Given a representation $\phi: SO(m) \to SO(q)$ write

$$J_\phi = J \circ \phi_*: \pi_r SO(m) \to \pi_{r+q}(S^q),$$

where J denotes the usual Hopf-Whitehead homomorphism. For example, if $q > m$ and ϕ is the inclusion then

$$(6.2) \qquad J_\phi = (-1)^{m-q} \Sigma_*^{m-q} J,$$

by (3.2) of [5] (cf. [8]). If $q = 2m$ and $\phi = 1 \oplus 1$ it is easily seen that

$$(6.3) \qquad J_\phi = 2(-1)^m \Sigma_*^m J.$$

Consider the function-space $N = N(S^p, S^q)$ of pointed maps $S^p \to S^q$. We identify $\pi_i(N)$ $(i=0, 1, \ldots)$ with $\pi_{i+p}(S^q)$ in the standard way (see [15]). Let G be a topological group and let

$$\phi: G \to SO(p), \quad \psi: G \to SO(q)$$

be representations of G. We regard S^p, S^q as pointed G-spaces using ϕ, ψ, respectively. Choose a principal G-bundle P over S^n with classifying element $\theta \in \pi_{n-1}(G)$, and take $E_1 = P_\#(S^p)$, $E_2 = P_\#(S^q)$. Then the operator D in our exact sequence is given

$$(6.4) \qquad D\alpha = \alpha \circ \Sigma_*^{r+p-q+1} J_\psi \theta - J_\phi \theta \circ \Sigma_*^{n+p-q-1} \alpha,$$

where $\alpha \in \pi_{r+p+1}(S^q)$. The case $r = 1$ of this result will be needed in §8 below.

7. The adjoint G-bundle

Let X be any space and let P be a principal G-bundle over X. We regard P as a (right) G-space in the usual way. By a *principal automorphism* we mean an equivariant fibre-preserving map of P into itself. By the *adjoint G-bundle* we mean the sectioned bundle $Q = P_\# G$, where G acts on itself by conjugation. Note that Q is a group ex-space since G is a group G-space. We can construct Q from $G \times P$ by identifying

$$(7.1) \qquad (gag^{-1}, b) \sim (a, bg) \qquad (a \in G, b \in P)$$

for all $g \in G$. The group ex-structure is given by

$$\{a_1, b\} \cdot \{a_2, b\} = \{a_1 \cdot a_2, b\} \quad (a_1, a_2 \in G),$$

where $\{\ ,\ \}$ denotes the equivalence class of $(\ ,\ )$. Every principal automorphism f of P determines a cross-section $f': X \to Q$ as follows.

Given $x \in X$ choose any $b \in P_x$; then $fb = bg$, for some $g \in G$, and we define $f'x = \{ g, b \}$. This correspondence establishes an isomorphism between the group of principal automorphisms of P and the group of cross-sections of Q.

Any element c of the centre of G determines a G-map $c_\#$ for any G-space A. Notice that $c_\#$ is a principal automorphism in the case of P and that the corresponding cross-section $c'_\#$ of Q is given by $c'_\# \{ b \} = \{ c, b \}$. When X is a sphere these central cross-sections of Q can be analysed as follows.

Take $X = S^n$ $(n \geqslant 2)$, so that P is a principal G-bundle over S^n. Let B^n denote the n-ball with boundary S^{n-1}. Choose a relative homeomorphism $(B^n, S^{n-1}) \to (S^n, x_0)$ and lift this to a map $k: (B^n, S^{n-1}) \to (P, G)$. The homotopy class $\theta \in \pi_{n-1}(G)$ of $l = k | S^{n-1}$ classifies the bundle according to clutching theory.

Let $c \in G$ be central and let $\lambda: I \to G$ be a path such that $\lambda(0) = e$, $\lambda(1) = c$. Consider the map $\Lambda: B^n \times I \to Q$ which is given by

$$\Lambda(y, t) = \{ \lambda(t), k(y) \} \qquad (y \in B^n, t \in I).$$

The boundary of $B^n \times I$ is the sphere

$$B^n \times 0 \cup S^{n-1} \times I \cup B^n \times 1,$$

and Λ maps $S^{n-1} \times I$ into $G \subset Q$ by

$$\Lambda(y, t) = (ly) . (\lambda t) . (ly)^{-1},$$

using (7.1). Let us compare this with the map Λ' of the boundary which agrees with Λ on $B^n \times \dot{I}$ but is given on $S^{n-1} \times I$ by $\Lambda'(y, t) = \lambda t$. Now λ can be regarded as a vertical homotopy of $e'_\#$ into $c'_\#$ over $\{ x_0 \}$ and Λ represents the obstruction

$$\delta = \delta(e'_\#, c'_\#; \lambda) \in \pi_n(G)$$

to extending this vertical homotopy over S^n. Since $\Lambda \mid (B^n \times I)^{\cdot}$ is nul-homotopic, however, it follows that δ is also represented by $d: \tilde{\Sigma} S^{n-1} \to G$, where

$$d(y, t) = (ly) . (\lambda t) . (ly)^{-1}.$$

For example, take $G = SO(m)$, with m even. Take $c = -e$ and

$$\lambda(t) = e \cos \pi t + b \sin \pi t \qquad (0 \leqslant t \leqslant 1),$$

where b denotes the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (m/2 \quad \text{summands}).$$

Then $\delta = F\theta$, by definition, where

$$F: \pi_{n-1}\, SO(m) \to \pi_n SO(m)$$

denotes the Bott suspension, as in [6].

Now let $A_i\,(i=1, 2)$ be a locally compact pointed G-space and write $E_i = P_{\#}A_i$. Recall that $N = N(A_1, A_2)$ denotes the function-space of pointed maps $A_1 \to A_2$. Given a pointed G-map $f: A_1 \to A_2$ we can construct an ex-map $P_{\#}f: E_1 \to E_2$ and a pointed G-map $\bar{f}: G \to N$, where $\bar{f}(g) = g_{\#} \circ f = f \circ g_{\#}$. I assert

PROPOSITION (7.2). *The ex-maps*

$$P_{\#}f, P_{\#}f \circ P_{\#}c: E_1 \to E_2$$

are ex-homotopic if and only if

$$\bar{f}^{\,*}\delta \in D\pi_1(N) \subset \pi_n(N),$$

where δ is as above.

Here D is the operator which occurs in the modified exact sequence of the evaluation fibration derived from the function-space bundle, as in §6. The proof of (7.2) is by naturality, as follows.

First observe that $\bar{f}$ extends to a fibre-preserving map $\hat{f}: Q \to M$, where $M = M_X(E_1, E_2)$ denotes the function-space bundle. To see this we note that f determines a pointed G-map $F: A_1 \times G \to A_2$, where

$$F(x, g) = f(xg) \qquad (x \in A_1, g \in G).$$

Hence $P_{\#}f: E_1 \times Q \to E_2$ is defined and we take $\hat{f}$ to be the adjoint.

We have $X = S^n$ so that the evaluation fibrations can be modified as in §6. Clearly

$$\Gamma_0(\hat{f}) \circ k \simeq l \circ \Omega^n(\bar{f}) \tag{7.3}$$

as shown below, where k is defined by subtracting the cross-section $e'_{\#}$ and l by subtracting $\hat{f} \circ e'_{\#}$.

$$\begin{array}{ccc} \Omega^n(G) & \xrightarrow{k} & \Gamma_0(Q) \\ {\scriptstyle \Omega^n(\bar{f})}\downarrow & & \downarrow{\scriptstyle \Gamma_0(\hat{f})} \\ \Omega^n(N) & \xrightarrow[l]{} & \Gamma_0(M) \end{array}$$

Hence we obtain a commutative diagram as follows, relating the modified exact sequences for Q and M.

$$\begin{array}{ccc} \pi_n(G) & \xrightarrow{u_*} & \pi(\Gamma(Q)) \\ {\scriptstyle \bar{f}_*}\big\downarrow & & \big\downarrow{\scriptstyle (\Gamma(\hat{f}))_*} \\ \pi_n(N) & \xrightarrow[v_*]{} & \pi(\Gamma(M)) \end{array}$$

Recall that δ is the obstruction to extending λ to a vertical homotopy of $e'_{\#}$ into $c'_{\#}$. Hence $\bar{f}_* \delta$ is the obstruction to extending $\bar{f} \circ \lambda$ to a vertical homotopy of $\hat{f} \circ e'_{\#}$ into $\hat{f} \circ c'_{\#}$. Hence it follows, as explained in the previous section, that $\hat{f} \circ e'_{\#}$ and $\hat{f} \circ c'_{\#}$ are vertically homotopic if and only if $\delta \in D\pi_1(N)$. Finally we use the correspondence between ex-maps and cross-sections to obtain (7.2) as stated.

8. Examples

Let X be a finite simply-connected complex and let P be a principal $SO(m)$-bundle over X. Consider the antipodal self-map a of S^{m-1}. The unreduced suspension $\hat{a}$ is a pointed $SO(m)$-map of S^m into itself. Hence $P_{\#}\hat{a}$ is an ex-map of $E = P_{\#}S^m$ into itself; let $\sigma \in \pi_X(E, E)$ denote the ex-homotopy class. Since $\hat{a}$ is of degree $(-1)^m$ we can apply (5.3) and obtain that

(8.1) $$2^r \Sigma_* \sigma = 2^r \qquad (m \text{ even}),$$

where $r = \operatorname{reg}(X)$. It follows at once that

(8.2) $$2^{r+1}[\iota_{\Sigma E}, \iota_{\Sigma E}] = 0 \qquad (m \text{ even}),$$

by (2.1) and (3.1), and hence from (3.3) that

(8.3) $$[\iota_{\Sigma E}, [\iota_{\Sigma E}, \iota_{\Sigma E}]] = 0 \qquad (m \text{ even}).$$

Here $\iota_{\Sigma E}$ denotes the ex-homotopy class of the identity on ΣE. Similar results, but under more restrictive conditions, have been obtained by Eggar [4]. It can also be shown that the quadruple Whitehead products

$$[[\iota_{\Sigma E}, \iota_{\Sigma E}], [\iota_{\Sigma E}, \iota_{\Sigma E}]], \; [\iota_{\Sigma E}, [\iota_{\Sigma E}, [\iota_{\Sigma E}, \iota_{\Sigma E}]]]$$

are trivial, whether m is even or odd.