

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 24 (1978)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SINGULAR INTEGRAL EQUATION CONNECTED WITH QUASICONFORMAL MAPPINGS IN SPACE

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Kapitel: 2. Definitions and notations

DOI: <https://doi.org/10.5169/seals-49703>

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A SINGULAR INTEGRAL EQUATION CONNECTED WITH QUASICONFORMAL MAPPINGS IN SPACE

by Lars V. AHLFORS ¹⁾

Dedicated to Albert Pfluger for his seventieth birthday

1. INTRODUCTION

This paper continues the author's investigation of two differential operators, S and S^* , which arise naturally in the study of infinitesimal quasiconformal mappings in n dimensions (see References). If Ω is open in \mathbf{R}^n the operator S acts on functions $f: \Omega \rightarrow \mathbf{R}^n$ and has values $Sf \in SM_n$ where SM_n is the space of symmetric $n \times n$ matrices with zero trace. Definitions are in Sec. 2.

A key question is the solvability of the inhomogeneous equation $Sf = v$. For $n = 2$, Sf can be identified with the complex derivative $f_{\bar{z}}$ of a complex-valued function, and the problem is that of recovering f from $f_{\bar{z}}$. As well known, this problem has always a solution, and it is given by the generalized Cauchy formula, also known as Pompeiu's formula. For $n > 2$ the right hand member v , an SM_n -valued function, must satisfy certain conditions, which are known in principle, as limiting cases of the Weyl-Schouten conditions of vanishing conformal curvature.

These conditions, although explicit, are quite intractable. It is therefore rather surprising that a necessary and sufficient condition for $Sf = v$ to be solvable can be expressed as a singular homogeneous integral equation satisfied by v . This integral equation can be treated by the methods of Calderon and Zygmund.

2. DEFINITIONS AND NOTATIONS

A quasiconformal homeomorphism $F: \Omega \rightarrow F(\Omega)$ is known to be differentiable almost everywhere. We denote its Jacobian matrix by DF . The normalized Jacobian is $XF = (\det DF)^{-1/n} DF$, and $MF = {}^t XF \cdot XF$

¹⁾ Supported by NSF Grant GP-38886.

is the normalized and symmetrized Jacobian; it carries the quasiconformal data of the mapping.

The Riemannian metric $ds^2 = {}^t dx (MF) dx$ is conformally flat, a condition expressed by the vanishing of the conformal curvature tensor. For $n = 3$ this tensor is identically zero, but there is instead an integrability condition.

Let $F(x, t)$ be a one-parameter family of homeomorphisms such that $F(x, 0) = x$, $\dot{F}(x, 0) = f(x)$. Under suitable regularity conditions $(DF)_0 = Df$, $(XF)_0 = Df - \frac{1}{n} \text{tr } Df \cdot 1_n$, and $(MF)_0 = Df + {}^t Df - \frac{2}{n} \text{tr } Df \cdot 1_n$. This motivates introducing the differential operator S defined by

$$(Sf)_{ij} = \frac{1}{2} (D_i f_j + D_j f_i) - \frac{1}{n} \delta_{ij} D_k f_k.$$

(The summation convention is in force in this paper). Note that Sf has values in SM_n .

There is a formal adjoint S^* which maps SM_n -valued functions on \mathbb{R}^n -valued functions. It is defined by

$$(S^* \varphi)_i = D_j \varphi_{ij},$$

and it satisfies

$$(1) \quad \int_{\Omega} Sf \cdot \varphi dx = - \int_{\Omega} f \cdot S^* \varphi dx$$

when either f or φ has compact support. ($Sf \cdot \varphi$ and $f \cdot S^* \varphi$ are the dot products $Sf_{ij} \varphi_{ij}$ and $f_i (S^* \varphi)_i$, respectively; dx is the euclidean volume element.)

Equation (1) defines Sf and $S^* \varphi$ as *distributions* even if f and φ are not differentiable. We are always assuming that f is continuous and φ locally integrable.

3. INVARIANCE PROPERTIES

In (1) we prefer to regard φdx as a matrix-valued measure, so that the pairing

$$\langle Sf, \varphi dx \rangle = \int_{\Omega} Sf \cdot \varphi dx$$

is between a function and a measure. Similarly, $S^*(\varphi dx) = (S^* \varphi) dx$ is a vector-valued measure.