3. Invariance properties

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is the normalized and symmetrized Jacobian; it carries the quasiconformal data of the mapping.

The Riemannian metric $ds^2 = {}^t dx\,(MF)\,dx$ is conformally flat, a condition expressed by the vanishing of the conformal curvature tensor. For $n = 3$ this tensor is identically zero, but there is instead an integrability condition.

Let $F(x, t)$ be a one-parameter family of homeomorphisms such that $F(x, 0) = x$, $\dot{F}(x, 0) = f(x)$. Under suitable regularity conditions $(DF)_0^{\cdot} = Df$, $(XF)_0^{\cdot} = Df - \frac{1}{n}\, tr\, Df \cdot 1_n$, and $(MF)_0^{\cdot} = Df + {}^t Df - \frac{2}{n}\, tr\, Df \cdot 1_n$. This motivates introducing the differential operator S defined by

$$(Sf)_{ij} = \frac{1}{2}(D_i f_j + D_j f_i) - \frac{1}{n}\,\delta_{ij}\,D_k f_k\,.$$

(The summation convention is in force in this paper). Note that Sf has values in SM_n.

There is a formal adjoint S^* which maps SM_n-valued functions on $\mathbf{R}^n$-valued functions. It is defined by

$$(S^*\varphi)_i = D_j \varphi_{ij}\,,$$

and it satisfies

$$(1) \qquad \int_\Omega Sf \cdot \varphi dx = - \int_\Omega f \cdot S^*\varphi\, dx$$

when either f or φ has compact support. ($Sf.\ \varphi$ and $f \cdot S^*\varphi$ are the dot products $Sf_{ij}\,\varphi_{ij}$ and $f_i\,(S^*\varphi)_i$, respectively; dx is the euclidean volume element.)

Equation (1) defines Sf and $S^*\varphi$ as *distributions* even if f and φ are not differentiable. We are always assuming that f is continuous and φ locally integrable.

3. Invariance properties

In (1) we prefer to regard $\varphi\, dx$ as a matrix-valued measure, so that the pairing

$$< Sf, \varphi dx > = \int_\Omega Sf \cdot \varphi\, dx$$

is between a function and a measure. Similarly, $S^*(\varphi dx) = (S^*\varphi)\, dx$ is a vector-valued measure.

Let A be a Möbius transformation. We define the *pull-backs* of vector- and SM_n-valued functions by

$$(A^*f)(x) = (DA)^{-1}f(Ax)$$
$$(A^*\varphi)(x) = (DA)^{-1}\varphi(Ax)\,DA$$

and for the corresponding measures by

$$A^*(f\,dx) = |\det A|\,{}^tDA\,f(Ax)\,dx$$
$$A^*(\varphi\,dx) = |\det A|\,(DA)^{-1}\varphi(Ax)\,DA\,.$$

These definitions are chosen so that the pairings are invariant:

$$< A^*f, A^*gdx > = < f, gdx >$$
$$< A^*v, A^*\varphi\,dx > = < v, \varphi\,dx > .$$

There is a basic identity

$$S(A^*f)(x) = (DA)^{-1}\,Sf(Ax)\,DA \tag{2}$$

which may be expressed as a commutativity relation $SA^* = A^*S$, applicable to functions, but not to measures. It implies the relation $S^*A^* = A^*S^*$, which is valid for measures in the sense that

$$S^*(A^*\varphi\,dx) = A^*(S^*\varphi\,dx)\,, \tag{3}$$

but not for functions. It should be noted that (2) and (3) are true only because A is conformal.

A function is transformed into a measure by multiplication with a fixed invariant measure $\rho\,dx$. The invariance means that $A^*(\rho\,dx) = \rho\,dx$, or $\rho(Ax)\,|\det DA| = \rho(x)$; we assume also that A leaves Ω invariant. In these circumstances it makes sense to consider the operator $S^*\rho S$ which takes f to $S^*[\rho(Sf)dx]$ and commutes with $A^*:(S^*\rho S)A^* = A^*(S^*\rho S)$.

There are three classical cases in which Ω is invariant under a transitive group $G(\Omega)$ of Möbius transformations:

(i) $\Omega = \mathbf{R}^n$. $G(\Omega)$ is the group of euclidean motions, and $\rho = 1$.

(ii) $\Omega = \mathrm{B}(1) = \{x : |x| < 1\}$. $G = G(B)$ is the group of non-euclidean motions, and $\rho = (1-|x|^2)^{-n}$.

(iii) Ω is the one-point compactification of $\mathbf{R}^n$, identified with S^n in $\mathbf{R}^{n+1}$. The group is formed by the rotations of the sphere, and $\rho = (1+|x|^2)^{-n}$.