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for all  $r$ . Therefore  $ISf(0) = 0$  and hence  $f(0) = 0$  by (8). If this result is applied to  $(T_y^{-1})^* f$  it follows that  $f(y) = 0$  for arbitrary  $y$ , so that  $f$  is indeed identically zero.

### 7. COMPUTATION OF $SIv$

It is easy to show that  $S_{ij,hk}(y) = [S\gamma_{ij,\cdot}(y)]_{hk}$  is a Calderon-Zygmund kernel for any choice of the indices; in other words, it is homogeneous of degree  $-n$ , and its mean-value over the unit sphere is 0. If  $v \in L^p$ ,  $1 < p < \infty$ , it follows by the Calderon-Zygmund theory that the principal value

$$\text{pr. v. } \int_B v_{ij}(x) S_{ij,hk}(x-y) dx$$

exists almost everywhere, and that it is the limit in  $L^p(B)$  of the corresponding truncated integrals. In view of (7) it follows that the integral

$$(9) \quad \Gamma v(y)_{hk} = \int_B v_{ij}(x) \Gamma_{ij,hk}(x,y) dx$$

will also exist as a principal value almost everywhere. One finds, however, that the remainder in (7) makes it possible to assert merely that the principal value is a limit in  $L^{p'}$  for any  $p' < p/n$ . In these circumstances it is natural to assume that  $v \in L^p(B)$  for all  $p \geq 1$ .

**THEOREM 2.** *If  $v \in L^p(B)$  with  $p > n$ , then  $SIv \in L^{p'}(B)$  for all  $1 \leq p' < p/n$ , and*

$$(10) \quad SIv = -b_n v + \Gamma v$$

where  $b_n = 4\omega_n/(n+2)$  and  $\Gamma v$  is defined by (9).

*Proof.* Let  $\varphi$  be an  $SM_n$ -valued test-function. The definition of  $SIv$  as a distribution leads to the following formal computation:

$$\begin{aligned} \int_B SIv(y)_{hk} \varphi(y)_{hk} dy &= - \int_B Iv(y)_k S^* \varphi(y)_k dy \\ &= - \int_B S^* \varphi(y)_k dy \int_B v_{ij}(x) \gamma_{ij,k}(x,y) dx \\ &= - \int_B v_{ij}(x) dx \int_B S^* \varphi(y)_k \gamma_{ij,k}(x,y) dy \\ &= - \int_B v_{ij}(x) dx [b_n \varphi_{ij}(x) - \int_B \varphi(y)_{hk} \Gamma_{ij,hk}(x,y) dy]. \end{aligned}$$

The justification, by means of the Zygmund-Calderon theory, is routine, and (10) follows.

Taken together, Theorems 1 and 2 lead to a very striking result:

**THEOREM 3.** *An  $SM_n$ -valued function  $v \in L^p(B)$ ,  $p > n$ , is of the form  $v = Sf$  with  $f = 0$  on  $S(1)$  if and only if it satisfies the homogeneous integral equation  $\Gamma v = -a_n v$  with  $a_n = c_n - b_n = 2(n-2)(n+1)\omega_n/n(n+2)$ .*

Indeed, if  $v$  is of this form, Theorem 1 implies  $c_n f = -Iv$ , hence  $c_n v = -SIv$ , and consequently  $\Gamma v = (b_n - c_n)v$  by Theorem 2. Conversely, if  $\Gamma v = -a_n v$  then  $SIv = -c_n v$  by (10), and  $f = Iv$  vanishes on  $S(1)$ .

The point of Theorem 3 is that the solvability of  $Sf = v$  (with an extra condition on  $f$ ) has been reduced to an integral equation.

**THEOREM 4.** *For any  $v \in L^p(B)$ ,  $p > n$ ,  $S^* \rho [\Gamma v + a_n v] = 0$ .*

*Proof.* Let  $f$  be a vector-valued test-function. Theorem 3 applies to  $Sf$ , and we obtain by use of Lemma 2

$$\begin{aligned} \int_B S^* \rho \Gamma v \cdot f \, dx &= - \int_B \rho(x) \Gamma v(x)_{ij} Sf(x)_{ij} \, dx \\ &= - \int_B \rho(x) Sf(x)_{ij} \, dx \int_B v(y)_{hk} \Gamma_{hk,ij}(y, x) \, dy \\ &= - \int_B \rho(y) v(y)_{hk} \, dy \int_B Sf(x)_{ij} \Gamma_{ij,hk}(x, y) \, dx \\ &= - \int_B \rho(y) v(y)_{hk} \Gamma Sf(y)_{hk} \, dy = a_n \int_B \rho(y) v(y)_{hk} Sf(y)_{hk} \, dy \\ &= - a_n \int_B S^* \rho v \cdot f \, dy \end{aligned}$$

and hence  $S^* \rho \Gamma v = -a_n S^* v$ .

**THEOREM 5.** *Every  $v$  which is in all  $L^p(B)$  has a unique representation in the form  $v = v' + v''$  where  $v'$  and  $v''$  are in all  $L^p(B)$  while  $v'$  is in the image of  $SI$  and  $v''$  is in the kernel of  $S^* \rho$ .*

As a consequence of Theorems 3 and 4 the representation is given by

$$c_n v = -SIv + (\Gamma v + a_n v).$$

It is unique, for if  $SI = \Gamma v + a_n v$ , then  $S^* \rho SIv = 0$  so that  $Iv$  is harmonic and 0 on  $S(1)$ , hence identically zero.