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# A SINGULAR INTEGRAL EQUATION CONNECTED WITH QUASICONFORMAL MAPPINGS IN SPACE

by Lars V. AHLFORS <sup>1</sup>)

Dedicated to Albert Pfluger for his seventieth birthday

# 1. INTRODUCTION

This paper continues the author's investigation of two differential operators, S and  $S^*$ , which arise naturally in the study of infinitesimal quasiconformal mappings in n dimensions (see References). If  $\Omega$  is open in  $\mathbb{R}^n$  the operator S acts on functions  $f: \Omega \to \mathbb{R}^n$  and has values  $Sf \in SM_n$  where  $SM_n$  is the space of symmetric  $n \times n$  matrices with zero trace. Definitions are in Sec. 2.

A key question is the solvability of the inhomogeneous equation Sf = v. For n = 2, Sf can be identified with the complex derivative  $f_{\bar{z}}$  of a complex-valued function, and the problem is that of recovering f from  $f_{\bar{z}}$ . As well known, this problem has always a solution, and it is given by the generalized Cauchy formula, also known as Pompeiu's formula. For n > 2 the right hand member v, an  $SM_n$ -valued function, must satisfy certain conditions, which are known in principle, as limiting cases of the Weyl-Schouten conditions of vanishing conformal curvature.

These conditions, although explicit, are quite intractable. It is therefore rather surprising that a necessary and sufficient condition for Sf = v to be solvable can be expressed as a singular homogeneous integral equation satisfied by v. This integral equation can be treated by the methods of Calderon and Zygmund.

# 2. Definitions and notations

A quasiconformal homeomorphism  $F: \Omega \to F(\Omega)$  is known to be differentiable almost everywhere. We denote its Jacobian matrix by *DF*. The normalized Jacobian is  $XF = (\det DF)^{-1/n} DF$ , and  $MF = {}^{t}XF \cdot XF$ 

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is the normalized and symmetrized Jacobian; it carries the quasiconformal data of the mapping.

The Riemannian metric  $ds^2 = {}^t dx (MF) dx$  is conformally flat, a condition expressed by the vanishing of the conformal curvature tensor. For n = 3 this tensor is identically zero, but there is instead an integrability condition.

Let F(x, t) be a one-parameter family of homeomorphisms such that F(x, 0) = x, F(x, 0) = f(x). Under suitable regularity conditions  $(DF)_0^{\cdot}$  = Df,  $(XF)_0^{\cdot} = Df - \frac{1}{n} tr Df \cdot 1_n$ , and  $(MF)_0^{\cdot} = Df + {}^tDf - \frac{2}{n} tr Df \cdot 1_n$ . This motivates introducing the differential operator S defined by

$$(Sf)_{ij} = \frac{1}{2} (D_i f_j + D_j f_i) - \frac{1}{n} \delta_{ij} D_k f_k .$$

(The summation convention is in force in this paper). Note that Sf has values in  $SM_n$ .

There is a formal adjoint  $S^*$  which maps  $SM_n$ -valued functions on  $\mathbb{R}^n$ -valued functions. It is defined by

$$(S^*\varphi)_i = D_j\varphi_{ij},$$

and it satisfies

(1)  $\int_{\Omega} Sf \cdot \varphi dx = - \int_{\Omega} f \cdot S^* \varphi dx$ 

when either f or  $\varphi$  has compact support. (Sf.  $\varphi$  and  $f \cdot S^* \varphi$  are the dot products  $Sf_{ij} \varphi_{ij}$  and  $f_i (S^* \varphi)_i$ , respectively; dx is the euclidean volume element.)

Equation (1) defines Sf and  $S^* \varphi$  as distributions even if f and  $\varphi$  are not differentiable. We are always assuming that f is continuous and  $\varphi$  locally integrable.

## 3. INVARIANCE PROPERTIES

In (1) we prefer to regard  $\varphi dx$  as a matrix-valued measure, so that the pairing

$$\langle Sf, \varphi dx \rangle = \int_{\Omega} Sf \cdot \varphi dx$$

is between a function and a measure. Similarly,  $S^*(\varphi dx) = (S^*\varphi) dx$  is a vector-valued measure.