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with

$$\Delta(x, y) = (1 - 2Q(y))(1 - 2Q(x - y^*)) = (1 - 2Q(y - x^*))(1 - 2Q(x)).$$

Observe that $\Delta(x, y) = {}^t\Delta(y, x)$ and $\Delta(x, y)^2 = 1_n$ so that $\Delta(x, y) \in O(n)$. The matrix $\Delta(x, y)$ generalizes the angle $\arg(1 - \bar{x}y)/(1 - \bar{y}x)$.

It is useful to note that $|Ax - Ay|^2 = |A'(x)| |A'(y)| |x - y|^2$ for any Möbius transformation A , and $[Ax, Ay]^2 = |A'(x)| |A'(y)| [x, y]^2$ if $A \in G$. There is an important relation between T_yx and $T_x y$ expressed by

$$(4) \quad T_yx = -\Delta(x, y) T_x y.$$

We refer to [2, 3, 4, 5] for the elementary proofs of these formulas.

5. FUNDAMENTAL SOLUTIONS

A continuous mapping $f: B \rightarrow \mathbf{R}^n$ will be called a *deformation*. In this paper we shall assume, mainly for simplicity, that f is continuous on the boundary $S(1)$, and that $x \cdot f(x) = 0$ on $S(1)$; this means that f maps B on itself when regarded as an infinitesimal mapping.

A deformation is *trivial* if $Sf = 0$. There are very few trivial deformations: a complete list is given in [3].

It is customary to say that f is a *quasiconformal* deformation if $\|Sf\| \in L^\infty(B)$; here $\|Sf\|$ is the function whose value at x is the square norm of the matrix $Sf(x)$. More generally, we shall also consider functions with $\|Sf\| \in L^p(B)$; we abbreviate to $Sf \in L^p$, and we denote the L^p -norm of the square norm by $\|Sf\|_p$. The same convention will prevail for all matrix-valued functions.

We shall say that f is *harmonic* if $S^* \rho Sf = 0$, $\rho = (1 - |x|^2)^{-n}$. Because of the invariance, if f is harmonic and $A \in G$, then $A^* f$ is also harmonic. Harmonicity in this sense is not the same as requiring the components to be harmonic with respect to the Poincaré metric.

There are n linearly independent solutions of the equation $S^* \gamma = 0$ which are homogeneous of degree $1 - n$. We denote them by $\gamma_{\dots, k}$, $k = 1, \dots, n$, the elements being

$$\gamma_{ij, k}(x) = |x|^{-n} (\delta_{ik} x_j + \delta_{jk} x_i - \delta_{ij} x_k) + (n - 2) |x|^{-n-2} x_i x_j x_k.$$

There is a unique vector-valued function $g_{\dots, k}(x)$ with components $g_{ik}(x)$ such that $g_{\dots, k}(x) = 0$ for $|x| = 1$ and $\rho Sg_{\dots, k} = \gamma_{\dots, k}$ so that

$S^* \rho S g_{.k} = 0$, or more precisely a Dirac distribution concentrated at 0. It is easy to see that $g = g_{ik}$, which we regard as a Green's matrix, will be of the form $g_{ik}(x) = a(|x|) \delta_{ik} + b(|x|) x_i x_k$; the explicit expressions for $a(r)$ and $b(r)$ are unimportant, except that g is of order $O((1 - |x|^2)^{n+1})$ for $|x| \rightarrow 1$ and $O(|x|^{-n+2})$ for $x \rightarrow 0$ (if $n = 2$ the latter is replaced by $O(\log 1/|x|)$).

If $U \in O(n)$ it is immediate that $g(Ux) = Ug(x)^t U$. If we replace x by $T_x y$ and U by $-\Delta(x, y)$ it follows with the help of (4) that

$$(5) \quad \Delta(y, x) g(T_y x) = g(T_x y) \Delta(y, x).$$

We now define the Green's matrix with singularity at y by

Definition 1.

$$(6) \quad g_{.k}(x, y) = (1 - |y|^2) (T_y^* g_{.k})(x) = (1 - |y|^2) T_y'(x)^{-1} g(T_y x) \\ = [x, y]^2 \Delta(y, x) g(T_y x).$$

It is clear that $(S^* \rho S)_1 g(x, y) = 0$ (the subscript indicates that the operator applies to the first variable). In view of (5) we can read off the symmetry property

$$\text{LEMMA 1. } g(x, y) = {}^t g(y, x).$$

This symmetry plays a prominent role in H. Weyl's classical paper [9] which has been a strong inspiration for this work.

If $A \in G$ it is an easy consequence of (6) that

$$g(Ax, Ay) = A'(x) g(x, y) {}^t A'(y)$$

or, in a more suggestive form,

$$A_1^* A_2^* g(x, y) = g(x, y),$$

where A_1^* is A^* applied to the first variable and the first index, and similarly for A_2^* .

Next we define

Definition 2.

$$\gamma_{\dots, k}(x, y) = \rho(x) S_1 g_{.k}(x, y) = (1 - |y|^2) \rho(x) (S_1 T_y^* g_{.k})(x).$$

It is evident by invariance that $S_1^* \gamma_{\dots, k}(x, y) = 0$. When x and y are transformed by the same $A \in G$ one finds

$$A_1^* A_2^* \gamma_{\dots, k}(x, y) dx = \gamma_{\dots, k}(x, y) dx$$

where A_1^* acts on x and the double index, A_2^* on y and the single index. For $A = T_y$ this leads to the explicit formula

$$\gamma_{\dots,k}(x, y) = \frac{(1 - |y|^2)^{n+1}}{[x, y]^{2n}} \Delta(y, x) \gamma_{\dots,k}(T_y x) \Delta(x, y).$$

We note that $\gamma_{\dots}(x, 0) = \gamma_{\dots}(x)$ and $\gamma_{\dots}(0, y) = -(1 - |y|^2)^{n+1} \gamma_{\dots}(y)$.

We shall need to apply S to either variable in $\gamma_{\dots}(x, y)$. For this purpose we introduce

Definition 3. $\Gamma_{ij,hk}(x, y) = [S_2 \gamma_{ij,\dots}(x, y)]_{hk}$.

Because differentiations with respect to x and y commute it is clear that $S_1^* \Gamma_{\dots,hk}(x, y) = 0$. Moreover, starting from the relation $g_{ik}(x, y) = g_{ki}(y, x)$ it is not difficult to derive the following symmetry property:

LEMMA 2. $\rho(y) \Gamma_{ij,hk}(x, y) = \rho(x) \Gamma_{hk,ij}(y, x)$.

It follows, in particular, that $S_2^* \rho(y) \Gamma_{ij,\dots}(x, y) = 0$.

It is also important to know the asymptotic behavior of $\Gamma_{ij,hk}(x, y)$ when $x - y \rightarrow 0$. We observe first that

$$\begin{aligned} \rho(y) \Gamma_{ij,hk}(0, y) &= -(1 - |y|^2)^{-n} [S(1 - |y|^2)^{n+1} \gamma_{ij,\dots}(y)]_{hk} \\ &= -S_{ij,hk}(y) + R_{ij,hk}(y) \end{aligned}$$

where $S_{ij,hk}(y) = [S \gamma_{ij,\dots}(y)]_{hk}$ is homogeneous of degree $-n$ and $R_{ij,hk}(y)$ is homogeneous of degree $2 - n$. The explicit expression for $\Gamma_{ij,hk}(x, y)$ reads

$$\Gamma_{ij,\dots}(x, y) = \frac{(1 - |y|^2)^n}{[x, y]^{2n}} \Delta(x, y) \Gamma_{ij,\dots}(0, T_x y) \Delta(y, x).$$

Elementary estimates show that

$$(7) \quad |\Gamma_{ij,hk}(x, y) + S_{ij,hk}(x - y)| \leq C_n |x - y|^{1-n} [x, y]^{-1}$$

with constant C_n .

6. POTENTIALS

Given an SM_n -valued function v on B we define its *potential* as the vector-valued function Iv with components

$$Iv(y)_k = \int_B v_{ij}(x) \gamma_{ij,k}(x, y) dx.$$