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where A_1^* acts on x and the double index, A_2^* on y and the single index. For $A = T_y$ this leads to the explicit formula

$$\gamma_{..,k}(x, y) = \frac{(1-|y|^2)^{n+1}}{[x, y]^{2n}} \Delta(y, x) \gamma_{..,k}(T_y x) \Delta(x, y) .$$

We note that $\gamma_{..,.}(x, 0) = \gamma_{..,.}(x)$ and $\gamma_{..,.}(0, y) = -(1-|y|^2)^{n+1} \gamma_{..,.}(y)$.

We shall need to apply S to either variable in $\gamma_{..,.}(x, y)$. For this purpose we introduce

Definition 3. $\Gamma_{ij,hk}(x, y) = [S_2 \gamma_{ij,.}(x, y)]_{hk}$.

Because differentiations with respect to x and y commute it is clear that $S_1^* \Gamma_{..,hk}(x, y) = 0$. Moreover, starting from the relation $g_{ik}(x, y) = g_{ki}(y, x)$ it is not difficult to derive the following symmetry property:

LEMMA 2. $\rho(y) \Gamma_{ij,hk}(x, y) = \rho(x) \Gamma_{hk,ij}(y, x)$.

It follows, in particular, that $S_2^* \rho(y) \Gamma_{ij,..}(x, y) = 0$.

It is also important to know the asymptotic behavior of $\Gamma_{ij,hk}(x, y)$ when $x - y \to 0$. We observe first that

$$\begin{aligned} \rho(y) \Gamma_{ij,hk}(0, y) &= -(1-|y|^2)^{-n} [S(1-|y|^2)^{n+1} \gamma_{ij,.}(y)]_{hk} \\ &= -S_{ij,hk}(y) + R_{ij,hk}(y) \end{aligned}$$

where $S_{ij,hk}(y) = [S \gamma_{ij,.}(y)]_{hk}$ is homogeneous of degree $-n$ and $R_{ij,hk}(y)$ is homogeneous of degree $2-n$. The explicit expression for $\Gamma_{ij,hk}(x, y)$ reads

$$\Gamma_{ij,..}(x, y) = \frac{(1-|y|^2)^n}{[x, y]^{2n}} \Delta(x, y) \Gamma_{ij,..}(0, T_x y) \Delta(y, x) .$$

Elementary estimates show that

$$(7) \qquad |\Gamma_{ij,hk}(x, y) + S_{ij,hk}(x-y)| \leqq C_n |x-y|^{1-n} [x, y]^{-1}$$

with constant C_n.

6. POTENTIALS

Given an SM_n-valued function v on B we define its *potential* as the vector-valued function $I v$ with components

$$I v(y)_k = \int_B v_{ij}(x) \gamma_{ij,k}(x, y) \, dx .$$

The integral converges if $v \in L^p(B)$ for some p with $n < p \leqq \infty$. In fact, one proves that

$$|I\,v(y)| \leqq C_{n,p} \,||\, v \,||_p (1 - |\, y \,|)^{1-n/p}$$

if $p < \infty$ and

$$|I\,v(y)| \leqq C_n \,||\, v \,||_\infty (1 - |\, y \,|)(1 + \log 1/|(1 - |\, y \,|)$$

if $p = \infty$. In any event $I\,v\,(y)$ vanishes at a fixed rate for $|\, y \,| \to 1$.

The forming of the potential is an invariant operation in the sense that $IA^*v = A^*I\,v$ for every $A \in G$. The potential is harmonic outside the support of v, for $(S^*\,\rho\,S)_2\,\gamma_{ij,.}(x, y) = 0$.

The following theorem serves to recover f from Sf and its boundary values:

THEOREM 1. *If* $Sf \in L^p(B), p > n,$ *then*

$$c_n f(y) = -ISf(y) + c_n Hf(y) \tag{8}$$

with

$$Hf(y) = \frac{1}{c_n} \int_{S(1)} \gamma_{ij,.}(x, y)\, x_j f_i d\sigma(x)\,.$$

Moreover, Hf *is the unique harmonic function with the same boundary values as* f, *and if* $x \cdot f = 0$ *on* $S(1)$ *it can also be written in the form*

$$Hf(y) = \frac{1}{c_n} \int_{S(1)} \frac{(1 - |\, y \,|^2)^{n+1}}{|\, x - y \,|^{2n}}\, \Delta(x, y) f(x)\, d\sigma(x)\,.$$

Remarks. $d\sigma$ refers to the $(n-1)$-dimensional measure on $S(1)$, and $c_n = 2\,(n-1)\,\omega_n/n$ where ω_n is the total measure of $S(1)$. We are assuming that f has a continuous extension to $S(1)$. Actually, this is automatically true if we assume the side condition in the form $x \cdot f(x) \to 0$ as $|\, x \,| \to 1$, for it can be shown that $Sf \in L^p$ forces f to satisfy a uniform Hölder condition.

The proof is a straight-forward application of Stokes' formula. The passage from the differentiable to the distributional case is elementary. The fact that a harmonic function is uniquely determined by its boundary values can be demonstrated as follows: Suppose that f is harmonic and zero on $S(1)$. It is readily shown that

$$\int_{S(r)} Sf(x)_{ij}\, \gamma_{ij,k}(x)\, d\sigma = 0$$

for all r. Therefore $ISf(0) = 0$ and hence $f(0) = 0$ by (8). If this result is applied to $(T_y^{-1})^* f$ it follows that $f(y) = 0$ for arbitrary y, so that f is indeed identically zero.

7. Computation of SIv

It is easy to show that $S_{ij,hk}(y) = [S\gamma_{ij,.}(y)]_{hk}$ is a Calderon-Zygmund kernel for any choice of the indices; in other words, it is homogeneous of degree $-n$, and its mean-value over the unit sphere is 0. If $v \in L^p$, $1 < p < \infty$, it follows by the Calderon-Zygmund theory that the principal value

$$\text{pr. v.} \int_B v_{ij}(x)\, S_{ij,hk}(x-y)\, dx$$

exists almost everywhere, and that it is the limit in $L^p(B)$ of the corresponding truncated integrals. In view of (7) it follows that the integral

$$\Gamma v(y)_{hk} = \int_B v_{ij}(x)\, \Gamma_{ij,hk}(x, y)\, dx \tag{9}$$

will also exist as a principal value almost everywhere. One finds, however, that the remainder in (7) makes it possible to assert merely that the principal value is a limit in $L^{p'}$ for any $p' < p/n$. In these circumstances it is natural to assume that $v \in L^p(B)$ for all $p \geqq 1$.

Theorem 2. *If $v \in L^p(B)$ with $p > n$, then $SIv \in L^{p'}(B)$ for all $1 \leqslant p' < p/n$, and*

$$SI\, v = -b_n v + \Gamma v \tag{10}$$

where $b_n = 4\,\omega_n/(n+2)$ and Γv is defined by (9).

Proof. Let φ be an SM_n-valued test-function. The definition of SIv as a distribution leads to the following formal computation:

$$\begin{aligned}
\int_B SI\, v(y)_{hk}\, \varphi(y)_{hk}\, dy &= -\int_B Iv(y)_k S^* \varphi(y)_k\, dy \\
&= -\int_B S^* \varphi(y)_k\, dy \int_B v_{ij}(x)\, \gamma_{ij,k}(x, y)\, dx \\
&= -\int_B v_{ij}(x)\, dx \int_B S^* \varphi(y)_k\, \gamma_{ij,k}(x, y)\, dy \\
&= -\int_B v_{ij}(x)\, dx \left[b_n \varphi_{ij}(x) - \int_B \varphi(y)_{hk}\, \Gamma_{ij,hk}(x, y)\, dy\right].
\end{aligned}$$