Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 24 (1978)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SINGULAR INTEGRAL EQUATION CONNECTED WITH QUASICONFORMAL MAPPINGS IN SPACE

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Kapitel: 5. FUNDAMENTAL SOLUTIONS

DOI: https://doi.org/10.5169/seals-49703

with

$$\Delta(x, y) = (1-2Q(y))(1-2Q(x-y^*)) = (1-2Q(y-x^*))(1-2Q(x)) .$$

Observe that $\Delta(x, y) = {}^t\Delta(y, x)$ and $\Delta(x, y)^2 = 1_n$ so that $\Delta(x, y) \in O(n)$. The matrix $\Delta(x, y)$ generalizes the angle arg $(1-\bar{x}y)/(1-\bar{y}x)$.

It is useful to note that $|Ax - Ay|^2 = |A'(x)|\,|A'(y)|\,|x-y|^2$ for any Möbius transformation A, and $[Ax, Ay]^2 = |A'(x)|\,|A'(y)|\,[x, y]^2$ if $A \in G$. There is an important relation between $T_y x$ and $T_x y$ expressed by

$$T_y x = -\Delta(x, y)\, T_x y . \tag{4}$$

We refer to [2, 3, 4, 5] for the elementary proofs of these formulas.

5. Fundamental solutions

A continuous mapping $f : B \to \mathbf{R}^n$ will be called a *deformation*. In this paper we shall assume, mainly for simplicity, that f is continuous on the boundary $S(1)$, and that $x \cdot f(x) = 0$ on $S(1)$; this means that f maps B on itself when regarded as an infinitesimal mapping.

A deformation is *trivial* if $Sf = 0$. There are very few trivial deformations: a complete list is given in [3].

It is customary to say that f is a *quasiconformal* deformation if $\| Sf \| \in L^\infty(B)$; here $\| Sf \|$ is the function whose value at x is the square norm of the matrix $Sf(x)$. More generally, we shall also consider functions with $\| Sf \| \in L^p(B)$; we abbreviate to $Sf \in L^p$, and we denote the L^p-norm of the square norm by $\| Sf \|_p$. The same convention will prevail for all matrix-valued functions.

We shall say that f is *harmonic* if $S^* \rho\, Sf = 0$, $\rho = (1-|x|^2)^{-n}$. Because of the invariance, if f is harmonic and $A \in G$, then $A^* f$ is also harmonic. Harmonicity in this sense is not the same as requiring the components to be harmonic with respect to the Poincaré metric.

There are n linearly independent solutions of the equation $S^* \gamma = 0$ which are homogeneous of degree $1 - n$. We denote them by $\gamma_{..,k}$, $k = 1, ..., n$, the elements being

$$\gamma_{ij,k}(x) = |x|^{-n}(\delta_{ik}x_j + \delta_{jk}x_i - \delta_{ij}x_k) + (n-2)\,|x|^{-n-2}\,x_i x_j x_k .$$

There is a unique vector-valued function $g_{.k}(x)$ with components $g_{ik}(x)$ such that $g_{.k}(x) = 0$ for $|x| = 1$ and $\rho\, Sg_{.k} = \gamma_{..,k}$ so that

$S^* \rho\, S g_{.k} = 0$, or more precisely a Dirac distribution concentrated at 0. It is easy to see that $g = g_{ik}$, which we regard as a Green's matrix, will be of the form $g_{ik}(x) = a(|x|)\,\delta_{ik} + b(|x|)\, x_i x_k$; the explicit expressions for $a(r)$ and $b(r)$ are unimportant, except that g is of order $O((1-|x|^2)^{n+1})$ for $|x| \to 1$ and $O(|x|^{-n+2})$ for $x \to 0$ (if $n = 2$ the latter is replaced by $O(\log 1/|x|)$).

If $U \in O(n)$ it is immediate that $g(Ux) = Ug(x)^t U$. If we replace x by $T_x y$ and U by $-\Delta(x, y)$ it follows with the help of (4) that

$$\Delta(y, x)\, g(T_y x) = g(T_x y)\, \Delta(y, x)\,. \tag{5}$$

We now define the Green's matrix with singularity at y by

Definition 1.

$$\begin{aligned} g_{.k}(x, y) &= (1 - |y|^2)(T_y^* g_{.k})(x) = (1 - |y|^2)\, T_y'(x)^{-1}\, g(T_y x) \\ &= [x, y]^2\, \Delta(y, x)\, g(T_y x)\,. \end{aligned} \tag{6}$$

It is clear that $(S^* \rho\, S)_1\, g(x, y) = 0$ (the subscript indicates that the operator applies to the first variable). In view of (5) we can read off the symmetry property

LEMMA 1. $g(x, y) = {}^t g(y, x)$.

This symmetry plays a prominent role in H. Weyl's classical paper [9] which has been a strong inspiration for this work.

If $A \in G$ it is an easy consequence of (6) that

$$g(Ax, Ay) = A'(x)\, g(x, y)^t A'(y)$$

or, in a more suggestive form,

$$A_1^* A_2^*\, g(x, y) = g(x, y)\,,$$

where A_1^* is A^* applied to the first variable and the first index, and similarly for A_2^*.

Next we define

Definition 2.

$$\gamma_{..,k}(x, y) = \rho(x)\, S_1 g_{.k}(x, y) = (1 - |y|^2)\, \rho(x)\, (S_1 T_y^* g_{.k})(x).$$

It is evident by invariance that $S_1^*\, \gamma_{..,k}(x, y) = 0$. When x and y are transformed by the same $A \in G$ one finds

$$A_1^* A_2^*\, \gamma_{..,.}(x, y)\, dx = \gamma_{..,.}(x, y)\, dx$$

where A_1^* acts on x and the double index, A_2^* on y and the single index. For $A = T_y$ this leads to the explicit formula

$$\gamma_{..,k}(x, y) = \frac{(1 - |y|^2)^{n+1}}{[x, y]^{2n}} \Delta(y, x) \gamma_{..,k}(T_y x) \Delta(x, y) .$$

We note that $\gamma_{..,.}(x, 0) = \gamma_{..,.}(x)$ and $\gamma_{..,.}(0, y) = -(1 - |y|^2)^{n+1} \gamma_{..,.}(y)$.

We shall need to apply S to either variable in $\gamma_{..,.}(x, y)$. For this purpose we introduce

Definition 3. $\Gamma_{ij,hk}(x, y) = [S_2 \gamma_{ij,.}(x, y)]_{hk}$.

Because differentiations with respect to x and y commute it is clear that $S_1^* \Gamma_{..,hk}(x, y) = 0$. Moreover, starting from the relation $g_{ik}(x, y) = g_{ki}(y, x)$ it is not difficult to derive the following symmetry property:

LEMMA 2. $\rho(y) \Gamma_{ij,hk}(x, y) = \rho(x) \Gamma_{hk,ij}(y, x)$.

It follows, in particular, that $S_2^* \rho(y) \Gamma_{ij,..}(x, y) = 0$.

It is also important to know the asymptotic behavior of $\Gamma_{ij,hk}(x, y)$ when $x - y \to 0$. We observe first that

$$\begin{aligned} \rho(y) \Gamma_{ij,hk}(0, y) &= -(1 - |y|^2)^{-n} [S(1 - |y|^2)^{n+1} \gamma_{ij,.}(y)]_{hk} \\ &= -S_{ij,hk}(y) + R_{ij,hk}(y) \end{aligned}$$

where $S_{ij,hk}(y) = [S \gamma_{ij,.}(y)]_{hk}$ is homogeneous of degree $-n$ and $R_{ij,hk}(y)$ is homogeneous of degree $2 - n$. The explicit expression for $\Gamma_{ij,hk}(x, y)$ reads

$$\Gamma_{ij,..}(x, y) = \frac{(1 - |y|^2)^n}{[x, y]^{2n}} \Delta(x, y) \Gamma_{ij,..}(0, T_x y) \Delta(y, x) .$$

Elementary estimates show that

$$(7) \qquad |\Gamma_{ij,hk}(x, y) + S_{ij,hk}(x - y)| \leqq C_n |x - y|^{1-n} [x, y]^{-1}$$

with constant C_n.

6. POTENTIALS

Given an SM_n-valued function v on B we define its *potential* as the vector-valued function $I v$ with components

$$I v(y)_k = \int_B v_{ij}(x) \gamma_{ij,k}(x, y) \, dx .$$