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A SINGULAR INTEGRAL EQUATION CONNECTED WITH QUASICONFORMAL MAPPINGS IN SPACE

by Lars V. AHLFORS ¹⁾

Dedicated to Albert Pfluger for his seventieth birthday

1. INTRODUCTION

This paper continues the author's investigation of two differential operators, S and S^* , which arise naturally in the study of infinitesimal quasiconformal mappings in n dimensions (see References). If Ω is open in \mathbf{R}^n the operator S acts on functions $f : \Omega \rightarrow \mathbf{R}^n$ and has values $Sf \in SM_n$, where SM_n is the space of symmetric $n \times n$ matrices with zero trace. Definitions are in Sec. 2.

A key question is the solvability of the inhomogeneous equation $Sf = v$. For $n = 2$, Sf can be identified with the complex derivative $f_{\bar{z}}$ of a complex-valued function, and the problem is that of recovering f from $f_{\bar{z}}$. As well known, this problem has always a solution, and it is given by the generalized Cauchy formula, also known as Pompeiu's formula. For $n > 2$ the right hand member v , an SM_n -valued function, must satisfy certain conditions, which are known in principle, as limiting cases of the Weyl-Schouten conditions of vanishing conformal curvature.

These conditions, although explicit, are quite intractable. It is therefore rather surprising that a necessary and sufficient condition for $Sf = v$ to be solvable can be expressed as a singular homogeneous integral equation satisfied by v . This integral equation can be treated by the methods of Calderon and Zygmund.

2. DEFINITIONS AND NOTATIONS

A quasiconformal homeomorphism $F : \Omega \rightarrow F(\Omega)$ is known to be differentiable almost everywhere. We denote its Jacobian matrix by DF . The normalized Jacobian is $XF = (\det DF)^{-1/n} DF$, and $MF = {}^tXF \cdot XF$

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is the normalized and symmetrized Jacobian; it carries the quasiconformal data of the mapping.

The Riemannian metric $ds^2 = {}^t dx (MF) dx$ is conformally flat, a condition expressed by the vanishing of the conformal curvature tensor. For $n = 3$ this tensor is identically zero, but there is instead an integrability condition.

Let $F(x, t)$ be a one-parameter family of homeomorphisms such that $F(x, 0) = x$, $\dot{F}(x, 0) = f(x)$. Under suitable regularity conditions $(DF)_0 = Df$, $(XF)_0 = Df - \frac{1}{n} \text{tr} Df \cdot 1_n$, and $(MF)_0 = Df + {}^t Df - \frac{2}{n} \text{tr} Df \cdot 1_n$. This motivates introducing the differential operator S defined by

$$(Sf)_{ij} = \frac{1}{2} (D_i f_j + D_j f_i) - \frac{1}{n} \delta_{ij} D_k f_k .$$

(The summation convention is in force in this paper). Note that Sf has values in SM_n .

There is a formal adjoint S^* which maps SM_n -valued functions on \mathbf{R}^n -valued functions. It is defined by

$$(S^* \varphi)_i = D_j \varphi_{ij} ,$$

and it satisfies

$$(1) \quad \int_{\Omega} Sf \cdot \varphi dx = - \int_{\Omega} f \cdot S^* \varphi dx$$

when either f or φ has compact support. ($Sf \cdot \varphi$ and $f \cdot S^* \varphi$ are the dot products $Sf_{ij} \varphi_{ij}$ and $f_i (S^* \varphi)_i$, respectively; dx is the euclidean volume element.)

Equation (1) defines Sf and $S^* \varphi$ as *distributions* even if f and φ are not differentiable. We are always assuming that f is continuous and φ locally integrable.

3. INVARIANCE PROPERTIES

In (1) we prefer to regard φdx as a matrix-valued measure, so that the pairing

$$\langle Sf, \varphi dx \rangle = \int_{\Omega} Sf \cdot \varphi dx$$

is between a function and a measure. Similarly, $S^*(\varphi dx) = (S^* \varphi) dx$ is a vector-valued measure.

Let A be a Möbius transformation. We define the *pull-backs* of vector- and SM_n -valued functions by

$$\begin{aligned}(A^* f)(x) &= (DA)^{-1} f(Ax) \\ (A^* \varphi)(x) &= (DA)^{-1} \varphi(Ax) DA\end{aligned}$$

and for the corresponding measures by

$$\begin{aligned}A^*(f dx) &= |\det A| {}^t D A f(Ax) dx \\ A^*(\varphi dx) &= |\det A| (DA)^{-1} \varphi(Ax) DA.\end{aligned}$$

These definitions are chosen so that the pairings are invariant:

$$\begin{aligned}\langle A^* f, A^* g dx \rangle &= \langle f, g dx \rangle \\ \langle A^* v, A^* \varphi dx \rangle &= \langle v, \varphi dx \rangle.\end{aligned}$$

There is a basic identity

$$(2) \quad S(A^* f)(x) = (DA)^{-1} S f(Ax) DA$$

which may be expressed as a commutativity relation $SA^* = A^* S$, applicable to functions, but not to measures. It implies the relation $S^* A^* = A^* S^*$, which is valid for measures in the sense that

$$(3) \quad S^*(A^* \varphi dx) = A^*(S^* \varphi dx),$$

but not for functions. It should be noted that (2) and (3) are true only because A is conformal.

A function is transformed into a measure by multiplication with a fixed invariant measure ρdx . The invariance means that $A^*(\rho dx) = \rho dx$, or $\rho(Ax) |\det DA| = \rho(x)$; we assume also that A leaves Ω invariant. In these circumstances it makes sense to consider the operator $S^* \rho S$ which takes f to $S^* [\rho(Sf)dx]$ and commutes with $A^* : (S^* \rho S) A^* = A^* (S^* \rho S)$.

There are three classical cases in which Ω is invariant under a transitive group $G(\Omega)$ of Möbius transformations:

- (i) $\Omega = \mathbf{R}^n$. $G(\Omega)$ is the group of euclidean motions, and $\rho = 1$.
- (ii) $\Omega = B(1) = \{x : |x| < 1\}$. $G = G(B)$ is the group of non-euclidean motions, and $\rho = (1 - |x|^2)^{-n}$.
- (iii) Ω is the one-point compactification of \mathbf{R}^n , identified with S^n in \mathbf{R}^{n+1} . The group is formed by the rotations of the sphere, and $\rho = (1 + |x|^2)^{-n}$.

4. NON-EUCLIDEAN MOTIONS

The euclidean case was dealt with in [3]. In the present paper we undertake a more detailed study of the hyperbolic case. The unit ball in \mathbf{R}^n is denoted by B , and G is the full group of Möbius transformations mapping B on itself. The Poincaré metric $ds = (1 - |x|^2)^{-1} |dx|$ and the non-euclidean volume element $\rho dx = (1 - |x|^2)^{-n} dx$ are invariant under G .

For $A \in G$ we prefer to denote the Jacobian by $A'(x)$ rather than $DA(x)$. We use $|A'(x)|$ for the linear rate of change, the same in all directions. This notation has the advantage of leading to formulas which are easily recognizable generalizations of the familiar formulas for $n = 2$ in complex notation. $|A'(x)|$ is also the square norm of the matrix $A'(x)$, and $|\det A'(x)| = |A'(x)|^n$.

Reflection in the unit sphere is denoted by $x^* = x/|x|^2$. Its Jacobian is $Dx^* = |x|^{-2} (1_n - 2Q(x))$ with $Q(x)_{ij} = x_i x_j / |x|^2$; note that $(1_n - 2Q(x))^2 = 1_n$.

For every $y \in B$ there is a unique $T_y \in G$ such that $T_y y = 0$ and $T'_y(y) = |T'_y(y)| \cdot 1_n$. The most general $A \in G$ is of the form $A = UT_y$ with $y = A^{-1}(0)$ and $U \in O(n)$.

For $n = 2$, in complex notation,

$$T_y x = \frac{x - y}{1 - \bar{y}x}$$

$$T'_y(x) = \frac{1 - |y|^2}{(1 - \bar{y}x)^2}.$$

The first formula can be rewritten as

$$T_y x = \frac{(x - y)(1 - |y|^2) - |x - y|^2 y}{|y|^2 |x - y^*|^2}.$$

In this form it makes sense for arbitrary n and is in fact the correct formula. The denominator $|y|^2 |x - y^*|^2$ corresponds to $|1 - \bar{y}x|^2$, and it is equal to $1 - 2(xy) + |x|^2 |y|^2$, where (xy) is the inner product. To emphasize the symmetry we shall use the notation $|y| |x - y^*| = |x| |y - x^*| = [x, y]$.

The expression for $T'_y(x)$ is

$$T'_y(x) = \frac{1 - |y|^2}{[x, y]^2} \Delta(x, y)$$

with

$$\Delta(x, y) = (1 - 2Q(y))(1 - 2Q(x - y^*)) = (1 - 2Q(y - x^*))(1 - 2Q(x)).$$

Observe that $\Delta(x, y) = {}^t\Delta(y, x)$ and $\Delta(x, y)^2 = 1_n$ so that $\Delta(x, y) \in O(n)$. The matrix $\Delta(x, y)$ generalizes the angle $\arg(1 - \bar{x}y)/(1 - \bar{y}x)$.

It is useful to note that $|Ax - Ay|^2 = |A'(x)| |A'(y)| |x - y|^2$ for any Möbius transformation A , and $[Ax, Ay]^2 = |A'(x)| |A'(y)| [x, y]^2$ if $A \in G$. There is an important relation between T_yx and $T_x y$ expressed by

$$(4) \quad T_yx = -\Delta(x, y) T_x y.$$

We refer to [2, 3, 4, 5] for the elementary proofs of these formulas.

5. FUNDAMENTAL SOLUTIONS

A continuous mapping $f: B \rightarrow \mathbf{R}^n$ will be called a *deformation*. In this paper we shall assume, mainly for simplicity, that f is continuous on the boundary $S(1)$, and that $x \cdot f(x) = 0$ on $S(1)$; this means that f maps B on itself when regarded as an infinitesimal mapping.

A deformation is *trivial* if $Sf = 0$. There are very few trivial deformations: a complete list is given in [3].

It is customary to say that f is a *quasiconformal* deformation if $\|Sf\| \in L^\infty(B)$; here $\|Sf\|$ is the function whose value at x is the square norm of the matrix $Sf(x)$. More generally, we shall also consider functions with $\|Sf\| \in L^p(B)$; we abbreviate to $Sf \in L^p$, and we denote the L^p -norm of the square norm by $\|Sf\|_p$. The same convention will prevail for all matrix-valued functions.

We shall say that f is *harmonic* if $S^* \rho Sf = 0$, $\rho = (1 - |x|^2)^{-n}$. Because of the invariance, if f is harmonic and $A \in G$, then $A^* f$ is also harmonic. Harmonicity in this sense is not the same as requiring the components to be harmonic with respect to the Poincaré metric.

There are n linearly independent solutions of the equation $S^* \gamma = 0$ which are homogeneous of degree $1 - n$. We denote them by $\gamma_{\dots, k}$, $k = 1, \dots, n$, the elements being

$$\gamma_{ij, k}(x) = |x|^{-n} (\delta_{ik} x_j + \delta_{jk} x_i - \delta_{ij} x_k) + (n - 2) |x|^{-n-2} x_i x_j x_k.$$

There is a unique vector-valued function $g_{\dots, k}(x)$ with components $g_{ik}(x)$ such that $g_{\dots, k}(x) = 0$ for $|x| = 1$ and $\rho Sg_{\dots, k} = \gamma_{\dots, k}$ so that

$S^* \rho S g_{.k} = 0$, or more precisely a Dirac distribution concentrated at 0. It is easy to see that $g = g_{ik}$, which we regard as a Green's matrix, will be of the form $g_{ik}(x) = a(|x|) \delta_{ik} + b(|x|) x_i x_k$; the explicit expressions for $a(r)$ and $b(r)$ are unimportant, except that g is of order $O((1 - |x|^2)^{n+1})$ for $|x| \rightarrow 1$ and $O(|x|^{-n+2})$ for $x \rightarrow 0$ (if $n = 2$ the latter is replaced by $O(\log 1/|x|)$).

If $U \in O(n)$ it is immediate that $g(Ux) = Ug(x)^t U$. If we replace x by $T_x y$ and U by $-\Delta(x, y)$ it follows with the help of (4) that

$$(5) \quad \Delta(y, x) g(T_y x) = g(T_x y) \Delta(y, x).$$

We now define the Green's matrix with singularity at y by

Definition 1.

$$(6) \quad g_{.k}(x, y) = (1 - |y|^2) (T_y^* g_{.k})(x) = (1 - |y|^2) T_y'(x)^{-1} g(T_y x) \\ = [x, y]^2 \Delta(y, x) g(T_y x).$$

It is clear that $(S^* \rho S)_1 g(x, y) = 0$ (the subscript indicates that the operator applies to the first variable). In view of (5) we can read off the symmetry property

$$\text{LEMMA 1. } g(x, y) = {}^t g(y, x).$$

This symmetry plays a prominent role in H. Weyl's classical paper [9] which has been a strong inspiration for this work.

If $A \in G$ it is an easy consequence of (6) that

$$g(Ax, Ay) = A'(x) g(x, y) {}^t A'(y)$$

or, in a more suggestive form,

$$A_1^* A_2^* g(x, y) = g(x, y),$$

where A_1^* is A^* applied to the first variable and the first index, and similarly for A_2^* .

Next we define

Definition 2.

$$\gamma_{\dots, k}(x, y) = \rho(x) S_1 g_{.k}(x, y) = (1 - |y|^2) \rho(x) (S_1 T_y^* g_{.k})(x).$$

It is evident by invariance that $S_1^* \gamma_{\dots, k}(x, y) = 0$. When x and y are transformed by the same $A \in G$ one finds

$$A_1^* A_2^* \gamma_{\dots, k}(x, y) dx = \gamma_{\dots, k}(x, y) dx$$

where A_1^* acts on x and the double index, A_2^* on y and the single index. For $A = T_y$ this leads to the explicit formula

$$\gamma_{\dots,k}(x, y) = \frac{(1 - |y|^2)^{n+1}}{[x, y]^{2n}} \Delta(y, x) \gamma_{\dots,k}(T_y x) \Delta(x, y).$$

We note that $\gamma_{\dots}(x, 0) = \gamma_{\dots}(x)$ and $\gamma_{\dots}(0, y) = -(1 - |y|^2)^{n+1} \gamma_{\dots}(y)$.

We shall need to apply S to either variable in $\gamma_{\dots}(x, y)$. For this purpose we introduce

Definition 3. $\Gamma_{ij,hk}(x, y) = [S_2 \gamma_{ij,\dots}(x, y)]_{hk}$.

Because differentiations with respect to x and y commute it is clear that $S_1^* \Gamma_{\dots,hk}(x, y) = 0$. Moreover, starting from the relation $g_{ik}(x, y) = g_{ki}(y, x)$ it is not difficult to derive the following symmetry property:

LEMMA 2. $\rho(y) \Gamma_{ij,hk}(x, y) = \rho(x) \Gamma_{hk,ij}(y, x)$.

It follows, in particular, that $S_2^* \rho(y) \Gamma_{ij,\dots}(x, y) = 0$.

It is also important to know the asymptotic behavior of $\Gamma_{ij,hk}(x, y)$ when $x - y \rightarrow 0$. We observe first that

$$\begin{aligned} \rho(y) \Gamma_{ij,hk}(0, y) &= -(1 - |y|^2)^{-n} [S(1 - |y|^2)^{n+1} \gamma_{ij,\dots}(y)]_{hk} \\ &= -S_{ij,hk}(y) + R_{ij,hk}(y) \end{aligned}$$

where $S_{ij,hk}(y) = [S \gamma_{ij,\dots}(y)]_{hk}$ is homogeneous of degree $-n$ and $R_{ij,hk}(y)$ is homogeneous of degree $2 - n$. The explicit expression for $\Gamma_{ij,hk}(x, y)$ reads

$$\Gamma_{ij,\dots}(x, y) = \frac{(1 - |y|^2)^n}{[x, y]^{2n}} \Delta(x, y) \Gamma_{ij,\dots}(0, T_x y) \Delta(y, x).$$

Elementary estimates show that

$$(7) \quad |\Gamma_{ij,hk}(x, y) + S_{ij,hk}(x - y)| \leq C_n |x - y|^{1-n} [x, y]^{-1}$$

with constant C_n .

6. POTENTIALS

Given an SM_n -valued function v on B we define its *potential* as the vector-valued function Iv with components

$$Iv(y)_k = \int_B v_{ij}(x) \gamma_{ij,k}(x, y) dx.$$

The integral converges if $v \in L^p(B)$ for some p with $n < p \leq \infty$. In fact, one proves that

$$|Iv(y)| \leq C_{n,p} \|v\|_p (1 - |y|)^{1-n/p}$$

if $p < \infty$ and

$$|Iv(y)| \leq C_n \|v\|_\infty (1 - |y|)(1 + \log 1/(1 - |y|))$$

if $p = \infty$. In any event $Iv(y)$ vanishes at a fixed rate for $|y| \rightarrow 1$.

The forming of the potential is an invariant operation in the sense that $IA^*v = A^*Iv$ for every $A \in G$. The potential is harmonic outside the support of v , for $(S^* \rho S)_2 \gamma_{ij..}(x, y) = 0$.

The following theorem serves to recover f from Sf and its boundary values:

THEOREM 1. *If $Sf \in L^p(B)$, $p > n$, then*

$$(8) \quad c_n f(y) = -ISf(y) + c_n Hf(y)$$

with

$$Hf(y) = \frac{1}{c_n} \int_{S(1)} \gamma_{ij..}(x, y) x_j f_i d\sigma(x).$$

Moreover, Hf is the unique harmonic function with the same boundary values as f , and if $x \cdot f = 0$ on $S(1)$ it can also be written in the form

$$Hf(y) = \frac{1}{c_n} \int_{S(1)} \frac{(1 - |y|^2)^{n+1}}{|x - y|^{2n}} \Delta(x, y) f(x) d\sigma(x).$$

Remarks. $d\sigma$ refers to the $(n-1)$ -dimensional measure on $S(1)$, and $c_n = 2(n-1)\omega_n/n$ where ω_n is the total measure of $S(1)$. We are assuming that f has a continuous extension to $S(1)$. Actually, this is automatically true if we assume the side condition in the form $x \cdot f(x) \rightarrow 0$ as $|x| \rightarrow 1$, for it can be shown that $Sf \in L^p$ forces f to satisfy a uniform Hölder condition.

The proof is a straight-forward application of Stokes' formula. The passage from the differentiable to the distributional case is elementary. The fact that a harmonic function is uniquely determined by its boundary values can be demonstrated as follows: Suppose that f is harmonic and zero on $S(1)$. It is readily shown that

$$\int_{S(r)} Sf(x)_{ij} \gamma_{ij,k}(x) d\sigma = 0$$

for all r . Therefore $ISf(0) = 0$ and hence $f(0) = 0$ by (8). If this result is applied to $(T_y^{-1})^* f$ it follows that $f(y) = 0$ for arbitrary y , so that f is indeed identically zero.

7. COMPUTATION OF SIv

It is easy to show that $S_{ij,hk}(y) = [S\gamma_{ij,\cdot}(y)]_{hk}$ is a Calderon-Zygmund kernel for any choice of the indices; in other words, it is homogeneous of degree $-n$, and its mean-value over the unit sphere is 0. If $v \in L^p$, $1 < p < \infty$, it follows by the Calderon-Zygmund theory that the principal value

$$\text{pr. v. } \int_B v_{ij}(x) S_{ij,hk}(x-y) dx$$

exists almost everywhere, and that it is the limit in $L^p(B)$ of the corresponding truncated integrals. In view of (7) it follows that the integral

$$(9) \quad \Gamma v(y)_{hk} = \int_B v_{ij}(x) \Gamma_{ij,hk}(x,y) dx$$

will also exist as a principal value almost everywhere. One finds, however, that the remainder in (7) makes it possible to assert merely that the principal value is a limit in $L^{p'}$ for any $p' < p/n$. In these circumstances it is natural to assume that $v \in L^p(B)$ for all $p \geq 1$.

THEOREM 2. *If $v \in L^p(B)$ with $p > n$, then $SIv \in L^{p'}(B)$ for all $1 \leq p' < p/n$, and*

$$(10) \quad SIv = -b_n v + \Gamma v$$

where $b_n = 4\omega_n/(n+2)$ and Γv is defined by (9).

Proof. Let φ be an SM_n -valued test-function. The definition of SIv as a distribution leads to the following formal computation:

$$\begin{aligned} \int_B SIv(y)_{hk} \varphi(y)_{hk} dy &= - \int_B Iv(y)_k S^* \varphi(y)_k dy \\ &= - \int_B S^* \varphi(y)_k dy \int_B v_{ij}(x) \gamma_{ij,k}(x,y) dx \\ &= - \int_B v_{ij}(x) dx \int_B S^* \varphi(y)_k \gamma_{ij,k}(x,y) dy \\ &= - \int_B v_{ij}(x) dx [b_n \varphi_{ij}(x) - \int_B \varphi(y)_{hk} \Gamma_{ij,hk}(x,y) dy]. \end{aligned}$$

The justification, by means of the Zygmund-Calderon theory, is routine, and (10) follows.

Taken together, Theorems 1 and 2 lead to a very striking result:

THEOREM 3. *An SM_n -valued function $v \in L^p(B)$, $p > n$, is of the form $v = Sf$ with $f = 0$ on $S(1)$ if and only if it satisfies the homogeneous integral equation $\Gamma v = -a_n v$ with $a_n = c_n - b_n = 2(n-2)(n+1)\omega_n/n(n+2)$.*

Indeed, if v is of this form, Theorem 1 implies $c_n f = -Iv$, hence $c_n v = -SIv$, and consequently $\Gamma v = (b_n - c_n)v$ by Theorem 2. Conversely, if $\Gamma v = -a_n v$ then $SIv = -c_n v$ by (10), and $f = Iv$ vanishes on $S(1)$.

The point of Theorem 3 is that the solvability of $Sf = v$ (with an extra condition on f) has been reduced to an integral equation.

THEOREM 4. *For any $v \in L^p(B)$, $p > n$, $S^* \rho [\Gamma v + a_n v] = 0$.*

Proof. Let f be a vector-valued test-function. Theorem 3 applies to Sf , and we obtain by use of Lemma 2

$$\begin{aligned} \int_B S^* \rho \Gamma v \cdot f \, dx &= - \int_B \rho(x) \Gamma v(x)_{ij} Sf(x)_{ij} \, dx \\ &= - \int_B \rho(x) Sf(x)_{ij} \, dx \int_B v(y)_{hk} \Gamma_{hk,ij}(y, x) \, dy \\ &= - \int_B \rho(y) v(y)_{hk} \, dy \int_B Sf(x)_{ij} \Gamma_{ij,hk}(x, y) \, dx \\ &= - \int_B \rho(y) v(y)_{hk} \Gamma Sf(y)_{hk} \, dy = a_n \int_B \rho(y) v(y)_{hk} Sf(y)_{hk} \, dy \\ &= - a_n \int_B S^* \rho v \cdot f \, dy \end{aligned}$$

and hence $S^* \rho \Gamma v = -a_n S^* v$.

THEOREM 5. *Every v which is in all $L^p(B)$ has a unique representation in the form $v = v' + v''$ where v' and v'' are in all $L^p(B)$ while v' is in the image of SI and v'' is in the kernel of $S^* \rho$.*

As a consequence of Theorems 3 and 4 the representation is given by

$$c_n v = -SIv + (\Gamma v + a_n v).$$

It is unique, for if $SI = \Gamma v + a_n v$, then $S^* \rho SIv = 0$ so that Iv is harmonic and 0 on $S(1)$, hence identically zero.

8. AUTOMORPHIC FUNCTIONS AND BELTRAMI DIFFERENTIALS

Although this aspect has not been emphasized it should be clear that the author is trying to develop a theory which is immediately applicable to the study of discrete subgroups of G . All the definitions have been chosen with this in mind, and the relevant theorems for subgroups follow effortlessly.

Let G^0 be a discrete subgroup of G . A vector-valued function f is *automorphic* with respect to G^0 if $A^* f = f$, or more explicitly $A'(x)^{-1} f(Ax) = f(x)$ for all $A \in G^0$. Similarly, an SM_n -valued function v will be called a *Beltrami differential* for G^0 if $A^* v = v$, or $A'(x)^{-1} v(Ax) A'(x) = v(x)$, for all $A \in G^0$. If v is a Beltrami differential, then $A^*(\rho v dx) = \rho v dx$ for all $A \in G^0$, and $\rho v dx$ is called an n th order differential. The terminology is borrowed from the corresponding notions for $n = 2$.

If v is Beltrami and in L^∞ , then it is also in $L^p(B)$ for all p , and Theorems 2-5 are applicable. They gain added significance from the fact that Iv is automatically automorphic with respect to G^0 (it is easy to show that $A^* Iv = IA^* v$ for all v and $A \in G$). As a consequence SIv is Beltrami, and by Theorem 2 the same is true of Γv . It follows that Theorems 2-5 may be interpreted as referring to the quotient space $G^0 \backslash B$, provided that we start from the hypothesis $v \in L^\infty$. In the conclusion we know, for instance, that

$$\int_B \|SI \gamma\|^p dx = \int_{G^0 \backslash B} \|SI v\|^p \rho_0 dx < \infty$$

where, by a theorem of Godement,

$$\rho_0(x) = \sum_{A \in G^0} |A'(x)|^n$$

is known to converge.

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