is of positive degree, some $\frac{\partial H}{\partial y_i} \neq 0$, It follows that the corresponding $H_i(x) \neq 0$, as H was chosen to be of minimal degree; i.e. not all H_i's $= 0$. We relabel indices so that $H_1, \ldots, H_s$, $1 \leqslant s \leqslant r$, are ideally independent (i.e. none of the H_i's is in the ideal generated by the others) and $H_{s+j} \in (H_1, \ldots, H_s)$. $1 \leqslant j \leqslant r - s$. Thus $H_{s+j} = \sum_{i=1}^{s} V_{ji} H_i$, $1 \leqslant j \leqslant r - s$, where each V_{ji} is a homogeneous polynomial of degree $d_i - d_{s+j}$ (V_{ji} is interpreted to be 0 if this degree is negative). Differentiating the relation $H(I_1(x), \ldots, I_r(x)) = 0$ with respect to x_k, we obtain

$$(2.1) \qquad \sum_{i=1}^{r} H_i \frac{\partial I_i}{\partial x_k} = \sum_{i=1}^{s} H_i \frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} H_{s+l} \frac{\partial I_{s+l}}{\partial x_k}$$

$$= \sum_{i=1}^{s} H_i \left[\frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k}\right] = 0.$$

Since

$$\frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k}$$

is homogeneous of degree $d_i - 1$, we conclude from Lemma 2.1 that

$$(2.2) \qquad \frac{\partial I_i}{\partial x_k} + \sum_{l=1}^{r-s} V_{li} \frac{\partial I_{s+l}}{\partial x_k} = \sum_{j=1}^{r} B_j I_j, \; 1 \leqslant i \leqslant s \, ,$$

where the B_j's are homogeneous and each term in (2.2) is homogeneous of degree $d_i - 1$. This forces $B_i = 0$. Multiply both sides of (2.2) by x_k and sum over k. We conclude, by Euler's identity for homogeneous polynomials,

$$(2.3) \qquad d_i I_i + \sum_{l=1}^{r-s} V_{li} \, d_{s+l} I_{s+l} = \sum_{j=1}^{r} A_j I_j \, ,$$

the A_j's being homogeneous with $A_i = 0$.

(2.3) shows that $I_i \in (I_1, ..., I_{i-1}, I_{i+1}, ..., I_r)$, contradicting the minimality of the basis $I_1, ..., I_r$. Hence $I_1, ..., I_r$ are algebraically independent and $r = n$.

2. The Theorem of Shephard and Todd

We obtain in this section a converse to Chevalley's Theorem, thereby obtaining an invariant theoretical characterization of finite reflection groups. We first prove several preliminary results.

Lemma 2.2. Let H be a finite group of linear transformations acting on the n-dimensional space V and fixing the $n-1$ dimensional hyperplane π. The elements of H have a common eigenvector $v \in V - \pi$. Let $\sigma(v) = \zeta(\sigma) v$, $\sigma \in H$. $\zeta(\sigma)$ is an isomorphism from H into the multiplicative group of the roots of unity in k. It follows that H is a cyclic group.

Remark. The above lemma is a consequence of Maschke's Theorem proven in section 2.3. We provide another proof below.

Proof. Let $\sigma_1 \in H$, $\sigma_1 \neq e$ (the identity of H). By the remark following Definition 2.1, there exists $v \in V - \pi$ such that $\sigma_1(v) = \zeta_1 v$, ζ_1 being a root of unity $\neq 1$. For $\sigma \in H$, let $\sigma(v) = \zeta(\sigma) v + p(\sigma)$, $\zeta(\sigma) \in k$ and $p(\sigma) \in \pi$. Let $\sigma^* = \sigma_1^{-1} \sigma^{-1} \sigma_1 \sigma$. Then $\sigma^*(v) = v + (1 - \zeta_1) p(\sigma)$. Since σ^* is of finite order, $(1 - \zeta_1) p(\sigma) = 0 \Rightarrow p(\sigma) = 0$. Hence $\sigma(v) = \zeta(\sigma) v$. $\zeta(\sigma)$ is clearly an isomorphism from H into U, the multiplicative group of the roots of unity in k. U is known to be cyclic ([22], Vol. 1, p. 112). It follows that $\zeta(H)$, a subgroup of U, is cyclic and so H is cyclic.

THEOREM 2.2. *Let G be a finite group acting on the n-dimensional space V. Let $I_1, \ldots, I_n$ be homogeneous polynomials forming a basis for the invariants of G. Let $d_1, \ldots, d_n$ be the respective degrees of $I_1, \ldots, I_n$. Then*

$$\prod_{i=1}^{n} d_i = |G|, \quad \sum_{i=1}^{n} (d_i - 1) = r \tag{2.4}$$

where r = number of reflections in G.

Proof. By Theorem 1.2, $I_1, \ldots, I_n$ are algebraically independent. Let $I(x)$ be a homogeneous invariant of degree m. Then I is a linear combination of the monomials $I_1^{a_1} \ldots I_n^{a_n}$ where $a_1 d_1 + \ldots a_n d_n = m$. Furthermore, these monomials are linearly independent over k, as $I_1, \ldots, I_n$ are algebraically independent over k. It follows that the dimension δ_m of homogeneous invariants of degree m = number of non-negative integer solutions to $a_1 d_1 + \ldots + a_n d_n = m$. Hence

$$\sum_{m=0}^{\infty} \delta_m t^m = \frac{1}{(1-t^{d_1}) \ldots (1-t^{d_n})}. \tag{2.5}$$

(1.9) and (2.5) yield

$$\frac{1}{|G|} \sum_{\sigma \varepsilon G} \frac{1}{(1-\omega_1(\sigma)t) \ldots (1-\omega_n(\sigma)t)} = \frac{1}{(1-t^{d_1}) \ldots (1-t^{d_n})} \tag{2.6}$$

Expand both sides of (2.6) in powers of $(1-t)$. Let $\mathcal{R}$ = set of reflections in G and $\zeta(\sigma)$ = eigenvalue of the reflection σ which $\neq 1$. We have

$$\frac{1}{|G|} \sum_{\sigma \varepsilon G} \frac{1}{(1-\omega_1)(\sigma)t) \ldots (1-\omega_n(\sigma)t)} \tag{2.7}$$

$$= \frac{1}{|G|} \frac{1}{(1-t)^n} + \frac{1}{|G|} \sum_{\sigma \varepsilon \mathcal{R}} \frac{1}{1-\zeta(\sigma)} \frac{1}{(1-t)^{n-1}} + \ldots$$

$$\frac{1}{(1-t^{d_1}) \ldots (1-t^{d_n})} = \prod_{i=1}^{n} \frac{1}{d_i(1-t) - \binom{d_i}{2}(1-t)^2 + \ldots \pm (1-t)^{d_i}} \tag{2.8}$$

$$= \frac{1}{\prod_{i=1}^{n} d_i} \frac{1}{(1-t)^n} + \frac{\frac{1}{2} \sum_{i=1}^{n} (d_i - 1)}{\prod_{i=1}^{n} d_i} \frac{1}{(1-t)^{n-1}} + \ldots$$

Equating coefficients of (2.7), (2.8), we get

$$\prod_{i=1}^{n} d_i = |G|, \quad \sum_{i=1}^{n} (d_i - 1) = 2 \sum_{\sigma \varepsilon \mathscr{R}} \frac{1}{1-\zeta(\sigma)}. \tag{2.9}$$

We evaluate the sum

$$\sum_{\sigma \varepsilon \mathscr{R}} \frac{1}{1-\zeta(\sigma)}:$$

Let π be any r.h. Let $H_\pi = \{\sigma \mid \sigma \in G \text{ and } \sigma \text{ fixes } \pi\}$. Thus H_π is the subgroup of G consisting of the identity and those reflections in G with r.h. π. Applying Lemma 2.2 to H_π, we conclude that there exists $v \notin \pi$ such that $\sigma(v) = \zeta(\sigma)\, v$ for $\sigma \in H_\pi$. Let $H'_\pi = H_\pi - \{e\}$. Since $\zeta(\sigma^{-1}) = (\zeta(\sigma))^{-1}$, we obtain

$$\sum_{\sigma \varepsilon H'_\pi} \frac{1}{1-\zeta(\sigma)} = \sum_{\sigma \varepsilon H'_\pi} \frac{1}{1-\zeta(\sigma^{-1})} \tag{2.10}$$

$$= \sum_{\sigma \varepsilon H'_\pi} \left(1 - \frac{1}{1-\zeta(\sigma)}\right) = |H'_\pi| - \sum_{\sigma \varepsilon H'_\pi} \frac{1}{1-\zeta(\sigma)}.$$

Hence

$$\sum_{\sigma \varepsilon H'_\pi} \frac{1}{1-\zeta(\sigma)} = \frac{|H'_\pi|}{2}. \tag{2.11}$$

Summing both sides of (2.11) over all r.h. π, we get

$$\sum_{\sigma \varepsilon \mathscr{R}} \frac{1}{1-\zeta(\sigma)} = \frac{r}{2}. \tag{2.12}$$

(2.9), (2.12) yield Theorem 2.2.

THEOREM 2.3. *Let $f_1, \ldots, f_n$ be polynomials in the variables $x_1, \ldots, x_n$. $f_1, \ldots, f_n$ are algebraically independent over $k \Leftrightarrow$*

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)} \neq 0.$$

Proof. Suppose that $f_1, \ldots, f_n$ are algebraically independent. Then $G(f_1, \ldots, f_n) = 0$ for some polynomial $G = G(y_1, \ldots, y_n)$. Assume that $G(y_1, \ldots, y_n)$ is of minimal positive degree. Differentiating this relation with respect to x_j, we get

$$\sum_{i=1}^{n} \frac{\partial G}{\partial y_i}(f_1, \ldots, f_n) \frac{\partial f_i}{\partial x_j} = 0, \ 1 \leqslant j \leqslant n \,. \tag{2.13}$$

(2.13) is a system of linear equations (with coefficients in $k(x_1, \ldots, x_n)$) in the unknowns $H_i(x) = \dfrac{\partial G}{\partial y_i}(f_1, \ldots, f_n)$, $1 \leqslant i \leqslant n$. $\dfrac{\partial G}{\partial y_i} \neq 0$ for some i, as G is not constant, and $\deg \dfrac{\partial G}{\partial y_i} < \deg G$. It follows that the corresponding $H_i(x) \neq 0$. Thus the linear system (2.13) has a non-zero solution, so that its determinant

$$\frac{\partial (f_1, \ldots, f_n)}{\partial (x_1, \ldots, x_n)} \neq 0 \,.$$

Conversely, let $f_1, \ldots, f_n$ be algebraically independent. For each i, $x_i, f_1, \ldots, f_n$ are algebraically dependent. Hence there exists a polynomial $G_i(x_i, y_1, \ldots, y_n)$ of minimal positive degree in x_i such that $G_i(x_i, f_1, \ldots, f_n) = 0$. Differentiating these relations with respect to x_k, we get

$$\sum_{j=1}^{n} \frac{\partial G_i}{\partial y_j}(x_i, f_1, \ldots, f_n) \frac{\partial f_j}{\partial x_k} + \frac{\partial G_i}{\partial x_k}(x_i, f_1, \ldots, f_n)\, \delta_{ik}, \ 1 \leqslant k \leqslant n \,, \tag{2.14}$$

δ_{ik} denoting the Kronecker symbol. (2.14) may be rewritten in matrix notation as

$$\left(\frac{\partial G_i}{\partial y_j}\right) \cdot \left(\frac{\partial f_i}{\partial x_j}\right) = D \tag{2.15}$$

where the entries of D are

$$- \delta_{ij} \frac{\partial G_i}{\partial x_j} \,.$$

$\det D \neq 0$, as x_i — degree of $\dfrac{\partial G_i}{\partial x_i} < x_i$ — degree of G_i, $1 \leqslant i \leqslant n$.

It follows from (2.15) that $\dfrac{\partial (f_1, \ldots, f_n)}{\partial (x_1, \ldots, x_n)} \neq 0$.

THEOREM 2.4. (Shephard and Todd [19]). *Let G be a finite group acting on the n-dimensional space V. Suppose there exists a basis of n homogeneous polynomials for the invariants of G. Then G is a finite reflection group.*

Proof. Let H be the subgroup of G generated by the reflections in G. By assumption G has n basic homogeneous invariants which, by Theorem 1.2, are algebraically independent. Since H is a finite reflection group, we conclude from Chevalley's Theorem that H has n basic homogeneous invariants $J_1, \ldots, J_n$ which are algebraically independent. Each I_i is invariant under H so that $I_i = I_i(J_1, \ldots, J_n)$, the latter quantity denoting a polynomial in the J_i's. We may assume that $I_i(J_1, \ldots, J_n)$ is a linear combination of monomials $J_1^{a_1} \ldots J_n^{a_n}$ whose x-degree $=$ deg I_i. We have

$$\text{(2.16)} \qquad \frac{\partial(I_1, \ldots, I_n)}{\partial(x_1, \ldots, x_n)} = \frac{\partial(I_1, \ldots, I_n)}{\partial(J_1, \ldots, J_n)} \cdot \frac{\partial(J_1, \ldots, J_n)}{\partial(x_1, \ldots, x_n)}$$

By Theorem 2.3,

$$\frac{\partial(I_1, \ldots, I_n)}{\partial(x_1, \ldots, x_n)} \neq 0$$

and (2.16) then shows that

$$\frac{\partial(I_1, \ldots, I_n)}{\partial(J_1, \ldots, J_n)} \neq 0 .$$

It follows that there is a rearrangement $k_1, \ldots, k_n$ of $1, \ldots, n$ so that

$$\frac{\partial I_{k_1}}{\partial J_1} \cdots \frac{\partial I_{k_n}}{\partial J_n} \neq 0 .$$

Hence $I_{k_i}(J_1, \ldots, J_n)$ is of positive degree in J_i and $\deg I_{k_i} \geqslant \deg J_i$, $1 \leqslant i \leqslant n$. Applying Theorem 2.2 both to G and H, we obtain

$$\text{(2.17)} \qquad \prod_{i=1}^{n} \deg J_i = |H|, \quad \prod_{i=1}^{n} \deg I_i = |G|$$

$$\text{(2.18)} \qquad \sum_{i=1}^{n} (\deg J_i - 1) = \sum_{i=1}^{n} (\deg I_i - 1) = r$$

where r = number of reflections in G = number of reflections in H.

Since $\deg I_{k_i} \geqslant \deg J_i$, $1 \leqslant i \leqslant n$, we conclude from (2.18) that $\deg I_{k_i} = \deg J_i$, $1 \leqslant i \leqslant n$. Hence $\prod_{i=1}^{n} \deg I_i = \prod_{i=1}^{n} \deg J_i$, and we conclude from (2.17) that $|G| = |H|$. Thus $G = H$ and G is a finite reflection group.

3. A Formula for $\dfrac{\partial (I_1, \ldots, I_n)}{\partial (x_1, \ldots, x_n)}$

We obtain a formula which shall be used in Chapter III.

THEOREM 2.5. *Let G be a finite reflection group acting on the n-dimensional space V. Let $I_1, \ldots, I_n$ be a basic set of homogeneous invariants for G. Let x be a coordinate system for V and $L_i(x) = 0$, $1 \leqslant i \leqslant r$, the r.h.'s for G, each L_i being linear and homogeneous. Then*

$$\frac{\partial (I_1, \ldots, I_n)}{\partial (x_1, \ldots, x_n)} = c \prod_{i=1}^{r} L_i(x) \tag{2.19}$$

c being a constant $\neq 0$.

Proof. Let J the left hand side of (2.19). We observe that J is a non-zero homogeneous polynomial of degree $\sum_{i=1}^{n} (d_i - 1)$. By Theorem 2.2, $\sum_{i=1}^{n} (d_i - 1) = r$, so that $\deg J = r$. If k is the real field R, we have the following simple proof of (2.19). $I_i = I_i(x_1, \ldots, x_n)$, $1 \leqslant i \leqslant n$, is a mapping from x-space to I-space. This mapping is not $1 - 1$ in any neighborhood of a point x lying in the r.h. $L_i(x) = 0$, as any point and its reflection get mapped into the same point I. It follows from the Implicit Function Theorem that $J(x) = 0$. whenever $L_i(x) = 0$. Thus $L_i \mid J$, $1 \leqslant i \leqslant r$, and so $\prod_{i=1}^{r} L_i \mid J$. Since J, $\prod_{i=1}^{r} L_i$ have the same degree r, we have $J = c \prod_{i=1}^{r} L_i$, $c \neq 0$.

For an arbitrary field k, the theorem is proven as follows. Let π be an r.h. with equation $L(x) = 0$ and H the subgroup of h elements in G fixing π. Thus there are $h - 1$ reflections in G with r.h. π. We show that $L^{h-1} \mid J$. By Lemma 2.2, H is a cyclic group generated by an element σ. Furthermore there exists $v \notin \pi$ and a primitive h-th root of 1 such that $\sigma(v) = \zeta v$. Choose a coordinate system $y = (y_1, \ldots, y_n)$ in V so that π has the equation $y_n = 0$ and $v = (0, \ldots, 0, 1)$ σ then becomes the transformation $(y_1, \ldots, y_{n-1}, y_n) \to (y_1, \ldots, y_{n-1}, \zeta y_n)$. Let $x = \tau y$ and $J_i(y) = I_i(\tau y)$, $1 \leqslant i \leqslant n$. We have

$$J_i(y_1, \ldots, y_{n-1}, \zeta y_n) = J_i(y_1, \ldots, y_{n-1}, y_n), \ 1 \leqslant i \leqslant n \tag{2.20}$$

Let $J_i = \sum A_m y_n^m$, the A_m's being polynomials in $y_1, \ldots, y_{n-1}$. (2.20) implies that $A_m = 0$ whenever $h \nmid m$, so that $A_m = 0, 0 \leqslant m \leqslant h - 1$. Since

$$\frac{\partial J_i}{\partial y_m} = \Sigma_m A_m y_n^{m-1},$$

we conclude

$$y_n^{h-1} \,\Big|\, \frac{\partial J_i}{\partial y_n}, \; 1 \leqslant i \leqslant n .$$

Hence

(2.21) $$y_n^{h-1} \,\Big|\, \frac{\partial (J_1, \ldots, J_n)}{\partial (y_1, \ldots, y_n)},$$

Since

$$\frac{\partial (J_1, \ldots, J_n)}{\partial (y_1, \ldots, y_n)} = J(x) \cdot \det \tau ,$$

(2.21) is equivalent to $L^{h-1}(x) \mid J(x)$. It follows that if $L_i(x) = 0$, $1 \leqslant i \leqslant r$, are the r.h.'s for G, then $\prod_{i=1}^{r} L_i \mid J$. But J, $\prod_{i=1}^{r} L_i$ have the same degree r, so that $J = c \prod_{i=1}^{r} L_i\, c \neq 0$.

4. Decomposition of Finite Reflection Groups

We shall decompose every finite reflection group into a direct product of irreducible ones and show that it suffices to study the invariant theory of the irreducible groups.

Definition 2.3. Let the group G act on V. G is said to be reducible iff there exists a proper subspace W invariant under G; i.e. $\sigma w \in W$ for $\sigma \in G$, $w \in W$. G is said to be completely reducible iff $V = V_1 \oplus V_2$, V_1 and V_2 being proper invariant subspaces. G is said to be irreducible iff it is not reducible.

Theorem 2.6. (Maschke [22], Vol. 2, p. 179). *Let G be a finite group acting on the vector space V. If G is reducible, then it is completely reducible.*

Proof. Let V_1 be a proper invariant subspace of V. Let V_2 be a complementary subspace. Thus for $v \in V$, we have a unique decomposition

$v = v_1 + v_2, v_i \in V_i \ (i=1, 2)$. Let $\eta \, v = v_2$ and set $\tau = \dfrac{1}{|G|} \sum_{\sigma \varepsilon G} \sigma \, \eta \, \sigma^{-1}$.

τ satisfies the following:

i) $\tau \sigma = \sigma \tau, \sigma \in G$. For $\sigma \tau = \dfrac{1}{|G|} \sum_{\sigma_1 \varepsilon G} \sigma \sigma_1 \eta \, (\sigma \sigma_1)^{-1} \sigma = \tau \sigma$

ii) $\tau v_1 = 0, v_1 \in V_1$. For $\sigma^{-1} v_1 \in V_1, \sigma \in G$, so that $\eta \, \sigma^{-1} v_1 = 0 \Rightarrow \tau v_1 = 0$

iii) $(1-\tau) \, v \in V_1, v \in V$, 1 denoting the identity of G. For $(1-\eta) \, v \in V_1$, so that $(1-\eta) \, \sigma^{-1} v \in V_1 \Rightarrow \sigma (1-\eta) \, \sigma^{-1} v \in V_1, \sigma \in G$. It follows that $(1-\tau) \, v = \dfrac{1}{|G|} \sum_{\sigma \varepsilon G} \sigma \, (1-\eta) \sigma^{-1} v \in V_1$.

Let $V_2' = \tau V$. V_2' is invariant under G as $\sigma (\tau v) = \tau (\sigma v)$. For any v, $v = \tau v + (1-\tau) \, v$. It follows from iii) that $V = V_1 + V_2'$. ii), iii) imply $\tau (1-\tau) = 0 \Leftrightarrow \tau = \tau^2$. Hence $\tau v_2' = v_2'$ for $v_2' \in V_2'$. Let $v_1 + v_2' = 0$, where $v_1 \in V_1, v_2' \in V_2'$. Applying τ to both sides, we get $v_2' = 0$ and so $v_1 = 0$. Hence $V = V_1 \oplus V_2'$.

Repeated application of Maschke's Theorem yields the

Corollary. *Let G be a finite group acting on the finite-dimensional vector space V. Then $V = V_1 \oplus \ldots \oplus V_s$, the V_i's being invariant subspaces of V and G acting irreducibly on each V_i.*

For finite reflection groups, we have

Theorem 2.7. *Let G be a finite reflection group acting on V. There exists a decomposition $V = V_1 \oplus \ldots \oplus V_s$ into invariant subspaces such that:*

1) *Let $G_i = G \,|_{V_i} =$ group of restrictions of elements of G to V_i. Then G is isomorphic to $G_1 \times \ldots \times G_s$*

2) *Each G_i, $1 \leqslant i \leqslant s$, is a reflection group acting irreducibly on V_i.*

Proof. By the corollary to Theorem 2.6, there exists a decomposition $V = V_1 \oplus \ldots \oplus V_s$, the V_i's being invariant subspaces and G_i irreducible for $1 \leqslant i \leqslant s$. We label the V_i's so that $V_1, \ldots, V_r$ are 1-dimensional and $G \,|_{V_i} =$ identity.

By the remark following Definition 2.1, for each reflection σ there exists an eigenvector $v \in V - \pi$, π being the r.h. for σ. Call v a root of G. We have

$$\dim (V_i + \pi) + \dim (V_i \cap \pi) = \dim V_i + \dim \pi \,. \tag{2.22}$$

If $V_i \not\subset \pi$, then $V_i + \pi = V$ and we conclude from (2.22) that $\dim V_i = \dim (V_i \cap \pi) + 1$. I.e. $V_i \cap \pi$ is a hyperplane in V_i and $\sigma |_{V_i}$ a reflection on V_i. Choose $u \in V_i - \pi$ so that u is an eigenvector of σ. u is a multiple of the root v, so that $v \in V_i$. Thus $\sigma |_{V_i}$ is a reflection of V_i if $v \in V_i$, and the identity if $v \notin V_i$. Furthermore, each root v is in some V_i, $r + 1 \leqslant i \leqslant s$, otherwise the corresponding reflection σ would have been the identity.

Let $\tilde{G}_i$ = subgroup generated by those reflections whose roots are in V_i, $1 \leqslant i \leqslant s$. It is readily checked that $G = \tilde{G}_1 \times \ldots \times \tilde{G}_s$, $G_i = \tilde{G}_i |_{V_i}$. If $\sigma \in \tilde{G}_i$ and $\sigma |_{V_i}$ = identity then σ = identity. The mapping $\sigma \to \sigma |_{V_i}$ is thus an isomorphism from $\tilde{G}_i$ onto G_i.

THEOREM 2.8. *Let G be a finite reflection group acting on V and decompose V as in Theorem 2.7. Every polynomial invariant under G is a polynomial in the invariant polynomials of $G_1, \ldots, G_s$.*

Proof. For each $v \in V$, write $v = v_1 + \ldots + v_s$, $v_i \in V_i$. By Theorem 2.7, for each $\sigma \in G$, we may write $\sigma v = \sigma_1 v_1 + \ldots + \sigma_s v_s$, $\sigma_i \in G_i$. For any polynomial function $p(v)$ on V, we have $p(v) = \sum_{i=1}^{N} p_{i1}(v_1) \ldots p_{is}(v_s)$ where $p_{ij}(v_j)$ is a polynomial function on V_j. If $p(v)$ is invariant under G, then

$$(2.23) \qquad p(v) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} p(\sigma v) = \sum_{i=1}^{N} I_{i1}(v_1) \ldots I_{is}(v_s)$$

where

$$(2.24) \qquad I_{ij}(v_j) = \frac{1}{|G_j|} \sum_{\sigma_j \varepsilon G_j} p_{ij}(\sigma_j v_j)$$

is an invariant of G_j.

CHAPTER III

THE DEGREES OF THE BASIC INVARIANTS

We determine the degrees of the basic homogeneous invariants in case G is a finite reflection group. We present two different methods. The first one (Theorem 3.8), restricts itself to the case where k is the real field and has the advantage of providing an effective method for computing the degrees. The second method (Theorem 3.14) is valid for an arbitrary field of characteristic 0, but is less effective than the first in the real case.

We first prove that the degrees of the basic invariants are independent of any particular basis.

THEOREM 3.1. *Let G a finite reflection group acting on the n-dimensional vector space V. Let $I_1, \ldots, I_n$ be homogeneous polynomials of respective degrees $d_1 \leqslant \ldots \leqslant d_n$ forming a basis for the invariants of G. $d_1, \ldots, d_n$ are independent of the chosen basis $I_1, \ldots, I_n$.*

Proof. Let $J_1, \ldots, J_n$ be another set of homogeneous invariants forming a basis for the invariants of G. Let $d_1' \leqslant \ldots \leqslant d_n'$ be the respective degrees of $J_1, \ldots, J_n$. We must show that $d_i' = d_i$, $1 \leqslant i \leqslant n$. If not, then let i_0 be the smallest i such that $d_{i_0}' \neq d_{i_0}$, say $d_{i_0}' < d_{i_0}$. Each J_i is a polynomial in those I_i's whose degree $\leqslant \deg J_i$. It follows that for $1 \leqslant i \leqslant i_0$, $J_i = P_i(I_1, \ldots, I_{i_0-1})$, $\mathrm{P}_i(y_1, \ldots, y_{i_0-1})$ being a polynomial in $y_1, \ldots, y_{i_0-1}$. Hence $J_1, \ldots, J_{i_0}$ are algebraically dependent over k ([22], Vol. 1, p. 181), contradicting that $J_1, \ldots, J_n$ are algebraically independent over k (Theorem 1.2). Thus $d_i' = d_i$, $1 \leqslant i \leqslant n$.

Theorem 3.1. shows that the numbers $d_1, \ldots, d_n$ are determined by G. We shall give an effective method for the computation of the d_i's in case the underlying field k is real. We first digress to discuss the classification of the finite real reflection groups.

1. THE CLASSIFICATION OF THE FINITE REAL REFLECTION GROUPS

These groups have been classified by Coxeter [6]. We give here a brief description of the theory, as we require it for the computation of the d_i's.

We first observe that we may assume G to be orthogonal.

THEOREM 3.2. *Let G be a finite group acting on the n-dimensional Euclidean space R^n. There exists a non-singular transformation τ on R^n such that the group $\tau^{-1} G \tau$ consists of orthogonal transformations.*

Proof. Let $P(x) = \sum_{\sigma \varepsilon G} (\sigma x, \sigma x)$ where $x = (x_1, \ldots, x_n)$ and (x, y) is the inner product of x and y. For $x \neq 0$, each $(\sigma x, \sigma x) > 0$ so that $P(x) > 0$. Furthermore for $\sigma_1 \in G$, $P(\sigma_1 x) = \sum_{\sigma \varepsilon G} (\sigma \sigma_1 x, \sigma \sigma_1 x) = \sum_{\sigma \varepsilon G} (\sigma x, \sigma x) = P(x)$. Thus $P(x)$ is a positive definite quadratic form invariant under G. Choose $x = \tau y$ so that $P(\tau y) = (y, y)$. We have $(\tau^{-1}\sigma\tau y, \tau^{-1}\sigma\tau y) = P(\sigma\tau y) = P(\tau y) = (y, y)$, $\sigma \in G$, so that the transformations $\tau^{-1}\sigma\tau$ are orthogonal.

Thus all transformations of G become orthogonal after a suitable linear change of variables. We assume from now on that G is orthogonal. If G is a finite reflection group, this condition is equivalent to demanding that all reflections of G are orthogonal. I.e. for any reflection σ, σ fixes all vectors in the r.h. π and $\sigma(v) = -v$, iff v is perpendicular to π. The two unit vectors perpendicular to π are called roots of G. The set of all roots is called the root system of G.

DEFINITION 3.1. Let F be a region of R^n, G a finite group acting on R^n. F is a fundamental region for G iff:

i) $\sigma_1 F \cap \sigma_2 F = \Phi$ whenever $\sigma_1 \neq \sigma_2$,

ii) $R^n = \bigcup_{\sigma \varepsilon G} \sigma \bar{F}$, $\bar{F}$ being the closure of F.

We remark that it suffices to know i) for $\sigma_1 = e$, the identity of G. For $\sigma_1 F \cap \sigma_2 F = \Phi$ iff $\sigma_1^{-1}(\sigma_1 F \cap \sigma_2 F) = F \cap \sigma_1^{-1}\sigma_2 F = \Phi$. If F is a fundamental region, then so is σF, $\sigma \in G$. The group G permutes these fundamental regions and acts transitively on them.

THEOREM 3.3. *Let G be a finite reflection group acting on R^n. Assume that the roots of G span R^n (G is then called a Coxeter group). The complement of the union of the r.h.'s of G consist of $|G|$ fundamental regions called the chambers of G. G permutes these chambers and acts transitively on them. Each chamber F is bounded by n r.h.'s called the walls of F. Let $r_1, \ldots, r_n$ be the n roots perpendicular to the n walls $W_1, \ldots, W_n$ and pointing into F, and let R_i be the reflection in W_i. The r_i's are linearly independent and $r_i \cdot r_j = -\cos \pi/p_{ij}$, $p_{ii} = 1$ and p_{ij} being an integer $\geqslant 2$ if $i \neq j$. The R_i's generate G.*

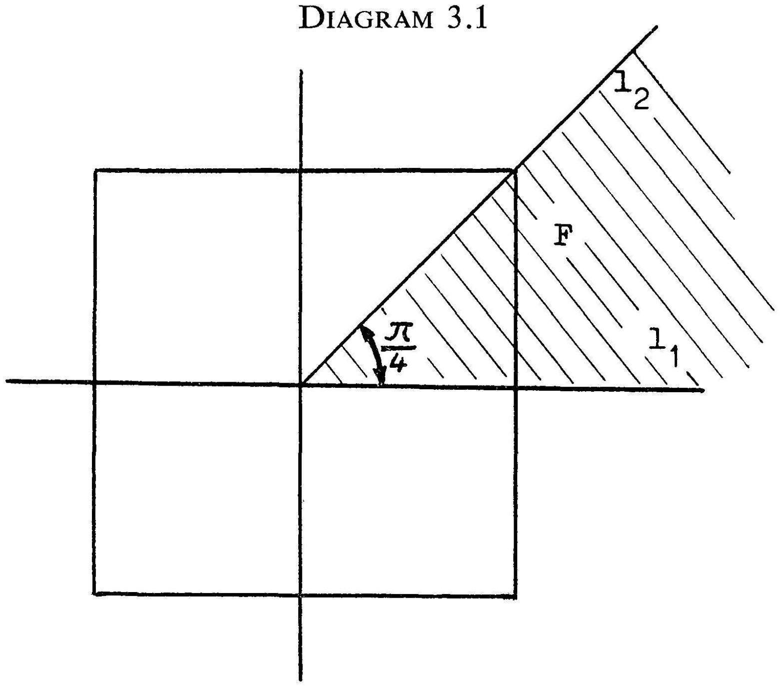
We have $F = \{x \mid x \cdot r_i > 0, 1 \leqslant i \leqslant n\}$. F may also be described as follows. Choose $\{r_1', \ldots, r_n'\}$ to be the dual basis to $\{r_1, \ldots, r_n\}$; i.e. $(r_i, r_j') = \delta_{ij}$. For any x, $x = \sum_{i=1}^{n} (x \cdot r_i) r_i'$. Thus

$$F = \{x \mid x = \sum_{i=1}^{n} \lambda_i r_i', \lambda_i > 0 \text{ for } 1 \leqslant i \leqslant n\}.$$

F is thus a wedge with n walls, the vectors r_i' lying along its edges. The angle between the walls W_i, W_j $(i \neq j)$ is readily seen to be π/p_{ij}. We refer to $\{r_1, \ldots, r_n\}$ as a fundamental system of roots and to $R_1, \ldots, R_n$ as a fundamental system of reflections.

As a simple illustration of the above concepts, we choose G to be the group of symmetries of a regular n-gon p_n. G is then called the dihedral group of order $2n$ and we denote it by H_2^n. Assume that the center of the polygon is at the origin. We choose in this case two rays l_1, l_2 emanating from the origin making an angle π/n, one of the rays passing through a vertex of p_n, the other through a mid-point of a side of p_n (see the diagram where $n = 4$). F is the wedge with sides l_1, l_2. The reflections in l_1, l_2 generate H_2^n.

DIAGRAM 3.1



For any Coxeter group G acting on R^n, we introduce the associated Coxeter graph $\mathscr{G}$ as follows. Let $\mathscr{G}$ consist of n points, called the nodes and label these as $1, \ldots, n$. We set up the $1-1$ correspondence $i \leftrightarrow r_i$, $r_1, \ldots, r_n$ being the fundamental root system of Theorem 3.3. The i-th and j-th node $(i \neq j)$ are joined by a branch iff $(r_i, r_j) \neq 0$. If this be the case then $p_{ij} \geqslant 3$; we mark the branch joining i to j by p_{ij} whenever $p_{ij} > 3$, and omit a mark if $p_{ij} = 3$. Eg. the graph associated with H_2^n is $\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$ for $n = 3$ and $\circ\overset{n}{-\!\!\!-\!\!\!-}\circ$ for $n \geqslant 4$.

The motivation for the rather artificial looking definition of $\mathscr{G}$ stems from the following facts.

THEOREM 3.4. *Let G be a Coxeter group acting on R^n. G is irreducible iff its corresponding graph is connected.*

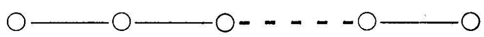
Proof. If the graph of G has more than one component, then the root system $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2$ where $\mathscr{R}_1, \mathscr{R}_2$ are disjoint and non-empty, the roots in $\mathscr{R}_1$ being perpendicular to those in $\mathscr{R}_2$. Let V be the span of the roots in $\mathscr{R}_1$. If σ is a reflection corresponding to a root in $\mathscr{R}_1$, then $\sigma\,|_V$ is a reflection of V. If σ is a reflection corresponding to a root in $\mathscr{R}_2$, then $\sigma\,|_V =$ identity. Since the reflections generate G, V is a proper invariant subspace.

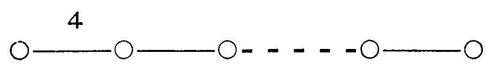
Conversely, let V be a proper invariant subspace of G. Then so is the orthogonal complement $V^{\perp}$. The proof of Theorem 2.7 shows that every root is either in Vor $V^{\perp}$. Since the roots span R^n, there are roots both in V and $V^{\perp}$. Since the roots in $\mathscr{R} \cap V$ are perpendicular to those of $\mathscr{R} \cap V^{\perp}$, the graph of G consists of at least two components.

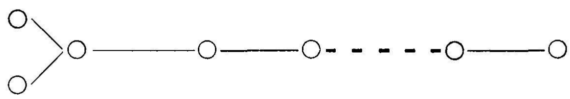
Coxeter has found all graphs corresponding to the irreducible Coxeter groups. We have the following classification.

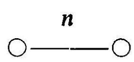
THEOREM 3.5. *Let $\mathscr{G}$ be a connected Coxeter graph. The following list exhausts the possibilities for $\mathscr{G}$.*

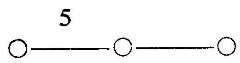
DIAGRAM 3.2

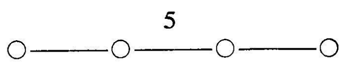
A_n $(n \geqslant 1)$ 

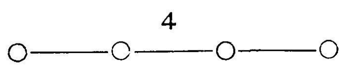
B_n $(n \geqslant 2)$ 

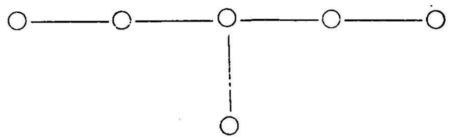
D_n $(n \geqslant 4)$ 

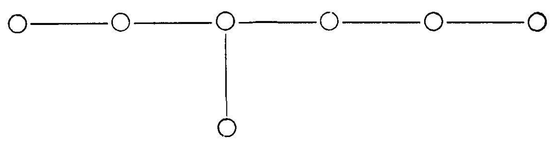
H_2^n $(n \geqslant 5)$ 

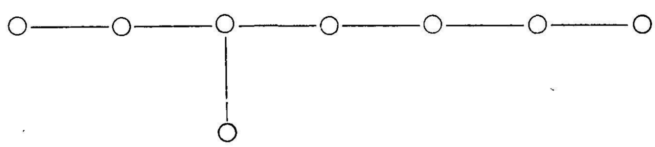
I_3 

I_4 

F_4 

E_6 

E_7 

E_8 

In each case the subscript denotes the number of nodes. The above list yields all irreducible Coxeter groups up to conjugacy. I.e. two irreducible groups which are conjugate subgroups of the orthogonal group have the same graph and conversely.

We give a brief description of the groups listed above.

A_n. Let S_{n+1} be the symmetric group of linear transformations $x_i' = x_{\sigma(i)}$, $1 \leqslant i \leqslant n+1$, $\sigma(i)$ being any permutation of $1, \ldots, n+1$. Let V $\{x \mid x_1 + \ldots + x_{n+1} = 0\}$ and $A_n = S_{n+1}|_V$. A_n is the group of symmetries of the regular n-simplex whose vertices are the permutations of $(-1, \ldots, -1, n)$.

B_n is the group of symmetries of the n cube with vertices $(\pm 1, \ldots, \pm 1)$. It consists of the $2^n n!$ linear transformations $x_i' = \pm x_{\sigma(i)}$, $1 \leqslant i \leqslant n$, the $\pm$ signs being chosen independently and $\sigma(i)$ an arbitrary permutation of $1, \ldots, n$.

D_n consists of the $2^{n-1} n!$ linear transformations $x_i' = \pm x_{\sigma(i)}$, $1 \leqslant i \leqslant n$, where $\sigma(i)$ is any permutation of $1, \ldots, n$ and the number of $-$ signs is even. It is readily checked that D_n is a subgroup of index 2 in B_n.

H_2^n is the dihedral group of $2n$ symmetries of the regular n-gon.

I_3 is the icosahedral group, i.e. the group of symmetries of the icosahedron.

I_4, F_4 are the groups of symmetries of certain 4-dimensional regular polytopes described in ([5], p. 156)

E_6, E_7, E_8 are the groups of symmetries of certain polytopes in R^6, R^7, R^8 known as Gosset's figures and described in ([5], p. 202)

An inspection of diagram 3.2 reveals that the graphs are of two types, those consisting of one chain and those consisting of three chains joined at a node. We refer to these graphs and their associated groups as being of types I and II. It can be shown that the groups of type I are precisely those which are the groups of symmetries of the regular polytopes ([5], p. 199).

The following theorem gives a complete description of all finite reflection groups acting on R^n.

THEOREM 3.6. *Let G be a finite reflection group acting on R^n. R^n is a direct sum of mutually orthogonal subspaces $V_0, V_1, \ldots, V_k$ with the following properties.*

1) *Let $G_i = G|_{V_i}$ = the restrictions of the elements of G to V_i. Then G is isomorphic to $G_0 \times G_1 \times \ldots \times G_k$.*
2) *G_0 consists only of the identity transformation on V_0.*

3) *Each* G_i, $1 \leqslant i \leqslant k$, *is one of the groups described in Theorem 3.5.* G *is a Coxeter group iff* $V_0 = 0$.

The proof of Theorem 3.6 is identical with that of Theorem 2.7. We simply observe that we may now choose the V_i's to be mutually orthogonal.

2. The Computation of the Degrees for Real Finite Reflection Groups

Let G be a finite irreducible orthogonal reflection group acting on the n-dimensional Euclidean space R^n. Let F be a fundamental region as described in Theorem 3.3 and $R_1, ..., R_n$ the n reflections in the walls of F. We shall relate the degrees $d_1, ..., d_n$ of the basic homogeneous invariants to the eigenvalues of $R_1 ... R_n$. We first prove

Theorem 3.7. *Let* $\sigma(i)$ *be any permutation of* $1, ..., n$. *Then* $R_1 ... R_n$ *is conjugate to* $R_{\sigma(1)} ... R_{\sigma(n)}$

Proof. Observe that $R_1 (R_1 ... R_n) R_1 = R_2 ... R_n R_1$ so that all cyclic permutations yield conjugate transformations. We may also permute any two adjacent R_i's for which the corresponding walls are orthogonal, as the R_i's then commute. Theorem 3.7 will then follow from the following

Lemma 3.1. Let $p_1, ..., p_n$ be nodes of a tree T. Any circular arrangement of $1, ..., n$ can be obtained from a sequence of interchanges of pairs i, j which are adjacent on the circle and for which p_i, p_j are not linked in T.

Proof of Lemma 3.1. We proceed by induction, the result being obvious for $n = 1$ or 2. We may assume that p_n is an end node of the tree, i.e. it links to precisely one other node. We first rearrange $1, ..., n-1$ as we wish. To show that this can be done, we just consider the possibility - - - *inj* - - - where p_i, p_j are not linked. If p_i, p_n are not linked, then we interchange first i, n and then i, j, obtaining - - - *nji* - - -. If p_j, p_n are not linked, then we first interchange j, n and then j, i, obtaining - - - *jin* - - -. We may therefore arrange $1, ..., n-1$ in the desired order. Shifting n in one direction, which is permissible as n just fails to commute with one element, we obtain the desired arrangement of $1, ..., n$.

In view of Theorem 3.7, the eigenvalues of $R_1 ... R_n$ are independent of the order in which the R_i's appear. They are also independent of the particularly chosen F. For let F' be another fundamental region as described in Theorem 3.3. Then $F' = \sigma F$, $\sigma \in G$. The reflections in the walls of F'

are given by $R_i' = \sigma R_i \sigma^{-1}$, $1 \leqslant i \leqslant n$, so that $R_1' \dots R_n' = \sigma R_1 \dots R_n \sigma^{-1}$. The main result of the present section is the following

THEOREM 3.8 (Coleman [8]). *Let* $R_1 \dots R_n$ *have order* h. *Let* $\zeta = e^{2\pi i/h}$. *The eigenvalues of* $R_1 \dots R_n$ *are given by* $\zeta^{(d_j - 1)}$, $1 \leqslant j \leqslant n$, *the* $d_j's$ *being the degrees of the basic homogeneous invariants of* G.

Theorem 3.8. was first obtained by Coxeter [7], who verified this fact for each group listed in Theorem 3.5. Coleman [8] supplied a general proof, using the fact that the number of reflections $= \frac{1}{2} nh$. This fact, which was at first known only by individual verification [7], was proven by Steinberg [20]. In view of Theorem 3.8, the numbers $m_j = d_j - 1$ are usually referred to as the exponents of the group G.

We begin by proving Steinberg's result, needed for the proof of Coleman's theorem. We require a preliminary lemma and employ the following terminology. Let $A = (a_{ij})$ be an $n \times n$ matrix with non-negative entries. We associate with A a graph $\mathscr{G}$ consisting of n nodes, connecting the nodes i, j iff $a_{ij} > 0$. A is said to be connected iff $\mathscr{G}$ is connected.

LEMMA 3.2. Let $A = (a_{ij})$ be a symmetric connected matrix. The largest eigenvalue λ of A is positive and a corresponding eigenvector e can be chosen all of whose entries are positive.

REMARK. The above is a special case of a theorem of Frobenius concerning the eigenvalues of matrices with non-negative entries [13]. Indeed the symmetry of A is not required. This extraneous assumption permits for a somewhat simpler proof and suffices for our purposes.

Proof. Let $Q(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ be the quadratic form associated with (a_{ij}). Then $\lambda = \operatorname{Max}_{\|x\|=1} Q(x) > 0$, where $\|x\|^2 = \sum_{i=1}^{n} x_i^2$. Choose $v = (v_1, \dots, v_n)$, $\|v\| = 1$, so that $Q(v) = \lambda$ and let $e = (e_1, \dots, e_n)$, where $e_i = |v_i|$, $1 \leqslant i \leqslant n$. Then $e_i \geqslant 0$, $1 \leqslant i \leqslant n$, and $\|e\| = 1$. As all $a_{ij} \geqslant 0$ and $\|e\| = 1$, we have $\lambda = Q(v) \leqslant Q(e) \leqslant \lambda$, so that $Q(e) = \lambda$. The latter implies $Ae = \lambda e$. It remains to show that each $e_i > 0$. Choose $e_j > 0$. Because of the connectivity assumption, we may choose $i_1, \dots, j_r = j$ so that $a_{ij_1}, a_{j_1 j_2}, \dots, a_{j_{r-1}, j}$ are all > 0. The relation $\lambda e_{j_{r-1}} = \sum_{k=1}^{n} a_{j_{r-1}, k} e_k$ shows that $e_{j_{r-1}} > 0$. Repeating this reasoning r times, we conclude that each $e_i > 0$.

THEOREM 3.9 (Steinberg [20]). *Let* $h =$ *order of* $R_1 \dots R_n$, $r =$ *number of reflections in* G. *Then* $r = \dfrac{nh}{2}$.

Proof. We may label the walls of the fundamental region F so that $W_1 \dots W_s$ are mutually perpendicular, and $W_{s+1}, \dots, W_n$ are mutually perpendicular (I.e. if the nodes corresponding to $W_1, \dots, W_s$ are black and those corresponding to $W_{s+1}, \dots, W_n$ are white, then each black node is linked only to white nodes and conversely). Let $E_1 = W_{s+1} \cap \dots \cap W_n$, $E_2 = W_1 \cap \dots \cap W_s$. Thus in terms of the dual basis $\{r_i'\}$, E_1 is the linear span of $r_1', \dots, r_s'$ and E_2 the linear span of $r_{s+1}', \dots, r_n'$. Let S $= R_{s+1} \dots R_n$, $T = R_1, \dots, R_s$ and denote the orthogonal complement of E_i, $i = 1, 2$, by $E_i^{\perp}$. The restriction of S to E_1, denoted by S_{E_1}, is the identity $r_{s+1}, \dots, r_n$ form a basis for $E_1^{\perp}$. Since they are orthogonal to each other, $R_i r_j = 0$ for $i \neq j$, $s + 1 \leqslant i, j \leqslant n$, so that $S_{E_1^{\perp}} = -$ identity. Similarly T_{E_2} = identity, $T_{E_2^{\perp}} = -$ identity. We require the following

LEMMA 3.3. Let G_0 be the $n \times n$ matrix $((r_i, r_j))$ and I the $n \times n$ identity matrix. $I - G_0$ is connected. Thus, by Lemma 3.2, $I - G_0$ has a biggest positive eigenvalue λ and a corresponding eigenvector e with positive entries. Let $\sigma = \sum_{i=1}^{s} e_i r_i'$, $\tau = \sum_{i=s+1}^{n} e_i r_i'$ [1]. The plane π, determined by σ and τ, has non-trivial intersection with $E_1^{\perp}$ and $E_2^{\perp}$. It follows that S_π (T_π) is a reflection of π in the line through σ (τ).

Proof. The entries of $I - G_0$ are $\geqslant 0$, as $(r_i, r_j) \leqslant 0$ whenever $i \neq j$. The irreducibility of G is equivalent to saying that $I - G_0$ is connected. Let

$$G_0 = \begin{pmatrix} I & A \\ A' & I \end{pmatrix}, \quad G_0^{-1} = \begin{pmatrix} B & C \\ C' & D \end{pmatrix},$$

where A, C are $s \times n - s$ matrices (we use I to denote the identity matrix for various degrees; here degree $I = s$). The relations $r_i = \sum_{j=1}^{n} (r_i, r_j) r_j'$, $r_i' = \sum_{i=1}^{n} (r_i', r_j') r_j$, $1 \leqslant i \leqslant n$, show that $G_0^{-1} = ((r_i', r_j'))$. Since $G_0^{-1} G_0 = I$, we have

$$BA + C = C' + DA' = 0 \tag{3.1}$$

Let e^1 be the vector consisting of the first s components of e, e^2 the vector

[1]) Geometrically, the directions of σ, τ are those in E_1, E_2 which produce the smallest angle. To prove this, one solves this minimum problem by the method of multipliers. Lagrange's equations lead to (3.2.).

consisting of the last $n - s$ components of e. The equation $(I - G_0)\, e = \lambda e$ becomes

(3.2) $$A\, e^2 + \lambda\, e^1 = A'\, e^1 + \lambda\, e^2 = 0\,.$$

(3.1), (3.2) imply

(3.3) $$\lambda B\, e^1 - C\, e^2 = \lambda D\, e^2 - C'\, e^1 = 0\,.$$

Let $\sigma = \sum_{1=1}^{s} e_i\, r_i'$, $\tau = \sum_{i=s+1}^{n} e_i\, r_i'$. (3.3) may be rewritten as

(3.4) $$\begin{aligned} r_i' \cdot (\lambda\sigma - \tau) &= 0, \quad 1 \leqslant i \leqslant s\,, \\ r_i' \cdot (\lambda\tau - \sigma) &= 0, \quad s+1 \leqslant i \leqslant n\,. \end{aligned}$$

The vectors $\lambda\,\sigma - \tau$, $\lambda\,\tau - \sigma$ are $\neq 0$ and in π. (3.4) states that $\lambda\,\sigma - \tau \in E_1^\perp$, $\lambda\,\tau - \sigma \in E_2^\perp$. Since $\sigma \in E_1$, $\sigma' = \lambda\,\sigma - \tau \in E_1^\perp$, we have $S(\sigma) = \sigma$, $S(\sigma') = -\sigma'$. I.e. S_π is a reflection in the line through σ. Similarly, T_π is a reflection in the line through τ.

We now return to the proof of Theorem 3.9. Let H be the subgroup generated by S, T. H_π is the group generated by S_π, T_π. Let

$$F_0 = \{v \mid v = x\,\sigma + y\,\tau, x, y > 0\} = F \cap \pi\,.$$

F_0 is a fundamental region for H_π. For let $\gamma \in H$, $\gamma_\pi \neq I$. Then $\gamma \neq I$ and we have $\gamma_\pi\, F \cap F = \gamma\, F \cap F \cap \pi = \Phi$. R_π is a rotation of π through twice the angle between σ and τ. We show that ord $R_\pi = h$. For let ord $R_\pi = k$. Since $R^h = I$, $R_\pi^h = I$, we have $k \leqslant h$. Choose $p \in F_0$. $R^k(p) = R_\pi^k(p) = p$ so that $R^k\, F \cap F \neq \Phi \Rightarrow R^k = I \Rightarrow h \leqslant k$. Thus $h = k$. It follows that F_0 is an angular wedge of angular width $\dfrac{2\pi}{h}$ and H_π is a dihedral group of order $2h$. The h transforms of σ are contained in precisely $(n-s)$ r.h.'s. The h transforms of τ are contained in precisely s r.h.'s. Every r.h. of G has a non-trivial intersection with π. Since each of the transforms of F_0 is contained in a chamber of G and each chamber is free of r.h.'s, these r.h.'s meet π only at the transforms of σ and τ. Counting the r.h.'s at the transforms of σ and τ, we obtain the count $h\,s + h\,(n-s) = h\,n$. Each r.h. is however counted twice, as it intersects π in a line and thus meets two of the σ and τ transforms. Hence $r = \dfrac{h\,n}{2}$.

As a by product of the above proof, we obtain the following result required to establish Theorem 3.8.

THEOREM 3.10. $\zeta = e^{2\pi i/h}$ *is an eigenvalue of* R. *Corresponding to* ζ, *we may choose an eigenvector* v *not lying in any r.h.* (Note: if v is complex, then v is said to lie in the r.h. π iff $L(v) = 0$, $L(x) = 0$ being the equation of π).

Proof. Assume first that the R_i's are labeled as in the proof of Theorem 3.9; i.e. the walls $W_1, \ldots, W_s$ are mutually perpendicular as are also $W_{s+1}, \ldots, W_n$. Let π be the plane of Lemma 3.3. We choose two orthonormal vectors v_1, v_2 in π such that v_1 is not contained in any r.h. of G and

$$(3.5)\qquad \begin{aligned} R(v_1) &= \cos\frac{2\pi}{h} v_1 + \sin\frac{2\pi}{h} v_2 \\ R(v_2) &= -\sin\frac{2\pi}{h} v_1 + \cos\frac{2\pi}{h} v_2 \end{aligned}$$

Let $v = v_1 - iv_2$. We conclude from (3.5) that $R(v) = e^{2i\pi/h} v$. Thus v is an eigenvector corresponding to the eigenvalue $\zeta = e^{2i\pi/h}$. v is not in any r.h. of G as v_1 is not in any r.h. of G.

For an arbitrary labeling of indices, choose a permutation $i_1, \ldots, i_n$ of $1, \ldots, n$ so that the above reasoning applies to $R' = R_{i_1} \ldots R_{i_n}$. By Theorem 3.7. $R = R_1 \ldots R_n = \sigma R' \sigma^{-1}$ for some $\sigma \in G$. Hence $R(\sigma v) = \zeta(\sigma v)$. Since the r.h.'s are permuted by σ, we conclude that σv is also not contained in any r.h. of G.

We also require

THEOREM 3.11. 1 *is not an eigenvalue of* R.

REMARK. In Theorem 3.12 we obtain the characteristic equation of R, from which we may obtain Theorem 3.11. The following proof is shorter and avoids any explicit matrix representation for R.

Proof. Let π be the r.h. corresponding to the root r and σ the reflection in π. Then $v' = \sigma v$ becomes

$$(3.6)\qquad v' = v - 2(v,r)\,r$$

Suppose that $R_1 \ldots R_n v = v$, $\Leftrightarrow R_2 \ldots R_n v = R_1 v$. Repeated application of (3.6) shows that $R_2 \ldots R_n v = v + \lambda_2 r_2 + \ldots + \lambda_n r_n$, $\lambda_2, \ldots, \lambda_n$ being real numbers depending on v. Hence

$$(3.7)\qquad v + \lambda_2 r_2 + \ldots + \lambda_n r_n = v - 2(v, r_1)\, r_1$$

Since $r_1, \ldots, r_n$ are linearly independent we must have $(v, r_1) = 0$ $\Leftrightarrow R_1 v = v$, so that $R_2 \ldots R_n v = v$. Repeating the reasoning, we conclude $(v, r_i) = 0$, $1 \leqslant i \leqslant n$, $\Rightarrow v = 0$. Thus 1 is not an eigenvalue of $R_1 \dots R_n$.

We can now provide the

Proof of Theorem 3.8. Let $v_1, \dots, v_n$ be linearly independent eigenvectors of R with v_1 chosenas in Theorem 3.10; i.e. v_1 corresponds to the eigenvalue $\zeta = e^{2i\pi/h}$ and does not lie in any r.h. of G. Let $x_1, \dots, x_n$ be a coordinate system adapted to $v_1, \dots, v_n$. As $R^h = I$, all eigenvalues of R are h-th roots of I. By Theorem 3.11, 1 is not an eigenvalue of R. Hence the eigenvalues of R are $\zeta^{m_1}, \dots, \zeta^{m_n}$ where $m_1 = 1$ and $1 \leqslant m_1 \leqslant \dots \leqslant m_n = h - 1$, $1 \leqslant i \leqslant n$. R is given by $x_i' = \zeta^{m_i} x_i$, $1 \leqslant i \leqslant n$.

Let $I_1, \dots, I_n$ be a basic set of homogeneous invariants of G of respective degrees $d_1 \leqslant \dots \leqslant d_n$. By Theorem 2.5,

$$J = \frac{\partial (I_1, \dots, I_n)}{\partial (x_1, \dots, x_n)} \neq 0$$

off the r.h.'s of G. Hence $J \neq 0$ whenever $x = (x_1, 0, \dots, 0)$, $x_1 \neq 0$. It follows that there exists a permutation $j = j(i)$ of 1 to n such that

$$\frac{\partial I_i}{\partial x_j} (x_1, 0, \dots, 0) \neq 0$$

for $x_1 \neq 0$ and $1 \leqslant i \leqslant n$. This means that the $x_1^{d_i - 1}$ coefficient of

$$\frac{\partial I_i}{\partial x_j} \neq 0 \Rightarrow x_1^{d_i - 1} x_j$$

coefficient of $I_i \neq 0$, $1 \leqslant i \leqslant n$. Hence each $x_1^{d_i - 1} x_j$ is invariant under R. I.e.

(3.8) $$(d_i - 1) + m_j \equiv 0 \pmod{h}, \ 1 \leqslant i \leqslant n$$

Rewrite (3.8) as

(3.9) $$d_i - 1 = (h - m_j) + \varepsilon_i h, \ 1 \leqslant i \leqslant n$$

where each ε_i is an integer $\geqslant 0$. Let $m_j' = h - m_j$. The eigenvalues of R occur in pairs, so that the set of numbers $\{m_j'\}$ is identical with $\{m_j\}$. Summing both sides of (3.9) from $i = 1$ to $i = n$, we get

(3.10) $$\sum_{i=1}^{n} (d_i - 1) = \sum_{j=1}^{n} m_j' + \Big(\sum_{i=1}^{n} \varepsilon_i\Big) h$$

By Theorem 2.2, $\sum_{i=1}^{n} (d_i - 1) = r$. Since

$$(3.11)\qquad \sum_{j=1}^{n} m_j' = \sum_{j=1}^{n} (h - m_j) = n\,h - \sum_{j=1}^{n} m_j',$$

we also have $\sum_{j=1}^{n} m_j' = \frac{n\,h}{2}.$ We conclude from Theorem 3.9 that $\sum_{i=1}^{n} (d_i - 1) = \sum_{j=1}^{n} m_j'$. (3.10) shows that $\sum_{i=1}^{n} \varepsilon_i = 0 \Rightarrow \varepsilon_i = 0, 1 \leqslant i \leqslant n.$ It follows from (3.9) that $d_i - 1 = m_i$, $1 \leqslant i \leqslant n$.

To make effective use of Coleman's Theorem, we need the explicit expression for the characteristic equation of R.

Theorem 3.12 (Coxeter [5], p. 218). *The characteristic equation of* $R = R_1 \ldots R_n$ *is given by*

$$(3.12)\qquad \begin{vmatrix} \frac{1+\lambda}{2} & \lambda\, a_{12} & \ldots & \lambda\, a_{1n} \\ a_{21} & \frac{1+\lambda}{2} & \lambda\, a_{23} \ldots & \lambda\, a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{n,n-1} & \frac{1+\lambda}{2} \end{vmatrix} = 0$$

where $a_{ij} = -\cos(\pi/p_{ij})$, $1 \leqslant i, j \leqslant n$.

Proof. Let $v = \sigma\, v'$ where σ is a reflection in the r.h. perpendicular to the root r.
Then

$$(3.13)\qquad v = v' - 2\,(v' \cdot r)\, r$$

We use (3.13) to obtain the matrix for R_j relative to the basis $r_1', \ldots, r_n'$. Let $v = \sum_{i=1}^{n} x_i\, r_i'$, $v' = \sum_{i=1}^{r} x_i'\, r_i'$. Then $v' \cdot r_j = x_j'$, $r_j = \sum_{i=1}^{n} a_{ij}\, r_i'$. Substituting into (3.13), we get

$$(3.14)\qquad v = R_j v' \Leftrightarrow x_i = x_i' - 2 a_{ij}\, x'_j,\ 1 \leqslant i \leqslant n$$

Let

$$v = R_1\, v^{(1)},\ v^{(1)} = R_2\, v^{(2)}, \ldots, v^{(n-1)} = R_n\, v^{(n)}$$

so that $v = R_1 \dots R_n v^{(n)}$. Suppose that $v^{(j)} = \sum_{i=1}^{n} x_i^{(j)} r_i', 1 \leqslant j \leqslant n.$
We conclude from (3.14) that

$$\text{(3.15)} \qquad \begin{cases} x_i = x_i' - 2a_{i1} x_1' \\ x_i' = x_i'' - 2a_{i2} x_2'' \\ \dots\dots\dots\dots\dots\dots, 1 \leqslant i \leqslant n \\ x_i^{(n-1)} = x_i^{(n)} - 2a_{in} x_n^{(n)} \end{cases}$$

Let $y_i = x^{(k)}$, $1 \leqslant i \leqslant n$. For each i we rewrite (3.15) as

$$\text{(3.16)} \begin{cases} x_i' - x_i = 2a_{i1} y_1 \\ x_i'' - x_i' = 2a_{i2} y_2 \\ \dots\dots\dots\dots \\ y_i - x_i^{(i-1)} = 2a_{ii} y_i \end{cases} \qquad \text{(3.17)} \begin{cases} x_i^{(i+1)} - y_i = 2a_{i,i+1} y_{i+1} \\ x_i^{(i+2)} - x_i^{(i+1)} = 2a_{i,i+2} y_{i+2} \\ \dots\dots\dots\dots\dots\dots\dots \\ x_i^{(n)} - x_i^{(n-1)} = 2a_{in} y_n \end{cases}$$

Adding up respectively the equations in (3.16), and (3.17), we obtain

$$\text{(3.18)} \qquad -x_i = \sum_{j=1}^{i-1} 2a_{ij} y_j + y_i, \; 1 \leqslant i \leqslant n$$

$$\text{(3.19)} \qquad x_i^{(n)} = \sum_{j=i+1}^{n} 2a_{ij} y_j + y_i, \; 1 \leqslant i \leqslant n$$

(3.18), (3.19) may be abbreviated as

$$\text{(3.20)} \qquad -x = Ay, x^{(n)} = A' y$$

where

$$\text{(3.21)} \quad A = \begin{bmatrix} 1 & & & & \\ 2a_{21} & & & & \\ . & 1 & & & \\ . & . & . & & \\ . & . & . & . & \\ 2a_{n1} & . & . & 2a_{n,\,n-1} & 1 \end{bmatrix}$$

the entries above the diagonal being zero.

Hence $x = -A(A')^{-1} x^{(n)}$, so that $-A(A')^{-1}$ is the matrix for $R = R_1 \dots R_n$ relative to the basis $r_1', \dots, r_n'$. The characteristic equation for R is thus given by

$$(3.22) \qquad |-A(A')^{-1} - \lambda I| = 0 \Leftrightarrow \left|\frac{A + \lambda A'}{2}\right| = 0$$

which is the same as (3.12).

We rewrite the characteristic equation in a more symmetric form. Suppose first that G is of type I. We label nodes of the graphs in diagram 3.2 from left to right as 1, ..., n. Thus $a_{ij} = 0$ whenever $|j - i| > 1$. Multiplying first the i-th row of the determinant in (3.12) by $\lambda^{(i-1)/2}$, $1 \leqslant i \leqslant n$, then the j-th column by $\lambda^{-j/2}$, $1 \leqslant j \leqslant n$, we get

$$(3.23) \qquad \begin{vmatrix} \Lambda & & & & \\ & \cdot & & a_{ij} & \\ & & \cdot & & \\ & & & \cdot & \\ a_{ij} & & & & \cdot \\ & & & & \Lambda \end{vmatrix} = 0$$

where $\Lambda = \dfrac{\lambda^{1/2} + \lambda^{-1/2}}{2}$

If G is of type II, then the nodes on the principal chain are labeled from left to right as 1 to $n - 1$, the remaining node being labeled n. The n^{th} node is linked to the q^{th} node. Let $i' = i, j' = j$, $1 \leqslant i, j \leqslant n - 1$, and $i' = j' = q + 1$ whenever i or $j = n$. Multiply first the i-th row of the determinant in (3.12) by $\lambda^{\frac{i'-1}{2}}$, $1 \leqslant i \leqslant n$, then the j-th column by $\lambda^{-j'/2}$. We obtain again (3.23). We have proven

COROLLARY. *The characteristic equation of R is given by* (3.23).

We illustrate the use of Coleman's Theorem by computing the d_i's for the icosahedral group I_3. In this case the characteristic equation (3.23) becomes

$$(3.24) \qquad \begin{vmatrix} \Lambda & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \Lambda & -\cos\frac{\pi}{5} \\ 0 & -\cos\frac{\pi}{5} & \Lambda \end{vmatrix} = 0$$

The roots of (3.24) are readily computed to be $\zeta = e^{\frac{2\pi i}{10}}$, ζ^5, ζ^9. It follows from Coleman's Theorem that $d_1 = 2$, $d_2 = 6$, $d_3 = 10$.

3. Tabulation of the Degrees

Theorem 3.8 can be used to compute the degrees of the basic homogeneous invariants of G, in case G is an irreducible reflection group acting on R^n. This has been done in [7], and we tabulate these degrees below

Group	$d_1, \ldots, d_n$
A_n $(n \geqslant 1)$	$2, \ldots, n+1$
B_n $(n \geqslant 2)$	$2, 4, \ldots, 2n$
D_n $(n \geqslant 4)$	$2, 4, \ldots, n, \ldots, 2n-4, 2n-2$
H_2^n $(n \geqslant 5)$	$2, n$
E_6	2, 5, 6, 8, 9, 12
E_7	2, 6, 8, 10, 12, 14, 18
E_8	2, 8, 12, 14, 18, 20, 24, 30
F_4	2, 6, 8, 12
I_3	2, 6, 10
I_4	2, 12, 20, 30

We observe that in each case, $d_1 = 2$. This can be seen as follows. Suppose that there existed a homogeneous invariant $I(x)$ of degree 1. Since $I(\sigma x) = I(x)$ whenever $\sigma \in G$, the hyperplane $\{x \mid I(x) = 0\}$ would be a proper invariant subspace of G, contradicting that the latter is irreducible. Hence there are no homogeneous invariants of degree 1 and $d_1 \geqslant 2$. On the other hand, $\sum_{i=1}^{n} x_i^2$ is invariant under G as G is orthogonal. It follows that $d_1 = 2$, with corresponding invariant $I_1 = \sum_{i=1}^{n} x_i^2$.

In applying Theorem 3.8, we must find the roots of the characteristic equation (3.23). In some cases, this is a rather tedious computation. For the groups A_n, B_n, D_n H_2^n we can exhibit a basis of homogeneous invariants without the use of Theorem 3.8. We require

THEOREM 3.13. *Let G be a finite reflection group acting on the n-dimensional vector space V over a given field k. Let $P_1, \ldots, P_n$ be homogeneous* *invariants of* G *of respective degrees* $k_1, \ldots, k_n$. $P_1, \ldots, P_n$ *form a basis for the invariants of* $G \Leftrightarrow k_1 \ldots k_{\cdot n} = |G|$ *and*

$$\Delta = \frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} \neq 0 .$$

Proof. By relabeling indices, we may assume $k_1 \leqslant \ldots \leqslant k_n$. The $\Rightarrow$ part of the theorem is contained in Theorems 1.2, 2.2, 2.3. Conversely, let $k_1 \ldots k_n = |G|$ and $\Delta \neq 0$. Thus $P_1, \ldots, P_n$ are algebraically independent. Let $I_1, \ldots, I_n$ be basic homogeneous invariants of respective degrees $d_1, \ldots, d_n$. Suppose $k_i = d_i$, $1 \leqslant i \leqslant i_0$, but $k_{i_0+1} < d_{i_0+1}$. Then $P_1, \ldots, P_{i_0+1}$ are polynomials in $I_1, \ldots, I_{i_0}$, implying that $P_1, \ldots, P_n$ are algebraically dependent, a contradiction. Hence $k_i \geqslant d_i$, $1 \leqslant i \leqslant n$. Since $\prod_{i=1}^{n} d_i = \prod_{i=1}^{n} k_i = |G|$, we must have $k_i = d_i$, $1 \leqslant i \leqslant n$.

Let $\delta_m = \dim \mathcal{J}_m$, $0 \leqslant m < \infty$, $\mathcal{J}_m$ being the space of homogeneous invariants of degree m. Then δ_m = number of non-negative integral solutions to $j_1 d_1 + \ldots + j_n d_n = m$. This number also equals the number of monomials $P_1^{j1} \ldots P_1^{jn}$ which are of degree m. The algebraic independence of $P_1, \ldots, P_n$ implies that these δ_m monomials are linearly independent over k. Thus $\mathcal{J}_m$ is spanned by these monomials for $0 \leqslant m < \infty$. We have shown that every homogeneous invariant is a polynomial in $P_1, \ldots, P_n$, so that the P_i's form a basis for the invariants of G.

We now obtain an explicit basis for the invariants of A_n, B_n, D_n, H_2^n. A_n: This group consists of the $(n+1)!$ permutations $x_i' = x_{\sigma(i)}$, $1 \leqslant i \leqslant n+1$, restricted to the subspace $V = \{x \mid x_1 + \ldots + x_{n+1} = 0\}$. We choose $x_1, \ldots, x_n$ as coordinates on V. Let $P_i = \sum_{j=1}^{n+1} x_j^{i+1}$, $1 \leqslant i \leqslant n$, where $x_{n+1} = -(x_1 + \ldots + x_n)$. P_i is a homogeneous invariant of degree $i+1$. We have $2 \cdot \ldots \cdot (n+1) = (n+1)! = |A_n|$.

We show that $\Delta \neq 0$. Now

$$\frac{\partial P_i}{\partial x_j} = (i+1) x_j^i - (i+1) x_{n+1}^i, \; 1 \leqslant i, j \leqslant n .$$

Hence $\Delta = (n+1)!\, D$ where D is the $n \times n$ determinant whose (ij)-th entry $= x_j^i - x_{n+1}^i$. To evaluate D, we introduce the Vandermonde determinant

$$\begin{vmatrix} 1 & \ldots\ldots & 1 \\ x_1 & \ldots\ldots & x_{n+1} \\ x_1^n & \ldots\ldots & x_{n+1}^n \end{vmatrix} = \prod_{1 \leqq i < j \leqq n+1} (x_j - x_i)$$

Subtracting the $(n+1)$-th column from the first n columns, the above determinant is readily seen to equal $(-1)^n D$. Thus

$$(3.25) \qquad \Delta = (-1)^{n+2}(n+1)! \prod_{1 \leqq i < j \leqq n+1} (x_j - x_i) =$$

$$(n+1)! \prod_{1 \leqq j \leqq n} (x_j - x_i) \cdot \prod_{i=1}^{n} (x_i + s)$$

where $s = x_1 + \ldots + x_n$. (3.25) shows that $\Delta \neq 0$. We conclude that $d_1 = 2, \ldots, d_n = n + 1$.

B_n: Let $P_i = \sum_{j=1}^{n} x_j^{2i}$, $1 \leqslant i \leqslant n$. P_i is a homogeneous invariant of degree $2i$. We have $2 \cdot \ldots \cdot 2n = 2^n n! = |B_n|$. A computation shows that $\Delta = 2^n n! \prod_{i=1}^{n} x_i \prod_{1 \leqq i < j \leqq n} (x_j^2 - x_i^2) \neq 0$. It follows that $d_1 = 2, \ldots, d_n = 2n$.

D_n: Let $P_1 = x_1 \ldots x_n$, $P_i = \sum_{j=1}^{n} x_j^{2(i-1)}$, $2 \leqslant i \leqslant n$. P_1 is a homogeneous invariant of degree n; P_i, $2 \leqslant i \leqslant n$, is a homogeneous invariant of degree $2(i-1)$. The product of the degrees $= n \cdot 2 \cdot 4 \cdot \ldots \cdot (2n-2) = 2^{n-1} n! = |D_n|$.

$$(3.26) \qquad \Delta = \begin{vmatrix} \dfrac{P_1}{x_1} & \cdots & \dfrac{P_1}{x_n} \\ 2x_1 & \cdots & 2x_n \\ \cdot & \cdots & \cdot \\ 2(n-1)x_1^{2n-3} & \cdots & 2(n-1)x_n^{2n-3} \end{vmatrix}$$

$$= 2^{n-1}(n-1)! \prod_{1 \leqq i < j \leqq n} (x_j^2 - x_i^2) \neq 0$$

It follows that $d_i, \ldots, d_n$ are identical with the numbers $2, 4, \ldots, n, \ldots, 2n-4, 2n-2$.

H_2^n: Let z be the complex coordinate $x_1 + i x_2$. H_2^n may be described as the group generated by the transformation $z \to \bar{z}$, $z \to \zeta z$, where $\zeta = e^{\frac{2\pi i}{n}}$. Let $P_1 = x_1^2 + x_2^2$, $P_2 = Re\, z^n$. P_1, P_2 are homogeneous invariants of respective degrees $2, n$. The product of these degrees $= 2n = |H_2^n|$. A computation yields

$$\frac{\partial(P_1, P_2)}{\partial(x_1, x_2)} = -2n\, Im\, z^n \neq 0 .$$

It follows that $d_1 = 2$, $d_2 = n$.

4. Solomon's Theorem

We present in this section another method for determining the degrees of the basic invariants, valid whenever the underlying field k has characteristic 0.

Theorem 3.14 (Solomon [18]). *Let G be a finite reflection group acting on the n-dimensional space V. Let g_r = number of elements of G which fix some r-dimensional subspace of V but do not fix a subspace of higher dimension. Let $d_1, \ldots, d_n$ be the degrees of the basic homogeneous invariants of G and set $m_j = d_j - 1$. Then*

$$(3.27) \qquad (t+m_1)\ldots(t+m_n) = g_0 + g_1 t + \ldots + g_n t^n$$

Equating the t^{n-1}-coefficients of both sides of (3.27), we obtain $g_1 = r = \sum_{i=1}^{n} m_i$. Setting $t = 1$ in (3.27), we obtain $\prod_{i=1}^{n} (m_i + 1) = \sum_{i=0}^{n} g_i = |G|$. Thus Theorem 3.14 generalizes Theorem 2.2.

To prove Theorem 3.14, we obtain an analog of Molien's formula for the invariant differential forms of G. We digress to a brief discussion of differential forms.

For $p > 0$, let $\omega = \sum_{i_1 < \ldots < i_p} r_{i_1 \ldots i_p}(x)\, dx_{i_1} \ldots dx_{i_p}$, where $r_{i_1 \ldots i_p}(x) \in k(x)$, the summation extending over all integer p-tuples satisfying $1 \leqslant i_1 < \ldots < i_p \leqslant n$. ω is called a differential p-form (or simply p-form). The elements of $k(x)$ are called the 0-forms. If $\eta = \sum_{i_1 < \ldots i_p} s_{i_1 \ldots i_p}(x)\, dx_{i_1} \ldots dx_{i_p}$ is another p-form, then we define

$$\omega + \eta = \sum_{i_1 < \ldots < i_p} (r_{i_1 \ldots i_p} + s_{i_1 \ldots i_p})\, dx_{i_1} \ldots dx_{i_p}.$$

Thus the p-forms constitute a vector space over $k(x)$ which we denote by $\mathscr{D}_p$. The elements $dx_{i_1} \ldots dx_{i_p}$ form a basis for $\mathscr{D}_p$, so that $\dim \mathscr{D}_p = \binom{n}{p}$, $0 \leqslant p \leqslant n$. We also define a multiplication between two forms as follows. Let $dx_i\, dx_j = -dx_j\, dx_i$; in particular $dx_i\, dx_i = 0$. The product $\omega\eta$ of any two forms ω, η is then obtained by the distributive law. We observe that for 1-forms, $\omega\eta = -\eta\omega$, so that $\omega\omega = 0$. It follows that $\mathscr{D}_p = 0$ for $p > n$. Finally, for any rational function r, we define the 1-form dr to be

$$\sum_{i=1}^{n} \frac{\partial r}{\partial x_i} dx_i .$$

It is then readily checked that for n rational functions, $r_1, \ldots, r_n$, we have

$$dr_1 \ldots dr_n = \frac{\partial (r_1, \ldots, r_n)}{\partial (x_1, \ldots, x_n)} dx_1 \ldots dx_n .$$

Let σ be a non-singular matrix with entries in k. We define

$$\sigma \omega = \sum_{i_1 < \cdots < i_p} r_{i_1 \ldots i_p} (\sigma^{-1} x) \, dx_{i_1} (\sigma^{-1} x) \ldots dx_{i_p} (\sigma^{-1} x)$$

Thus σ becomes a linear transformation on each $\mathscr{D}_p$, interpreting the latter as a vector space over k. Let k^n be the space of n-tuples with entries in k. If G is a group of linear transformations acting on k^n, then ω is said to be invariant under G provided $\sigma\omega = \omega$, $\forall\, \sigma \in G$.

We shall prove Theorem 3.14 describing the invariant differential forms with polynomial coefficients. G is assumed throughout to be a finite reflection group acting on k^n.

LEMMA 3.4. Let $I_1, \ldots, I_n$ be basic homogeneous invariants for G. Let

$$\Pi (x) = \frac{\partial (I_1, \ldots, I_n)}{\partial (x_1, \ldots, x_n)} .$$

The polynomial $p(x)$ satisfies $\sigma p = (\det \sigma) p$, for every $\sigma \in G$ (in which case, we say p is skew) iff $p = \Pi i$ where i is a polynomial invariant under G.

Proof. Let $y = \sigma x$. Then

$$\Pi (x) = \frac{\partial (I_1 (y), \ldots, I_n(y))}{\partial (x_1, \ldots, x_n)} \tag{3.28}$$

$$= \frac{\partial (I_1 (y), \ldots, I_n (y))}{\partial (y_1, \ldots, y_n)} \det \sigma = \Pi (\sigma x) \det \sigma$$

which shows that Π is skew. Hence Πi is skew for every invariant polynomial i.

Conversely, let $p(x)$ be skew. Let π be an r.h. of G with equation $L(x) = 0$. By Lemma 2.2, we may choose $v \notin \pi$, so that v is a common eigenvector to all reflections in G with r.h. π. Choose $x = Ty$, $\det T \neq 0$, so that in the y coordinates the equation of π becomes $y_n = 0$ and v becomes $(0, \ldots, 0, 1)$. Let $q(y) = p(Ty)$. Let H be the subgroup of G which fixes π. By Lemma 2.2, H is a cyclic group. Let σ generate H and $h = \operatorname{ord} H$. If ζ is the eigenvalue of σ which is a primitive h-th root of 1, then

$q(y_1, \ldots, y_{n-1}, \zeta y_n) = \zeta^{-1} q(y_1, \ldots, y_n)$. Writing $q = \sum q_i y_n^i$, the q_i's being polynomials in $y_1, \ldots, y_{n-1}$, we obtain

$$\Sigma q_i \zeta^{i+1} y_n^i = \Sigma q_i y_n^i \tag{3.29}$$

Equating coefficients in (3.29), we conclude $q_i = 0$ whenever $h \nmid i+1$. Thus $q_i = 0$ for $i < h-1 \Rightarrow y_n^{h-1} \mid q \Rightarrow L^{h-1} \mid p$. Repeating this argument for all r.h.'s of G and using Theorem 2.5, we conclude that $P = \Pi i$, where i is a polynomial. $\sigma i = \sigma P / \sigma \Pi = \dfrac{P}{\Pi} = i$ shows that i is invariant under G.

LEMMA 3.5. Let σ be a non-singular matrix with entries in k. Let $r \in k(x)$. Then $\sigma(dr) = d(\sigma r)$.

Proof. By definition

$$\sigma(dr) = \sum_{i=1}^{n} \frac{\partial r}{\partial x_i}(\sigma^{-1}x)\, dx_i(\sigma^{-1}x), \quad d(\sigma r) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(r(\sigma^{-1}x)\big)\, dx_i \tag{3.30}$$

Let $\sigma^{-1} = (a_{ij})$. Then $x_i(\sigma^{-1}x) = \sum_{j=1}^{n} a_{ij} x_j$ and $\dfrac{\partial x_i}{\partial x_j}(\sigma^{-1}x) = a_{ij}$.

Hence

$$dx_i(\sigma^{-1}x) = \sum_{j=1}^{n} a_{ij}\, dx_j \tag{3.31}$$

Applying the chain rule,

$$\frac{\partial}{\partial x_i}\big(r(\sigma^{-1}x)\big) = \sum_{j=1}^{n} \frac{\partial r}{\partial x_j}(\sigma^{-1}x)\, a_{ji} \tag{3.32}$$

Inserting (3.31), (3.32) into (3.30), we get $\sigma(dr) = d(\sigma r)$.

THEOREM 3.15. *Every invariant p-form with polynomial coefficients may be expressed uniquely as*

$$\sum_{i_1 < \cdots < i_p} a_{i_1 \ldots i_p}\, dI_{i_1} \ldots dI_{i_p},\ a_{i_1 \ldots i_p} \in k[I_1, \ldots, I_n].$$

Proof. By Lemma 3.5, $\sigma(dI_k) = dI_k$, so that $dI_1, \ldots, dI_n$ are invariant forms. Since $\sigma(\omega\eta) = \sigma(\omega)\sigma(\eta)$ for any two forms ω, η, we conclude that $\sum_{i_1 < \ldots < i_p} a_{i_1 \ldots i_p}\, dI_{i_1} \ldots dI_{i_p}$ is invariant whenever $a_{i_1 \ldots 1_p} \in k(I_1, \ldots, I_n)$.

We show that the $\binom{n}{p}$ forms $dI_{i_1} \ldots dI_{i_p}$ are linearly independent over $k(x)$, so that they form a basis for $\mathscr{D}_p$ over $k(x)$. Suppose that

$$\sum_{i_1 < \cdots < i_p} k_{i_1 \ldots i_p}\, dI_{i_1} \ldots dI_{i_p} = 0,\ k_{i_1 \ldots i_p} \in k(x).$$

Multiply this relation by $dI_{i_{p+1}} \dots dI_{i_n}$, where $i_{p+1}, \dots, i_n$ are the indices complementary to $i_1, \dots, i_p$. We obtain

$$k_{i_1 \dots i_p} dI_1 \dots dI_n = k_{i_1 \dots i_p} \Pi(x)\, dx_1 \dots dx_n = 0 \Rightarrow k_{i_1 \dots i_p} = 0$$

for all $i_1, \dots, i_p$. Hence the $\binom{n}{p}$ forms $dI_{i_1} \dots dI_{i_p}$ are linearly independent over $k(x)$. It follows that every p-form ω may be expressed uniquely as

$$\omega = \sum_{i_1 < \dots < i_p} a_{i_1 \dots i_p} dI_{i_1} \dots dI_{i_p}, \; a_{i_1 \dots i_p} \in k(x).$$

If ω is invariant, then the group averaging argument shows that $a_{i_1 \dots i_p} \in k(I_1, \dots, I_n)$. Multiply both sides of the above relation by $dI_{i_{p+1}} \dots dI_{i_n}$. We get

$$\omega\, dI_{i_{p+1}} \dots dI_{i_n} = \pm \Pi a_{i_1 \dots i_p} dx_1 \dots dx_n . \tag{3.33}$$

Let ω be a p-form with polynomial coefficients. We conclude from (3.33) that $\Pi a_{i_1 \dots i_p}$ is a polynomial. Since $\Pi a_{i_1 \dots i_p}$ is skew, Lemma 3.4 implies that $\Pi a_{i_1 \dots i_p} = \Pi i$, i being an invariant polynomial. Hence $a_{i_1 \dots i_p} \in k[I_1, \dots, I_n]$ for all $i_1, \dots, i_p$, thus proving Theorem 3.11.

THEOREM 3.16. *Let $\sigma_p(x_1, \dots, x_n)$ be the p-th elementary symmetric function in $x_1, \dots, x_n$ (σ_0 is interpreted to be 1). Let $\omega_1(\gamma), \dots, \omega_n(\gamma)$ be the eigenvalues of γ, $\gamma \in G$. Then*

$$\frac{\sigma_p(t^{m_1}, \dots, t^{m_n})}{(1-t^{m_1+1}) \dots (1-t^{m_n+1})} \tag{3.34}$$
$$= \frac{1}{|G|} \sum_{\gamma \varepsilon G} \frac{\sigma_p(\omega_1(\gamma), \dots, \omega_n(\gamma))}{(1-\omega_1(\gamma)t) \dots (1-\omega_n(\gamma)t)}, \; 0 \leqslant p \leqslant n$$

REMARK. For $p = 0$, the above becomes formula (2.5) of Chapter II.

Proof. Let $\mathscr{D}_{pm}$ = space of p-forms whose coefficients are homogeneous polynomials of degree m. $\mathscr{D}_{pm}$ is a finite dimensional vector space over k. Let $\mathscr{I}_{pm}$ = space of invariant forms in $\mathscr{D}_{pm}$ and $d_{pm} = \dim \mathscr{I}_{pm}$. For $0 \leqslant p \leqslant n$, let $\mathfrak{p}_p(t) = \sum_{m=0}^{\infty} d_{pm} t^m$. We obtain two formulas for $\mathfrak{p}_p(t)$ by computing d_{pm} in two different ways. By Theorem 3.15, the differentials

$$I_1^{k_1} \dots I_n^{k_n} dI_{i_1} \dots dI_{i_p}, \; m = k_1(m_1+1) \dots + k_n(m_n+1)$$
$$+ m_{i_1} + \dots + m_{i_p},$$

form a basis for $\mathscr{J}_{pm}$, so that

$$(3.35) \qquad \mathfrak{p}_p(t) = \frac{\sigma_p(t^{m_1}, \ldots, t^{m_n})}{(1-t^{m_1+1})\ldots(1-t^{m_n+1})}$$

Let $\tilde{k}$ = algebraic closure of k. Define $\tilde{\mathscr{D}}_{pm}$, $\tilde{\mathscr{J}}_{pm}$, analogously to $\mathscr{D}_{pm}$, $\mathscr{J}_{pm}$, replacing k by $\tilde{k}$. For $\gamma \in G$, γ acts both on $\mathscr{D}_{pm}$ and $\tilde{\mathscr{D}}_{pm}$. Let $(\mathrm{Tr}\ \gamma)_{pm}$ = trace of γ as a transformation on $\mathscr{D}_{pm}$ = trace of γ as a transformation on $\tilde{\mathscr{D}}_{pm}$. By Lemma 1.2

$$(3.36) \qquad d_{pm} = \frac{1}{|G|} \sum_{\gamma \varepsilon G} (Tr\,\gamma)_{pm}$$

Choose T so that $T\sigma T^{-1} = D$, D being diagonal with diagonal entries $\omega_1(\gamma), \ldots, \omega_n(\gamma)$. The elements $x^a\, d\,x_{i_1} \ldots d\,x_{i_p}$, $|a| = m$ and $1 \leqslant i_1 < \ldots < i_p \leqslant n$, form a basis for $\tilde{\mathscr{D}}_{pm}$. Since

$$(3.37) \qquad D(x^a dx_i \ldots dx_{i_p}) = [\omega(\gamma^{-1})]^a\, \omega_{i_1}(\gamma^{-1}) \ldots \omega_{i_p}(\gamma^{-1}),$$

we have

$$(3.38) \qquad (TrD)_{pm} = \sum_{|a|=m} [\omega(\gamma^{-1})]^m \sigma_p(\omega(\gamma^{-1}))$$

(3.36), (3.38) yield

$$(3.39) \qquad d_{pm} = \frac{1}{|G|} \sum_{\gamma \varepsilon G} \sum_{|a|=m} [\omega(\gamma)]^a \sigma_p[\omega(\gamma)]$$

so that

$$(3.40) \qquad \mathfrak{p}_p(t) = \frac{1}{|G|} \sum_{m=0}^{\infty} \sum_{r \varepsilon G} \sum_{|a|=m} [\omega(\gamma)]^a \sigma_p(\omega(\gamma))\, t^m$$

$$= \frac{1}{|G|} \sum_{\gamma \varepsilon G} \frac{\sigma_p(\omega(\gamma))}{(1-\omega_1(\gamma)t)\ldots(1-\omega_n(\gamma)t)}$$

(3.34) follows from (3.35) and (3.40).

We derive from (3.34) the following identity.

THEOREM 3.17. *For* $1 \leqslant p \leqslant n$,

$$(3.41) \qquad \sum_{i_1<\ldots<i_p} \frac{t^{m_{i_1}+\ldots+m_{i_p}}}{(1-t^{m_{i_1}+1})\ldots(1-t^{m_{i_p}+1})}$$

$$= \frac{1}{|G|} \sum_{\gamma \varepsilon G} \sum_{i_1<\ldots<i_p} \frac{\omega_{i_1}(\gamma)\ldots\omega_{i_p}(\gamma)}{(1-\omega_{i_1}(\gamma)t)\ldots(1-\omega_{i_p}(\gamma)t)}$$

Proof. One verifies readily, for $1 \leqslant p \leqslant n$, the identity

$$(3.42) \qquad \sum_{i_1<\ldots<i_p} \frac{u_{i_1} \ldots u_{i_p}}{(1-u_{i_1}t)\ldots(1-u_{i_p}t)}$$

$$= \frac{h_{p1}(t)\,\sigma_1(u_1,\ldots,u_n) + \ldots + h_{pn}(t)\,\sigma_n(u_1,\ldots,u_n)}{(1-u_it)\ldots(1-u_nt)}$$

the u_i's being indeterminates and the h_{pi}'s being polynomials in t. Substitute for u_i, $\omega_i(\gamma)$ and average over the group. By Theorem 3.16, the group average becomes expression (3.42), u_i being replaced by t^{m_i}, thus proving (3.41).

We can now provide the

Proof of Theorem 3.14. Expand both sides of (3.41) in powers of $1-t$ and equate the coefficients of $(1-t)^{-p}$. For the left side this coefficient is

$$\sum_{i_1<\ldots<i_p} \frac{1}{(m_{i_1}+1)\ldots(m_{i_p}+1)}$$

Let γ be an element which fixes an r dimensional subspace, but does not fix a higher dimensional subspace. This means that precisely r of the eigenvalues of γ equal 1. γ contributes to the coefficient of $(1-t)^{-p}$ on the right side of (3.41) iff $r \geqslant p$, the contribution being $\binom{r}{p}$. It follows that for the right side, the $(1-t)^{-p}$ coefficient is $\frac{1}{|G|}\sum_{r=0}^{n} \binom{r}{p} g_r$. Since $\prod_{i=1}^{n} (m_i+1) = |G|$, we conclude that

$$(3.43) \qquad \sum_{r=0}^{n} \binom{r}{p} g_r = \sum_{i_1<\ldots<i_{n-p}} (m_{i_1}+1)\ldots(m_{i_{n-p}}+1),\ 1 \leqslant p \leqslant n$$

Note that for $p = 0$, (3.43) becomes $|G| = (m_1+1)\ldots(m_n+1)$. Hence (3.43) also holds for $p = 0$.

The left and right side of (3.43) equal respectively $\frac{1}{p!}$ (p-th derivative at $t = 1$) of $g_0 + \ldots + g_n t^n$, $(t+m_1)\ldots(t+m_n)$. Thus $(t+m_1)\ldots(t+m_n) = g_0 + \ldots + g_n t^n$.

CHAPTER IV

PARTIAL DIFFERENTIAL EQUATIONS AND MEAN VALUE PROPERTIES

1. INVARIANT PARTIAL DIFFERENTIAL EQUATIONS

We study in the present chapter a certain system of partial differential equations invariant under a finite reflection group G and related mean value properties. We assume throughout that the underlying field k is real (this permits us to introduce the methods of analysis) and that G is orthogonal, which can always be achieved after a linear change of variables. We rely on the invariant theory of the previous chapters to establish the forthcoming results. Conversely, we shall see that the problems studied in this chapter lead to a natural set of basic invariants for G. In the sequel, let R denote the ring of polynomials $k[x_1, ..., x_n]$. For any polynomial $p(x)$, $p(\partial)$ denotes the partial differential operator obtained by replacing $x = (x_1, ..., x_n)$ by the symbol

$$\partial = \partial_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right).$$

We shall use the following result.

THEOREM 4.1 (Fischer [9]). *Let* $\mathfrak{a}$ *be a homogeneous ideal of* R *(I.e. if* $p \in \mathfrak{a}$, *then each homogeneous block of* $p \in \mathfrak{a}$*). Let* S *be the space of polynomial solutions of* $a(\partial) f = 0$, $a \in \mathfrak{a}$. *Then* $\mathfrak{a}$, S, R *are vector spaces over* k *and* $R = \mathfrak{a} \otimes S$.

Proof. Let R_m = vector space of homogeneous polynomials of degree m, $0 \leqslant m < \infty$, $\mathfrak{a}_m = R_m \cap \mathfrak{a}$, $S_m = R_m \cap S$. We have $R = \sum_{m=0}^{\infty} \oplus R_m$, with similar expressions for $\mathfrak{a}$ and S. For any two polynomials P, Q, define $(P, Q) = P(\partial)\, Q\,|_{x=0}$. It is readily verified that (P, Q) is an inner product on R with $R_m \perp R_p$ whenever $m \neq p$. We show that $\mathfrak{a}_m$, S_m are orthogonal complements in R_m. Hence $R_m = \mathfrak{a}_m \oplus S_m$, $0 \leqslant m < \infty$, and so $R = \mathfrak{a} \oplus S$. $Q \in S$, $P \in \mathfrak{a}_m \Rightarrow P(\partial)\, Q(x) = 0 \Rightarrow (P, Q) = 0$. Hence $S_m \in \mathfrak{a}_m^{\perp}$. Let $Q \in \mathfrak{a}_m^{\perp}$. We show that $Q \in S_m$. It suffices to check that for any homogeneous $a \in \mathfrak{a}$ of degree $\leqslant m$, $a(\partial)\, Q(x) = 0 \Leftrightarrow b(\partial)\,[a(\partial) Q] = 0$ for all homogeneous b of degree $(m - \deg a)$. Now $b(\partial)\,[a(\partial) Q] = (ba, Q)$. Since $b\,a \in \mathfrak{a}_m$

and $Q \in \mathfrak{a}_m^{\perp}$, we conclude $b(\partial)[a(\partial)Q] = 0$. Thus $Q \in S_m$, so that $\mathfrak{a}_m^{\perp} \subset S_m$. It follows that $S_m = \mathfrak{a}_m^{\perp}$.

The following lemma will be required for the proof of Theorem 4.2.

LEMMA 4.1. *Let $i(x)$ be an invariant of G and $\sigma \in G$. Let $f(x)$ be C^{∞} on an n-dimensional region $\mathscr{R}$. Then $i(\partial) f(\sigma x) = [i(\partial)f](\sigma x)$, provided $x, \sigma x \in \mathscr{R}$.*

Proof. An application of the chain rule yields

$$i(\partial) f(\sigma x) = [i(\sigma^{-1}\partial)](\sigma x),$$

for any polynomial $i(x)$. If $i(x)$ is invariant under G, then $i(\sigma^{-1}x) = i(x)$, so that $i(\partial) f(\sigma x) = [i(\partial)f](\sigma x)$.

THEOREM 4.2. (Steinberg [21]). *Let* $\Pi(x) = \prod_{i=1}^{r} L_i(x)$, *where* $L_i(x) = 0$ *are the r.h.'s of* G, *and* $D\,\Pi =$ *linear span of partial derivatives of* $\Pi(x)$. *Let* S *be the solution space of* C^{∞} *functions on the* n*-dimensional region* $\mathscr{R}$ *satisfying* (4.1) $a(\partial) f = 0, x \in \mathscr{R}$ *and* $a \in \mathscr{I}$, $\mathscr{I}$ *being the ideal generated by all homogeneous invariants of* G *of positive degree. Then* $S = D\,\Pi$.

REMARK. If $O(n)$ is the orthogonal group acting on R^n, then it can easily be shown that $x_1^2 + \ldots x_n^2$ is a basis for the invariants of $O(n)$, i.e. each invariant polynomial is a polynomial in $x_1^2 + \ldots + x_n^2$. If we replace G by $O(n)$, then (4.1) reduces to Laplace's equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right) f = 0 .$$

Because of this, it is natural to refer to the elements in S as the harmonic functions for G. Theorem 4.2 describes these harmonic functions.

Proof of Theorem 4.2. The inclusion $D\,\Pi \subset S$ clearly follows from $a(\partial)\,\Pi = 0, a \in \mathscr{I}$. It suffices to prove the latter for a homogeneous invariant of positive degree. By Lemma 3.4, $\Pi(\sigma x) = \det \sigma.\ \Pi(x), \sigma \in G$. By Lemma 4.1, $[a(\partial)\Pi](\sigma x) = a(\partial)\,\Pi(\sigma x) = \det \sigma\,[a(\partial)\Pi]$. Thus $a(\partial)\,\Pi$ is skew. Again by Lemma 3.4, $\Pi \mid a(\partial)\,\Pi$. Since $\deg [a(\partial)\Pi] < \deg \Pi$, we must have $a(\partial)\,\Pi = 0$.

We now show that $S \subset D\,\Pi$. Let $f \in S$. We prove first that f is a polynomial x_i, $1 \leqslant i \leqslant n$, is a root of $P(X) = \prod_{\sigma \varepsilon G} [X - x_i(\sigma x)] = X^{|G|}$

$+ a_1 X^{|G|-1} + \ldots + a_{|G|}$, where the a_i's are homogeneous invariants of positive degree. Thus $x_i^{|G|} = -a_1 x_i^{|G|-1} \ldots a_{|G|} \in \mathscr{I}$, $1 \leqslant i \leqslant n$. The latter implies that every homogeneous polynomial $a(x)$ of degree $\geqslant n\,|\,G\,|$ is in $\mathscr{I}$. Hence $a(\partial) f = 0$, whenever $a(x)$ is homogeneous of degree $\geqslant n\,|\,G\,| \Rightarrow f$ is a polynomial of degree $< n\,|\,G\,|$. S is therefore a finite dimensional space of polynomials. In view of Fischer's Theorem $S \subset D\Pi \Leftrightarrow (D\Pi)^\perp \subset S^\perp$. A polynomial $P(x) \in (D\Pi)^\perp \Leftrightarrow (P, Q(\partial)\Pi) = 0$ $\forall$ polynomials $Q \Leftrightarrow Q(\partial)\,(P(\partial)\Pi)\,|_{x=0}$ $\forall$ polynomials $Q \Leftrightarrow P(\partial)\,\Pi = 0$. We must therefore show that $P(\partial)\,\Pi = 0 \Rightarrow P \in \mathscr{I}$.

It suffices to prove this for homogeneous P. The result holds for deg $P \geqslant n\,|G|$. Suppose that it holds for deg $P = m + 1$. We show that it holds for deg $P = m$ and, by induction, for arbitrary degree. Let $L(x) = 0$ be an r.h. of G. Then $L(\partial)\,P(\partial)\,\Pi(x) = 0$. By the induction hypothesis $L P \in \mathscr{I}$, so that

$$L(x)\,P(x) = \sum_{k=1}^{n} A_k(x)\,I_k(x) \tag{4.2}$$

where the A_k's are polynomials and $I_1, \ldots, I_n$ are a basic set of homogeneous invariants for G. Let σ be the reflection in the r.h. $L(x) = 0$. Substituting σx for x in (4.2) and subtracting the resulting equation from (4.1), we get

$$L(x)\,(P(x) + P(\sigma x)) = \sum_{k=1}^{n} (A_k(x) - A_k(\sigma x))\,I_k(x) \tag{4.3}$$

Each $[A_k(x) - A_k(\sigma x)] = 0$ whenever $L(x) = 0$. Thus

$$L(x)\mid [A_k(x) - A_k(\sigma x)]\,,$$

and

$$P(x) + P(\sigma x) = \sum_{k=1}^{n} \left[\frac{A_k(x) - A_k(\sigma x)}{L(x)}\right] I_k(x) \tag{4.4}$$

shows that $P(x) \equiv -P(\sigma x) \pmod{\mathscr{I}}$. Since the reflections in G generate G, we conclude from the latter that $P(x) \equiv \det \sigma\, P(\sigma x) \pmod{\mathscr{I}}$. Averaging over G, we obtain $P(x) \equiv P^*(x) \pmod{\mathscr{I}}$, where $P^*(x) = \dfrac{1}{|\,G\,|} \sum\limits_{\sigma \varepsilon G} \det \sigma \cdot P(\sigma x)$. We claim that $P^*(x)$ is skew. For if $\sigma_1 \in G$, then

$$\begin{aligned} P^*(\sigma_1 x) &= \frac{1}{|\,G\,|} \sum_{\sigma \varepsilon G} \det \sigma \cdot P(\sigma\sigma_1 x) \\ &= \frac{1}{\det \sigma_1} \sum_{\sigma \varepsilon G} \det \sigma\sigma_1\, P(\sigma\sigma_1 x) = \det \sigma_1\, P^*(x)\,. \end{aligned} \tag{4.5}$$

By lemma 3.4 $P^*(x) = \Pi(x)\, i(x)$, where i is a homogeneous invariant. If $\deg i > 0$, then $P^* \in \mathscr{I} \Rightarrow P \in \mathscr{I}$. Otherwise $P^* = c\,\Pi$, c a constant. By assumption $P(\partial)\,\Pi = 0$, while $a(\partial)\,\Pi = 0$ for $a \in \mathscr{I}$. It follows that $P^*(\partial)\,\Pi = c\,(\Pi, \Pi) \Rightarrow c = 0$, so that $P \equiv 0 \pmod{\mathscr{I}}$.

2. Mean Value Properties

We prove the equivalence of system (4.1) and a certain mean value property.

Theorem 4.3 (Steinberg [21]). *Let $f(x) \in C$ in the n-dimensional region $\mathscr{R}$ and let it satisfy the mean value property (m.v.p.)*

$$f(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} f(x+\sigma y), \quad x \in \mathscr{R} \text{ and } \|y\| < \varepsilon_x, \tag{4.6}$$

where $\inf_{x \in K} \varepsilon_x > 0$ for any compact subset K of $\mathscr{R}$ and $\|y\|^2 = \sum_{i=1}^{n} y_i^2$. This m.v.p. is equivalent to having $f \in C^\infty$ and satisfying (4.1). It follows from Theorem 4.2 that the space S of continuous solutions to (4.6) $= D\,\Pi$.

Remark. The harmonic functions on $\mathscr{R}$ are characterized as the continuous functions on $\mathscr{R}$ satisfying the m.v.p. $f(x) = \int f(x+y)\, d\sigma(y)$, $x \in \mathscr{R}$ and $\|y\| < \varepsilon_x$, where $d\sigma(y)$ is the normalized Haar measure on the orthogonal group $O(n)$. (4.6) is just the G-analog of this m.v.p.

Proof of Theorem 4.3. Suppose first that $f(x)$ is C^∞ on $\mathscr{R}$ and satisfies (4.6). Let $a(x)$ be any homogeneous invariant of positive degree. Apply the operator $a(\partial_y)$ to both sides of (4.6). In view of Lemma 4.1, we get

$$\begin{aligned} 0 = a(\partial_y) f(x) &= \frac{1}{|G|} \sum_{\sigma \varepsilon G} a(\partial_y) f(x+\sigma y) \\ &= \frac{1}{|G|} \sum_{\sigma \varepsilon G} \left[a(\partial_y) f(x+y)\right](\sigma y) \end{aligned} \tag{4.7}$$

Use $a(\partial_y) f(x+y) = a(\partial_x) f(x+y)$ and set $y = 0$. We obtain $a(\partial_x) f(x) = 0$, $x \in \mathscr{R}$ and a any homogeneous invariant of positive degree. Hence $a(\partial_x) f(x) = 0$, $x \in \mathscr{R}$ and $a \in \mathscr{I}$. Since $\sum_{i=1}^{n} x_i^2 \in \mathscr{I}$, we conclude in particular that $f(x)$ is harmonic on $\mathscr{R}$.

Suppose next that $f(x)$ is C on $\mathscr{R}$ and satisfies (4.6). Let $\{\delta_k\}$ be a sequence of C^∞ functions on R^n such that $\int \delta_k(x)\,dx = 1$, support of $\delta_k = \left\{x \mid \|x\| \leqslant \frac{1}{k}\right\}$, $\delta_k(x) \geqslant 0$ for all x and k. Let

$$f_k(x) = \int f(x-y)\,\delta_k(y)\,dy = \int f(y)\,\delta_k(x-y)\,dy\,.$$

It is readily checked that for any compact subset S of $\mathscr{R}$, $f_k(x) \in C^\infty$ on Int S (= interior of S) and satisfies (4.6) with $\mathscr{R}$ replaced by Int S, provided k is sufficiently large, and $f_k \to f$ uniformly on S as $k \to \infty$. For k sufficiently large, f_k is harmonic on Int S. It follows from Harnack's Theorem ([15], p. 248) that $f(x)$ is harmonic on $\mathscr{R}$. Hence $f(x)$ is real analytic on $\mathscr{R}$ ([15], p. 251) and so certainly C^∞ on $\mathscr{R}$.

Conversely let $f \in C^\infty$ on $\mathscr{R}$ and $a(\partial) f = 0$, $x \in \mathscr{R}$ and $a \in \mathscr{I}$. Then f is harmonic and so real analytic on $\mathscr{R}$. Hence there exists $\varepsilon_x > 0$ such that

$$f(x+y) = \sum_{m=0}^{\infty} \frac{1}{m!}(\partial_x, y)^m f(x),\ x \in \mathscr{R}$$

and $\|y\| < \varepsilon_x$. It follows that

$$\frac{1}{|G|}\sum_{\sigma \varepsilon G} f(x+\sigma y) = \sum_{m=0}^{\infty} \frac{P_m(\partial_x, y)}{m!} f(x),\ x \in \mathscr{R} \tag{4.8}$$

and $\|y\| < \varepsilon_x$ where

$$P_m(x, y) = \frac{1}{|G|}\sum_{\sigma \varepsilon G}(x, \sigma y)^m = \frac{1}{|G|}\sum_{\sigma \varepsilon G}(\sigma x, y)^m. \tag{4.9}$$

From (4.9), we see that for fixed y, each $P_m(x, y)$ is a homogeneous invariant polynomial in x of degree m. It follows that $P_m(\partial_x, y) f(x) = 0$, $x \in \mathscr{R}$ and $m \leqslant 1$, and (4.8) reduces to (4.6).

The solution space to either (4.1) or (4.6) is the finite dimensional vector space $D\,\Pi$. The following result gives further information on $D\,\Pi$.

THEOREM 4.4 (Chevalley [4]). *Let S_m = vector space of homogeneous polynomials of degree m in $D\,\Pi$, $0 \leqslant m < \infty$, so that $D\,\Pi = \sum_{m=0}^{\infty} \oplus S_m$. Let $d_1, \ldots, d_n$ be the degrees of the basic homogeneous invariants for G. Then*

$$(4.10) \qquad \sum_{m=0}^{\infty} (\dim S_m)\, t^m = \prod_{i=1}^{n} \frac{1-t^{d_i}}{1-t}$$

and $\dim D\Pi = |G|$.

We prove first the preliminary

LEMMA 4.2. Let $R = k[x_1, ..., x_n]$ = ring of polynomials in $x_1, ..., x_n$ with coefficients from k, k being any field of characteristic 0. Let G be a finite reflection group acting on k^n and $\mathscr{I}$ the ideal generated by homogeneous invariants of positive degree. For any polynomial P, let $\bar{P}$ be its residue class in the residue class ring $R/\mathscr{I}$. Suppose that $P_1, ..., P_s$ are homogeneous polynomials such that $\bar{P}_1, ..., \bar{P}_s$ are linearly independent over $R/\mathscr{I}$ (the latter is a vector space over k). Then $P_1, ..., P_s$ are linearly independent over $k(I)$, the field obtained by adjoining the set I of all invariant polynomials to k.

Proof. Suppose $\sum_{i=1}^{s} V_i P_i = 0$ where $V_i \in k(I)$, $1 \leqslant i \leqslant s$. We may suppose that the V_i's are homogeneous and $[\deg V_i + \deg P_i]$ is the same for all i. Let $I_1, ..., I_n$ be a basic set of homogeneous invariants of positive degree. Let S_j, $0 \leqslant j < \infty$, be the different monomials in $I_1 \dots I_n$ arranged by increasing x-degree, with $s_0 = 1$. Let $V_i = \sum_{j=0}^{\infty} k_{ij} S_j$, $1 \leqslant i \leqslant s$, the k_{ij}'s being elements of k, and define k_{i0} to be 0. We have

$$(4.11) \qquad \sum_{i=1}^{s} V_i P_i = \sum_{j=0}^{\infty} \Big[\sum_{i=1}^{s} k_{ij} P_i\Big] S_j = 0$$

Assume, as induction hypothesis, that $k_{ij} = 0$ for $j < l$. Thus $\sum_{j=l}^{\infty} [\sum_{i=1}^{s} k_{ij} P_i] S_j = 0$. $S_l \notin$ ideal generated by the S_j's, $j > l$, as $I_1, ..., I_n$ are algebraically independent. It follows from Lemma 2.1 that $\sum_{i=1}^{s} k_{il} P_i \in \mathscr{I} \Leftrightarrow \sum_{i=1}^{s} k_{il} \bar{P}_i = 0 \Leftrightarrow k_{il} = 0$, $1 \leqslant i \leqslant s$. Hence all $k_{ij} = 0$ and $V_i = 0$, $1 \leqslant i \leqslant s$. I.e. $P_1, ..., P_s$ are linearly independent over $k(I)$.

We now return to the proof of Theorem 4.4. Let $A_1, ..., A_q$ be homogeneous polynomials such that $\bar{A}_1, ..., \bar{A}_q$ form a basis for $R/\mathscr{I}$. By induction on the degree, we see that every polynomial P may be expressed as

$$(4.12) \qquad P = \sum_{i=1}^{q} J_i A_i$$

where the J_i's are invariant polynomials. Lemma 4.2 shows that this representation is unique. Let R_m = set of homogeneous polynomials of degree m, $I_m = I \cap R_m$, $(R/\mathscr{I})_m$ = vector space spanned by those $\bar{A}_i'$s for which degree $A_i = m$. Let

$$\mathfrak{p}_R(t) = \sum_{n=0}^{\infty} (\dim R_m)\, t^m, \quad \mathfrak{p}_I(t) = \sum_{m=0}^{\infty} (\dim I_m)\, t^m,$$

$$\mathfrak{p}_{R\mathscr{I}}(t) = \sum_{m=0}^{\infty} \dim (R/\mathscr{I})_m\, t^m.$$

In view of the uniqueness of the representation (4.12), we have

$$\mathfrak{p}_R(t) = \mathfrak{p}_I(t)\, \mathfrak{p}_{R/\mathscr{I}}(t) \tag{4.13}$$

Now

$$\mathfrak{p}_I(t) = \frac{1}{\prod_{i=1}^{n} (1 - t^{d_i})} \qquad \text{(formula (2.5))}$$

while

$$\mathfrak{p}_R(t) = \frac{1}{(1-t)^n}$$

(as $\dim R_m = \binom{m+n-1}{m}$). By Fischer's Theorem $R/\mathscr{I}$ may be identified with $D\,\Pi$, so that $\mathfrak{p}_{R/\mathscr{I}}(t) = \sum_{m=0}^{\infty} (\dim S_m)\, t^m$. Thus (4.13) becomes (4.10).

Set $t = 1$ in (4.10). The left side becomes $\sum_{m=0}^{\infty} \dim S_m = \dim D\,\Pi$. Since

$$\frac{1 - t^{d_i}}{1 - t} = 1 + t + \ldots + t^{d_i - 1} = d_i$$

at $t = 1$, the right side becomes $\prod_{i=1}^{n} d_i = |G|$ (by Theorem 2.2). Thus $\dim D\,\Pi = |G|$.

We now describe the solution space to (4.6) when we restrict the direction of y. For simplicity, we restrict ourselves to irreducible groups (the reducible case is discussed in [12]).

THEOREM 4.5. *Let $f(x) \in C$ in the n-dimensional region $\mathscr{R}$ and satisfy the m.v.p.*

$$f(x) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} f(x + t\sigma y), \; x \in \mathscr{R} \text{ and } 0 < t < \varepsilon_x, \tag{4.14}$$

$\inf_{x \in K} \varepsilon_x > 0$ for any compact subset K of $\mathscr{R}$ and y denoting a fixed vector $\neq 0$. This m.v.p. is equivalent to having $f \in C^\infty$ on $\mathscr{R}$ and $P_m(\partial_x, y) f = 0$, $x \in \mathscr{R}$ and $1 \leqslant m < \infty$, P_m being defined by (4.9).

Proof. Suppose first that $f \in C^\infty$ on $\mathscr{R}$ and satisfies (4.14). Using the finite Taylor expansion for $f(x + t\sigma y)$, we get for each integer $N \geqslant 0$

$$(4.15) \qquad 0 = \sum_{m=1}^{N} \left[\frac{P_m(\partial_x, y) f}{m\,!} \right] t^m + O(t^{N+1}) \text{ as } t \to 0\,.$$

Dividing by successive powers of t and letting $t \to 0$, we conclude $P_m(\partial_x, y) f = 0$, $x \in \mathscr{R}$ and $1 \leqslant m < \infty$. If $f \in C$, then we argue as in the proof of Theorem 4.3, introducing the functions f_k. For any compact subset S of $\mathscr{R}$ and k sufficiently large, the f_k's will be C^∞ on Int S and satisfy there $P_m(\partial_x, y) f = 0$, $1 \leqslant m < \infty$. $P_2(x, y)$ is a non-zero homogeneous invariant of degree 2. For irreducible G, there is up to a multiplicative constant, only one such invariant, namely $\sum_{i=1}^{n} x_i^2$. Thus $P_2(x, y) = c(y) \sum_{i=1}^{n} x_i^2$, where $c(y) \neq 0$ is a constant depending on y. Thus for k sufficiently large, $f_k(x)$ is harmonic on Int S. Since $f_k \to f$ uniformly on compact subsets of $\mathscr{R}$, $f(x)$ is harmonic on $\mathscr{R}$ and hence certainly C^∞ on $\mathscr{R}$.

Conversely, let $P_m(\partial_x, y) f = 0$, $x \in \mathscr{R}$ and $1 \leqslant m < \infty$. Since $P_2(\partial_x, y) f = 0$, f is harmonic and so real analytic on $\mathscr{R}$. It follows that there exists $\varepsilon_x > 0$ such that

$$(4.16) \qquad \frac{1}{|G|} \sum_{\sigma \varepsilon G} f(x + t\sigma y) = \sum_{m=0}^{\infty} \left[\frac{P_m(\partial_x, y) f}{m\,!} \right] t^m, \; x \in \mathscr{R}$$

and $0 < t < \varepsilon_x$.

Since $P_m(\partial_x, y) f = 0$, $x \in \mathscr{R}$ and $1 \leqslant m < \infty$, (4.16) reduces to (4.14).

We shall describe the solution space to $P_m(\partial_x, y) f = 0, 1 \leqslant m < \infty$, y being a fixed vector $\neq 0$. We first prove some preliminary lemmas.

Lemma 4.3. Let $\mathscr{C}$ be a collection of homogeneous polynomials in $k[x_1 \ldots, x_n]$ of positive degree, k being a field of characteristic 0. Let G be a finite reflection group acting on k^n. The following conditions are equivalent.

i) $\mathscr{C}$ is a basis for the invariants of G

ii) $\mathscr{C}$ is a basis for the ideal $\mathscr{I}$ generated by the homogeneous invariants of positive degree.

iii) Let $d_1, \ldots, d_n$ be the degrees of the basic homogeneous invariants of G.

For each d_i there exists a polynomial $P_i \in \mathscr{C}$ of degree d_i such that

$$\frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} \neq 0 .$$

Proof. Let $\mathscr{I}(\mathscr{C})$ = ideal generated by $\mathscr{C}$, so that $\mathscr{I}(\mathscr{C}) \subset \mathscr{I}$. If i) holds, then $\mathscr{I}(\mathscr{C})$ contains every homogeneous invariant of positive degree, so that $\mathscr{I} \subset \mathscr{I}(\mathscr{C}) \Rightarrow \mathscr{I} = \mathscr{I}(\mathscr{C})$.
Thus i) $\Rightarrow$ ii).

Suppose ii) holds. Choose in $\mathscr{C}$ a minimal basis for $\mathscr{I}$. The proof of Chevalley's Theorem shows that this minimal basis consists of n homogeneous invariants $P_1, \ldots, P_n$ which are algebraically independent

$$\Leftrightarrow \frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} \neq 0 .$$

According to Theorem 3.1, these degrees must be $d_1, \ldots, d_n$. Thus ii) $\Rightarrow$ iii).

Finally, the implication iii) $\Rightarrow$ i) is contained in Theorem 3.13.

Lemma 4.4. Let G be a finite reflection group acting on k^n. Let $I_1, \ldots, I_n$ be a basic set of homogeneous invariants of respective positive degrees $d_1, \ldots, d_n$ which are assumed distinct; i.e. $d_1 < d_2 < \ldots < d_n$. Let $P_1, \ldots, P_n$ be another set of homogeneous invariants of respective degrees $d_1, \ldots, d_n$. Thus

$$\begin{aligned} P_i(x) &= F_i\big(I_1(x), \ldots, I_{i-1}(x)\big) + c_i I_i(x) \\ &= F_i(x) + c_i I_i(x),\ 1 \leqslant i \leqslant n \end{aligned} \tag{4.17}$$

where $F_i(x)$ is homogeneous of degree m_i, with $F_1 = 0$, and c_i a constant. Then

$$\frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} = c_1 \ldots c_n \frac{\partial (I_1, \ldots, I_n)}{\partial (x_1, \ldots, x_n)} \tag{4.18}$$

Proof. We have

$$\frac{\partial (P_1, \ldots, P_n)}{\partial (x_1, \ldots, x_n)} = \frac{\partial (F_1, \ldots, F_n)}{\partial (I_1, \ldots, I_n)} \frac{\partial (I_1, \ldots, I_n)}{\partial (x_1, \ldots, x_n)}$$

The matrix $\left[\frac{\partial F_i}{\partial I_j}\right]$ is triangular and $\frac{\partial F_i}{\partial I_i} = c_i$, $1 \leqslant i \leqslant n$, so that

$$\frac{\partial (F_1, \ldots, F_n)}{\partial (x_1, \ldots, x_n)} = c_1 \ldots c_n.$$

THEOREM 4.6 (Flatto and Wiener [10]). i) *Let* S_y *be space of continuous functions on the* n*-dimensional region* $\mathscr{R}$ *satisfying the mean value property (4.14).* $S_y = D\Pi$ *iff* $G \neq D_{2n}$, $2 \leqslant n < \infty$, *and*

$$\frac{\partial (P_{d_1}, \ldots, P_{d_n})}{\partial (x_1, \ldots, x_n)} \neq 0 .$$

ii) *For* $G \neq D_{2n}$, $2 \leqslant n < \infty$, *we have*

$$\frac{\partial (P_{d_1}, \ldots, P_{d_n})}{\partial (x_1, \ldots, x_n)} = J_1(y) \ldots J_n(y)\, \Pi(x) \tag{4.19}$$

the J*'s being a basic set of homogeneous invariants for* G*. Hence*

$$S_y = D\Pi \text{ iff } J_1(y) \ldots J_n(y) \neq 0 .$$

Proof. According to Theorem 4.5, S is the solution space of

$$f \in C^\infty \text{ and } p(\partial) f = 0,\ x \in \mathscr{R} \text{ and } p \in \mathscr{P}_y. \tag{4.20}$$

where $\mathscr{P}_y = \left(P_1(x, y), \ldots, P_m(x, y), \ldots\right)$. It follows from Theorems 4.1, 4.2 that $S_y = D\Pi$ iff $\mathscr{P}_y = \mathscr{I}$. By Lemma 4.3, $\mathscr{P}_y = \mathscr{I}$ iff the degrees $d_1, \ldots, d_n$ are distinct and

$$\frac{\partial (P_{d_1}, \ldots, P_{d_n})}{\partial (x_1, \ldots, x_n)} \neq 0$$

An inspection of the table in section 3.3 reveals that the d_i's are distinct except when $G = D_{2n}$, $2 \leqslant n < \infty$, in which case two d_i's equal $2n$. ii) For each n-tuple $a = (a_1, \ldots, a_n)$ of non-negative integers, let $J_a(x) = \frac{1}{|G|} \sum_{\sigma \in G} (\sigma x)^a$. We have

$$P_m(x, y) = \frac{1}{|G|} \sum_{\sigma \varepsilon G} (\sigma x, y)^m = \frac{1}{|G|^2} \sum_{\sigma_1 \varepsilon G} \sum_{\sigma_2 \varepsilon G} (\sigma_1 x, \sigma_2 y)^m =$$

$$\frac{1}{|G|^2} \sum_{|a| = m} \sum_{\sigma_1 \varepsilon G} \sum_{\sigma_2 \varepsilon G} \frac{m!}{a!} (\sigma_1 x)^a (\sigma_2 y)^a = \sum_{|a| = m} \frac{m!}{a!} J_a(x)\, J_a(y) \tag{4.21}$$

Let $I_1, \ldots, I_n$ be a basic set of homogeneous invariants of respective degrees $d_1, \ldots, d_n$. Let $|a| = d_i$, $1 \leqslant i \leqslant n$. Then

$$(4.22) \qquad J_a(x) = F_a(I_1(x), \ldots, I_{i-1}(x)) + c_a I_i(x) = F_a(x) + c_a I_i(x)$$

where $F_a(x)$ is homogeneous of degree d_i with $F_a(x) = 0$ for $i = 1$, and c_a is a constant. (4.21), (4.22) give

$$(4.23) \quad P_{d_i}(x, y) = \sum_{|a|=d_i} \frac{d_i!}{a!} J_a(y) F_a(x) + J_i(y) I_i(x), \ 1 \leqslant i \leqslant n$$

where

$$(4.24) \qquad J_i(y) = \sum_{|a|=d_i} \frac{d_i!}{a!} c_a J_a(y), \ 1 \leqslant i \leqslant n$$

(4.19) follows from (4.23) and Lemma 4.4. J_i is homogeneous of degree d_i. We show that $J_1, \ldots, J_n$ are algebraically independent and thus conclude from Lemma 4.3 that $J_1, \ldots, J_n$ form a basis for the invariants of G. Now the J_a's form a basis for the invariants of G (see Noether's proof of Theorem 1.1). Hence, by Lemma 4.3, there exists $n\, J_a'$s of respective degrees $d_1, \ldots, d_n$ which are algebraically independent. By Lemma 4.4, for each of these J_a's, $c_a \neq 0$. (4.22), (4.24) give

$$(4.25) \quad J_i(y) = \sum_{|a|=d_i} \frac{d_i!}{a!} c_a F_a(y) + \Big(\sum_{|a|=m_i} \frac{d_i}{a!} c_a^2 \Big) I_i(y), \ 1 \leqslant i \leqslant n$$

For each $1 \leqslant i \leqslant n$, there exists an a such that $|a| = d_i$ and $c_a \neq 0$, so that the n constants $\sum_{|n|=d_i} \frac{d_i}{a!} c_a^2$ are all $\neq 0$. It follows from (4.25) and Lemma 4.4, that $J_1, \ldots, J_n$ are algebraically independent.

The following theorem yields an algebraic characterization of the J_i's.

THEOREM 4.7 [12]. $J_1(x) = c \sum_{i=1}^{n} x_i^2$, $c \neq 0$. *For* $2 \leqslant i \leqslant n$, $J_i(x)$ *is determined up to a constant as the homogeneous invariant of degree* d_i *which satisfies the differential equations* $J_k(\partial) J_i(x) = 0$, $1 \leqslant k < i$.

Proof. $J_1(x)$ is a non-zero homogeneous invariant of degree 2 and must therefore be a non-zero multiple of $\sum_{i=1}^{n} x_i^2$. Let $2 \leqslant i \leqslant n$ and $1 \leqslant k < d_i$. Let $Q(x)$ be an arbitrary homogeneous invariant polynomial of degree k. We have

$$(4.26) \qquad Q(\partial_y) P_m(x, y) = Q(\partial_y) \Big[\frac{1}{|G|} \sum_{\sigma \varepsilon G} (y, \sigma x)^m \Big]$$

$$= m(m-1) \ldots (m-k+1) P_{m-k}(x, y) \ Q(x)$$

From (4.23), we obtain

$$(4.27)\qquad Q(\partial_y)P_{d_i}(x,y) = \sum_{|a|=d_i}\frac{d_i!}{a!}\left[Q(\partial)J_a(y)\right]F_a(x) + \left[Q(\partial)J_i(y)\right]I_i(x), \quad 1 \leqslant i \leqslant n$$

so that

$$(4.28)\qquad d_i(d_i-1)\cdots(d_i-k+1)P_{d_i-k}(x,y)\,Q(x) = \sum_{|a|=d_i}\frac{d_i!}{a!}\left[Q(\partial)J_a(y)\right]F_a(x) + \left[Q(\partial)J_i(y)\right]I_i(x), \quad 1 \leqslant i \leqslant n$$

Suppose that $Q(\partial)J_i(y) \neq 0$. Choose y_0 so that $Q(\partial)J_i(y) \neq 0$ at y_0. Let $y = y_0$ in (4.28). The polynomial $P_{d_i-k}(x, y_0)$ has degree $< d_i$ and thus is a polynomial in $I_1, (x), \ldots, I_{i-1}(x)$. Each F_a is also a polynomial in $I_1, \ldots, I_{i-1}$. We conclude from (4.28) that $I_1, \ldots, I_i$ are algebraically dependent, a contradiction. Hence $Q(\partial)J_k(y) = 0$, so that $J_k(\partial)J_i(x) = 0$, $1 \leqslant k < i$.

The conditions of Theorem 4.7 determine J_i up to a constant. For let V_i = space of homogeneous invariants of degree d_i, W_i = space of homogeneous invariants of degree d_i spanned by the monomials in $I_1, \ldots, I_{i-1}$. Then $\dim V_i = \dim W + 1$. For any $J \in V_i$, the conditions $J_k(\partial)J(x) = 0$, $1 \leqslant k < i$, are equivalent to $J \in W_i^{\perp}$. Since $\dim W_i^{\perp} = \dim V_i - \dim W_i = 1$, we conclude that J_i is determined up to a constant.

COROLLARY. The manifold $\mathscr{M} = \{y \mid J_1(y)\cdots J_n(y) = 0\}$ contains real points $y \neq 0$. I.e. there exists $y \in R^n$ such that $S \neq D\,\Pi$.

Proof. For $2 \leqslant i \leqslant n$, $J_1(\partial)J_i(x) = 0$. Since $J_1(x) = c\sum_{i=1}^{n} x_i^2$, $c \neq 0$, this means that $J_i(x)$ is harmonic. By the mean value property for harmonic functions, the average value of $J_i(y)$ on a sphere of radius $r > 0 = J_i(0) = 0$. Thus $J_i(y)$ must change sign on this sphere and a connectedness argument yields the existence of a $y \neq 0$ for which $J_i(y) = 0$.

In view of Theorem 4.6, we call $\mathscr{M}$ the "exceptional manifold" for G and the non-zero vectors y of $\mathscr{M}$, the "exceptional directions" for G. A geometric description of $\mathscr{M}$ is given in [24] for the groups H_2^n and A_3. There remains the problem of describing the solution space S_y to the m.v.p. (4.14) in case y is an exceptional direction, as $D\,\Pi$ is then a proper subspace of S_y. This seems to be a difficult problem. In [11], it is solved for the groups H_2^n, A_3.

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