

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 23 (1977)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ALTERNATIVE HOMOTOPY THEORIES
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Kapitel: 8. Examples
DOI: <https://doi.org/10.5169/seals-48928>

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Hence we obtain a commutative diagram as follows, relating the modified exact sequences for Q and M .

$$\begin{array}{ccc} \pi_n(G) & \xrightarrow{u_*} & \pi(\Gamma(Q)) \\ \bar{f}_* \downarrow & & \downarrow (\Gamma(\hat{f}))_* \\ \pi_n(N) & \xrightarrow{v_*} & \pi(\Gamma(M)) \end{array}$$

Recall that δ is the obstruction to extending λ to a vertical homotopy of $e'_{\#}$ into $c'_{\#}$. Hence $\bar{f}_* \delta$ is the obstruction to extending $\bar{f} \circ \lambda$ to a vertical homotopy of $\hat{f} \circ e'_{\#}$ into $\hat{f} \circ c'_{\#}$. Hence it follows, as explained in the previous section, that $\hat{f} \circ e'_{\#}$ and $\hat{f} \circ c'_{\#}$ are vertically homotopic if and only if $\delta \in D\pi_1(N)$. Finally we use the correspondence between ex-maps and cross-sections to obtain (7.2) as stated.

8. EXAMPLES

Let X be a finite simply-connected complex and let P be a principal $SO(m)$ -bundle over X . Consider the antipodal self-map a of S^{m-1} . The unreduced suspension \hat{a} is a pointed $SO(m)$ -map of S^m into itself. Hence $P_{\#} \hat{a}$ is an ex-map of $E = P_{\#} S^m$ into itself; let $\sigma \in \pi_X(E, E)$ denote the ex-homotopy class. Since \hat{a} is of degree $(-1)^m$ we can apply (5.3) and obtain that

$$(8.1) \quad 2^r \Sigma_* \sigma = 2^r \quad (m \text{ even}),$$

where $r = \text{reg}(X)$. It follows at once that

$$(8.2) \quad 2^{r+1} [l_{\Sigma E}, l_{\Sigma E}] = 0 \quad (m \text{ even}),$$

by (2.1) and (3.1), and hence from (3.3) that

$$(8.3) \quad [l_{\Sigma E}, [l_{\Sigma E}, l_{\Sigma E}]] = 0 \quad (m \text{ even}).$$

Here $l_{\Sigma E}$ denotes the ex-homotopy class of the identity on ΣE . Similar results, but under more restrictive conditions, have been obtained by Eggar [4]. It can also be shown that the quadruple Whitehead products

$$[[l_{\Sigma E}, l_{\Sigma E}], [l_{\Sigma E}, l_{\Sigma E}]], [l_{\Sigma E}, [l_{\Sigma E}, [l_{\Sigma E}, l_{\Sigma E}]]]$$

are trivial, whether m is even or odd.

In particular, let X be a sphere. For m even (8.1) shows that $2\Sigma_*\sigma = 2$ and (8.2) that $4[l_{\Sigma E}, l_{\Sigma E}] = 0$. However, more precise results can be obtained by using the methods of §7, as follows. Take $X = S^n$ ($n \geq 2$) and let $\theta \in \pi_{n-1} SO(m)$ be the classifying element of P . We apply (7.2) with $f = 1$ and, using (6.4), obtain

THEOREM (8.4). *Let m be even. Then $\Sigma_*\sigma = 1$ if and only if $\Sigma_*JF\theta$ is contained in the image of*

$$D: \pi_{m+2}(S^{m+1}) \rightarrow \pi_{n+m+1}(S^{m+1}),$$

where

$$D\alpha = \Sigma_*J\theta \circ \Sigma_*^{n-1}\alpha - \alpha \circ \Sigma_*^2J\theta.$$

In the stable range, where $m > n$, the homomorphism D is trivial and $F\theta = \theta \circ \eta$, as in §6 of [6], where η generates the 1-stem. In this range it does not matter whether we deal with ex-maps or over-maps, and so (8.4) agrees with (4.5) of [8].

Now let l_m denote the pointed $SO(m)$ -homotopy class of the identity on S^m , so that $l_{\Sigma E} = P_{\#}\Sigma_*l_m$. Represent the Whitehead square

$$w(\Sigma S^m) = [\Sigma_*l_m, \Sigma_*l_m]$$

by a pointed $SO(m)$ -map $f: \Sigma(S^m \wedge S^m) \rightarrow \Sigma S^m$. Then $P_{\#}f$ represents the Whitehead square

$$w(\Sigma E) = P_{\#}w(\Sigma S^m) = [l_{\Sigma E}, l_{\Sigma E}].$$

We apply (7.2) again and, using (6.2)-(6.4), obtain

THEOREM (8.5). *Let m be even. Then $2w(\Sigma E) = 0$ if and only if $w_{m+1} \circ \Sigma_*^{m+1}JF\theta$ lies in the image of*

$$D: \pi_{2m+2}(S^{m+1}) \rightarrow \pi_{n+2m+1}(S^{m+1}),$$

where $D\alpha = \alpha \circ \Sigma_*^{m+2}J\theta - 2\Sigma_*J\theta \circ \Sigma_*^{n-1}\alpha$.

Here $w_{m+1} \in \pi_{2m+1}(S^{m+1})$ denotes the ordinary Whitehead square of the generator of $\pi_{m+1}(S^{m+1})$. Unless $m = 2$ or 6 we have $w_{m+1} \neq 0$ and (8.5) determines the order of $w(\Sigma E)$. When $m = 2$ or 6 it would be interesting to know when $w(\Sigma E) = 0$, i.e. when ΣE is a Hopf ex-space, but unfortunately our method does not apply.

For some examples where the order of $w(\Sigma E)$ is (precisely) 4, consider the transgression $\Delta: \pi_n(S^m) \rightarrow \pi_{n-1} SO(m)$ in the homotopy exact

sequence of the standard fibration of $SO(m+1)$. Take $\theta = \Delta\phi$, where $\phi \in \Sigma_*\pi_{n-1}(S^{m-1})$. Then $\Sigma_*J\theta = 0$ and so D is trivial. However it follows from (4.1) and (6.3) of [6] that

$$\Sigma_*JF\theta = [\Sigma_*l_m, \Sigma_*\phi].$$

This Whitehead product is non-zero if, for example, $m = n$ and $\phi = l_m$ with $m \neq 2, 6$. Of course E is trivial as a bundle, in these examples, although not as a sectioned bundle.

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(Reçu le 6 mai 1977)

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