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Autor: Mac Lane, Saunders
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ORIGINS OF THE COHOMOLOGY OF GROUPS [1]

by Saunders Mac Lane

1. The Historical Questions

This paper is a small essay on the history of contemporary mathematics. It will examine the questions: What were the origins of the theory of the cohomology of groups? What were the essential steps in the development of this theory? What were the effects of this development in related fields of mathematics? These questions seem appropriate to a conference in Zurich, because major steps in the development of this subject took place here at the Eidgenössische Technische Hochschule. These questions may also be typical of questions that can be put about the development of other parts of mathematics in the twentieth century. Here are some of these questions: How does the interchange of ideas between different fields of mathematics come about? Which ideas (or, which research papers) are of essential novelty or originality and which are derivative? Do some ideas arrive before their time, and so are neglected? What are the differences between mathematical developments seen beforehand, or seen after the fact—and is there not a third perspective, that of mathematical ideas as they are in process of development?

2. Fundamental Group and 2nd Betti Group

On September 12, 1941 Heinz Hopf communicated to the *Commentarii Mathematicii Helvetici* his paper "Fundamentalgruppe und zweite Bettische Gruppe". This paper proved the

Theorem. Each group G determines, by an algebraic process, a group G_1^* which is not generally zero. If G is the fundamental group of a complex K with second Betti group $B^2 = H_2(K, \mathbf{Z})$, and if S^2 is the spherical subgroup of B^2, then

$$B^2/S^2 \cong G_1^* \tag{1}$$

[1]) Presented at the Colloquium on Topology and Algebra, April 1977, Zurich.

In this theorem, a homology class in B^2 belongs to the subgroup S^2 of spherical cycles when it can be represented by a continuous image of a 2-sphere. The algebraic construction process used in this theorem was the following: Represent the fundamental group G as $G \cong F/R$, where F is a free group and R a subgroup of F, form the subgroup $[F, R]$ generated by the commutators $frf^{-1}r^{-1}$ for $f \in F$ and $r \in R$ and the corresponding commutator subgroup $[F, F]$. Then the factor group

$$G_1^* = R \cap [F, F]/[F, R] \tag{2}$$

is independent of the choice of the representation $G = F/R$ of G as a quotient group of a free group. This G_1^* is the algebraic construction used in (1) above to measure the influence of the fundamental group on the second homology group. For example, if G is a free abelian group of rank p, then G_1^* is free abelian of rank $p(p-1)/2$, so this last integer is a lower bound for the second Betti number of a complex with fundamental group G. In general, as Hopf observed, this "lower bound" G_1^* cannot be improved; for every finitely presented group G one can readily construct a complex K with G as fundamental group and with $S^2 = 0$, so that $H_2(K, Z)$ is exactly G_1^*.

The essential originality in this theorem of Hopf resides in its use of a non-obvious purely group-theoretic construction (2) in order to express the dependence of one topological invariant (here B^2/S^2) upon another, the fundamental group. What had been known before this? It had long been known that the fundamental group G determined the 1-dimensional Betti group B^1 as the factor commutator group $B^1 = G/[G, G]$. This was a fact which had a direct geometric interpretation and involved only an evident—and evidently invariant—construction on G. Hopf's construction of G_1^* was much more subtle, and required a proof that the result is independent of the choice of the representation $G = F/R$. Actually, this group construction (2) had been known before—it is exactly the Schur multiplicator of the group G. This multiplicator had been introduced by Schur in his study of the projective representations of groups. Hopf, while a student in Berlin, had been an assistant to Schur, but his 1942 paper does not mention the connection with the multiplicator. Instead, his motivation seems to have come more from his earlier studies of the homology of Lie groups. As Eckmann pointed out to me, Hopf described this connection in his 1946 (written 1941) "Report on some new results in topology". He says of his theorem above that "The proofs rest on the idea that systems of curves which represent certain finite systems of elements of the fundamental group span surfaces in the complex, whose contribution to B^2 can be specified... in some cases, the surface is a torus, as in the case of the Pontrjagin product in a group manifold".

In his 1942 paper, Hopf did not mention a connection with the higher homotopy groups, but this connection soon played an important part.

Hurewicz introduced the higher homotopy groups in 1935. In 1936 he proved for an aspherical complex (one with all higher homotopy groups zero) that the fundamental group did determine all the Betti groups: This meant that two such complexes K and K' with isomorphic fundamental groups would have isomorphic Betti groups $B^n \cong B'^n$ in all dimensions n. In particular, it showed for a complex K with $S^2 = 0$ that B^2 would depend only on the fundamental group G. Hurewicz did not determine the fashion of this dependence, though according to Freudenthal [1946] he did raise this question in conversations. In effect, Hopf's paper provided the answer to the question of Hurewicz for $n = 2$.

Hopf's 1942 paper was the starting point for the cohomology and homology of groups; indeed this Hopf group G_1^* is simply our present second homology group $H_2(G, \mathbf{Z})$. This idea and this paper were indirectly the starting point for several other developments: Invariants of group presentations; cohomology of other algebraic systems; functors and duality; transfer and Galois cohomology; spectral sequences; resolutions; Eilenberg-Mac Lane spaces; derived functors and homological algebra; and other ideas as we will indicate below.

After the fact, we can view Hopf's paper as the first decisive step in the development of group cohomology and homological algebra. Beforehand, it appears differently, as a specific answer to a question implicit in the work of Hurewicz: Exactly how does the fundamental group affect the second Betti group? During the process, it was soon apparent from Hopf's paper that something exciting was going on. The review by Hassler Whitney, in *Math. Reviews*, Vol. 3 (1942), p. 316 says in its first paragraph:

"This paper is, in the reviewer's mind, one of the most important contributions to combinatorial topology in recent years. It gives far reaching results concerning the relations between the fundamental group, the first and second homology and cohomology groups, and the products between these groups, with beautiful and simple methods. The work is based on some new constructions in groups which are undoubtedly of real significance by themselves. The paper is in three main parts: the group theory; determination of the second homology group B^2 (all groups are with integer coefficients) modulo the group S^2 of "spherical homology classes" [see below]; and the determination of the products for dimensions not greater than 2 (omitting considerations of torsion). In each case, the formulas are in terms of the fundamental group G, and pure group-theoretic constructions, but with geometric meanings."

3. Homology and Cohomology of Groups

As Whitney's review does suggest, Hopf's paper had immediate influence. His description of the second integral homology group of a group G was followed by four independent studies, two of which described the higher homology groups $H_n(G, \mathbf{Z})$ and two the higher cohomology group $H^n(G, A)$ for an abelian group A or, more generally, for a G-module A. Each of these papers explicitly recognizes the starting point provided by the paper of Hopf. In chronological order, these four studies are as follows:

Eilenberg and Mac Lane [1942] had been applying methods of group extensions to the universal coefficient theorem in cohomology, so they knew the group Ext (G, A) of all abelian extensions of the abelian group A by the abelian group G. They knew that a representation of G as F/R, with F and R free abelian, would give an exact sequence

$$0 \to \hom(G, A) \to \hom(F, A) \to \hom(R, A) \to \operatorname{Ext}(G, A) \to 0$$

(though they expressed this fact differently, writing Ext (G, A) as a suitable quotient of hom (R, A)). Moreover, they had heard of the Schur multiplicator through Mac Lane's work on class field theory. Furthermore, Eilenberg was very familiar with homotopy groups. Hence, as soon as they saw the Hopf 1942 paper, they decided that more group extensions must be hidden in Hopf's G_1^*, and they set about to find out how.

On April 7, 1943 Eilenberg and Mac Lane submitted to the *Proceedings* of the National Academy of Sciences an announcement "Relations between homology and homotopy groups". Given a group G, they constructed a chain complex $K(G)$, whose second homology group is exactly Hopf's group G_1^*. Their complex $K(G)$—now called the Eilenberg-Mac Lane complex $K(G, 1)$—had as generators in dimension n the cells $[x_1, ..., x_n]$ for entries $x_i \in G$, with boundary

$$\partial[x_1, \ldots, x_n] = [x_2, \ldots, x_n] + \sum_{i=1}^{n-1} (-1)^i [x_1, \ldots, x_i x_{i+1}, \ldots, x_n] + (-1)^n [x_1, \ldots, x_{n-1}] .$$