Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 24 (1978)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SINGULAR INTEGRAL EQUATION CONNECTED WITH QUASICONFORMAL MAPPINGS IN SPACE

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Kapitel: 2. Definitions and notations

DOI: https://doi.org/10.5169/seals-49703

A SINGULAR INTEGRAL EQUATION CONNECTED WITH QUASICONFORMAL MAPPINGS IN SPACE

by Lars V. AHLFORS [1])

Dedicated to Albert Pfluger for his seventieth birthday

1. INTRODUCTION

This paper continues the author's investigation of two differential operators, S and S^*, which arise naturally in the study of infinitesimal quasiconformal mappings in n dimensions (see References). If Ω is open in $\mathbf{R}^n$ the operator S acts on functions $f : \Omega \to \mathbf{R}^n$ and has values $Sf \in SM_n$ where SM_n is the space of symmetric $n \times n$ matrices with zero trace. Definitions are in Sec. 2.

A key question is the solvability of the inhomogeneous equation $Sf = v$. For $n = 2$, Sf can be identified with the complex derivative $f_{\bar{z}}$ of a complex-valued function, and the problem is that of recovering f from $f_{\bar{z}}$. As well known, this problem has always a solution, and it is given by the generalized Cauchy formula, also known as Pompeiu's formula. For $n > 2$ the right hand member v, an SM_n-valued function, must satisfy certain conditions, which are known in principle, as limiting cases of the Weyl-Schouten conditions of vanishing conformal curvature.

These conditions, although explicit, are quite intractable. It is therefore rather surprising that a necessary and sufficient condition for $Sf = v$ to be solvable can be expressed as a singular homogeneous integral equation satisfied by v. This integral equation can be treated by the methods of Calderon and Zygmund.

2. DEFINITIONS AND NOTATIONS

A quasiconformal homeomorphism $F : \Omega \to F(\Omega)$ is known to be differentiable almost everywhere. We denote its Jacobian matrix by DF. The normalized Jacobian is $XF = (\det DF)^{-1/n} DF$, and $MF = {}^tXF \cdot XF$

[1]) Supported by NSF Grant GP-38886.

is the normalized and symmetrized Jacobian; it carries the quasiconformal data of the mapping.

The Riemannian metric $ds^2 = {}^t dx\,(MF)\,dx$ is conformally flat, a condition expressed by the vanishing of the conformal curvature tensor. For $n = 3$ this tensor is identically zero, but there is instead an integrability condition.

Let $F(x, t)$ be a one-parameter family of homeomorphisms such that $F(x, 0) = x$, $\dot{F}(x, 0) = f(x)$. Under suitable regularity conditions $(DF)^{\cdot}_0 = Df$, $(XF)^{\cdot}_0 = Df - \frac{1}{n}\, tr\, Df \cdot 1_n$, and $(MF)^{\cdot}_0 = Df + {}^t Df - \frac{2}{n}\, tr\, Df \cdot 1_n$. This motivates introducing the differential operator S defined by

$$(Sf)_{ij} = \frac{1}{2}(D_i f_j + D_j f_i) - \frac{1}{n}\,\delta_{ij}\, D_k f_k\ .$$

(The summation convention is in force in this paper). Note that Sf has values in SM_n.

There is a formal adjoint S^* which maps SM_n-valued functions on $\mathbf{R}^n$-valued functions. It is defined by

$$(S^*\varphi)_i = D_j \varphi_{ij}\ ,$$

and it satisfies

$$\int_\Omega Sf \cdot \varphi dx = - \int_\Omega f \cdot S^*\varphi\, dx \tag{1}$$

when either f or φ has compact support. ($Sf.\ \varphi$ and $f \cdot S^*\ \varphi$ are the dot products $Sf_{ij}\ \varphi_{ij}$ and $f_i\,(S^*\ \varphi)_i$, respectively; dx is the euclidean volume element.)

Equation (1) defines Sf and $S^*\ \varphi$ as *distributions* even if f and φ are not differentiable. We are always assuming that f is continuous and φ locally integrable.

3. Invariance properties

In (1) we prefer to regard $\varphi\, dx$ as a matrix-valued measure, so that the pairing

$$< Sf, \varphi dx > = \int_\Omega Sf \cdot \varphi\, dx$$

is between a function and a measure. Similarly, $S^*\,(\varphi dx) = (S^*\varphi)\, dx$ is a vector-valued measure.