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for all r. Therefore $ISf(0) = 0$ and hence $f(0) = 0$ by (8). If this result is applied to $(T_y^{-1})^* f$ it follows that $f(y) = 0$ for arbitrary y, so that f is indeed identically zero.

7. Computation of SIv

It is easy to show that $S_{ij,hk}(y) = [S\gamma_{ij,.}(y)]_{hk}$ is a Calderon-Zygmund kernel for any choice of the indices; in other words, it is homogeneous of degree $-n$, and its mean-value over the unit sphere is 0. If $v \in L^p$, $1 < p < \infty$, it follows by the Calderon-Zygmund theory that the principal value

$$\text{pr. v.} \int_B v_{ij}(x)\, S_{ij,hk}(x-y)\, dx$$

exists almost everywhere, and that it is the limit in $L^p(B)$ of the corresponding truncated integrals. In view of (7) it follows that the integral

$$\Gamma v(y)_{hk} = \int_B v_{ij}(x)\, \Gamma_{ij,hk}(x, y)\, dx \tag{9}$$

will also exist as a principal value almost everywhere. One finds, however, that the remainder in (7) makes it possible to assert merely that the principal value is a limit in $L^{p'}$ for any $p' < p/n$. In these circumstances it is natural to assume that $v \in L^p(B)$ for all $p \geqq 1$.

THEOREM 2. *If* $v \in L^p(B)$ *with* $p > n$, *then* $SIv \in L^{p'}(B)$ *for all* $1 \leqslant p' < p/n$, *and*

$$SI\, v = -\, b_n v + \Gamma v \tag{10}$$

where $b_n = 4\,\omega_n/(n+2)$ *and* Γv *is defined by* (9).

Proof. Let φ be an SM_n-valued test-function. The definition of SIv as a distribution leads to the following formal computation:

$$\begin{aligned}
\int_B SI\, v(y)_{hk}\, \varphi(y)_{hk}\, dy &= -\int_B Iv(y)_k S^* \varphi(y)_k\, dy \\
&= -\int_B S^* \varphi(y)_k\, dy \int_B v_{ij}(x)\, \gamma_{ij,k}(x, y)\, dx \\
&= -\int_B v_{ij}(x)\, dx \int_B S^* \varphi(y)_k\, \gamma_{ij,k}(x, y)\, dy \\
&= -\int_B v_{ij}(x)\, dx \left[b_n \varphi_{ij}(x) - \int_B \varphi(y)_{hk}\, \Gamma_{ij,hk}(x, y)\, dy\right].
\end{aligned}$$

The justification, by means of the Zygmund-Calderon theory, is routine, and (10) follows.

Taken together, Theorems 1 and 2 lead to a very striking result:

THEOREM 3. *An* SM_n*-valued function* $v \in L^p(B)$, $p > n$, *is of the form* $v = Sf$ *with* $f = 0$ *on* $S(1)$ *if and only if it satisfies the homogeneous integral equation* $\Gamma v = -a_n v$ *with* $a_n = c_n - b_n = 2(n-2)(n+1)\omega_n / n(n+2)$.

Indeed, if v is of this form, Theorem 1 implies $c_n f = -Iv$, hence $c_n v = -SIv$, and consequently $\Gamma v = (b_n - c_n) v$ by Theorem 2. Conversely, if $\Gamma v = -a_n v$ then $SIv = -c_n v$ by (10), and $f = Iv$ vanishes on $S(1)$.

The point of Theorem 3 is that the solvability of $Sf = v$ (with an extra condition on f) has been reduced to an integral equation.

THEOREM 4. *For any* $v \in L^p(B)$, $p > n$, $S^* \rho\, [\Gamma v + a_n v] = 0$.

Proof. Let f be a vector-valued test-function. Theorem 3 applies to Sf, and we obtain by use of Lemma 2

$$\int_B S^* \rho\, \Gamma v \cdot f\, dx = - \int_B \rho(x)\, \Gamma v(x)_{ij}\, Sf(x)_{ij}\, dx$$

$$= - \int_B \rho(x)\, Sf(x)_{ij}\, dx \int_B v(y)_{hk}\, \Gamma_{hk,ij}(y,x)\, dy$$

$$= - \int_B \rho(y)\, v(y)_{hk} dy \int_B Sf(x)_{ij}\, \Gamma_{ij,hk}(x,y)\, dx$$

$$= - \int_B \rho(y)\, v(y)_{hk}\, \Gamma Sf(y)_{hk} dy = a_n \int_B \rho(y)\, v(y)_{hk}\, Sf(y)_{hk} dy$$

$$= - a_n \int_B S^* \rho v \cdot f\, dy$$

and hence $S^* \rho\, \Gamma v = - a_n S^* v$.

THEOREM 5. *Every* v *which is in all* $L^p(B)$ *has a unique representation in the form* $v = v' + v''$ *where* v' *and* v'' *are in all* $L^p(B)$ *while* v' *is in the image of* SI *and* v'' *is in the kernel of* $S^* \rho$.

As a consequence of Theorems 3 and 4 the representation is given by

$$c_n v = - SIv + (\Gamma v + a_n v)\,.$$

It is unique, for if $SI = \Gamma v + a_n v$, then $S^* \rho\, SIv = 0$ so that Iv is harmonic and 0 on $S(1)$, hence identically zero.