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Given a representation  $\phi: SO(m) \rightarrow SO(q)$  write

$$J_\phi = J \circ \phi_*: \pi_r SO(m) \rightarrow \pi_{r+q}(S^q),$$

where  $J$  denotes the usual Hopf-Whitehead homomorphism. For example, if  $q > m$  and  $\phi$  is the inclusion then

$$(6.2) \quad J_\phi = (-1)^{m-q} \Sigma_*^{m-q} J,$$

by (3.2) of [5] (cf. [8]). If  $q = 2m$  and  $\phi = 1 \oplus 1$  it is easily seen that

$$(6.3) \quad J_\phi = 2(-1)^m \Sigma_*^m J.$$

Consider the function-space  $N = N(S^p, S^q)$  of pointed maps  $S^p \rightarrow S^q$ . We identify  $\pi_i(N)$  ( $i=0, 1, \dots$ ) with  $\pi_{i+p}(S^q)$  in the standard way (see [15]). Let  $G$  be a topological group and let

$$\phi: G \rightarrow SO(p), \quad \psi: G \rightarrow SO(q)$$

be representations of  $G$ . We regard  $S^p, S^q$  as pointed  $G$ -spaces using  $\phi, \psi$ , respectively. Choose a principal  $G$ -bundle  $P$  over  $S^n$  with classifying element  $\theta \in \pi_{n-1}(G)$ , and take  $E_1 = P_\#(S^p)$ ,  $E_2 = P_\#(S^q)$ . Then the operator  $D$  in our exact sequence is given

$$(6.4) \quad D\alpha = \alpha \circ \Sigma_*^{r+p-q+1} J_\psi \theta - J_\phi \theta \circ \Sigma_*^{n+p-q-1} \alpha,$$

where  $\alpha \in \pi_{r+p+1}(S^q)$ . The case  $r = 1$  of this result will be needed in §8 below.

## 7. THE ADJOINT $G$ -BUNDLE

Let  $X$  be any space and let  $P$  be a principal  $G$ -bundle over  $X$ . We regard  $P$  as a (right)  $G$ -space in the usual way. By a *principal automorphism* we mean an equivariant fibre-preserving map of  $P$  into itself. By the *adjoint  $G$ -bundle* we mean the sectioned bundle  $Q = P_\#G$ , where  $G$  acts on itself by conjugation. Note that  $Q$  is a group ex-space since  $G$  is a group  $G$ -space. We can construct  $Q$  from  $G \times P$  by identifying

$$(7.1) \quad (gag^{-1}, b) \sim (a, bg) \quad (a \in G, b \in P)$$

for all  $g \in G$ . The group ex-structure is given by

$$\{a_1, b\} \cdot \{a_2, b\} = \{a_1 \cdot a_2, b\} \quad (a_1, a_2 \in G),$$

where  $\{ \ , \ }$  denotes the equivalence class of  $( \ , \ )$ . Every principal automorphism  $f$  of  $P$  determines a cross-section  $f': X \rightarrow Q$  as follows.

Given  $x \in X$  choose any  $b \in P_x$ ; then  $fb = bg$ , for some  $g \in G$ , and we define  $f'x = \{g, b\}$ . This correspondence establishes an isomorphism between the group of principal automorphisms of  $P$  and the group of cross-sections of  $Q$ .

Any element  $c$  of the centre of  $G$  determines a  $G$ -map  $c_{\#}$  for any  $G$ -space  $A$ . Notice that  $c_{\#}$  is a principal automorphism in the case of  $P$  and that the corresponding cross-section  $c'_{\#}$  of  $Q$  is given by  $c'_{\#} \{b\} = \{c, b\}$ . When  $X$  is a sphere these central cross-sections of  $Q$  can be analysed as follows.

Take  $X = S^n$  ( $n \geq 2$ ), so that  $P$  is a principal  $G$ -bundle over  $S^n$ . Let  $B^n$  denote the  $n$ -ball with boundary  $S^{n-1}$ . Choose a relative homeomorphism  $(B^n, S^{n-1}) \rightarrow (S^n, x_0)$  and lift this to a map  $k: (B^n, S^{n-1}) \rightarrow (P, G)$ . The homotopy class  $\theta \in \pi_{n-1}(G)$  of  $l = k|_{S^{n-1}}$  classifies the bundle according to clutching theory.

Let  $c \in G$  be central and let  $\lambda: I \rightarrow G$  be a path such that  $\lambda(0) = e$ ,  $\lambda(1) = c$ . Consider the map  $A: B^n \times I \rightarrow Q$  which is given by

$$A(y, t) = \{ \lambda(t), k(y) \} \quad (y \in B^n, t \in I).$$

The boundary of  $B^n \times I$  is the sphere

$$B^n \times 0 \cup S^{n-1} \times I \cup B^n \times 1,$$

and  $A$  maps  $S^{n-1} \times I$  into  $G \subset Q$  by

$$A(y, t) = (ly) \cdot (\lambda t) \cdot (ly)^{-1},$$

using (7.1). Let us compare this with the map  $A'$  of the boundary which agrees with  $A$  on  $B^n \times I$  but is given on  $S^{n-1} \times I$  by  $A'(y, t) = \lambda t$ . Now  $\lambda$  can be regarded as a vertical homotopy of  $e'_{\#}$  into  $c'_{\#}$  over  $\{x_0\}$  and  $A$  represents the obstruction

$$\delta = \delta(e'_{\#}, c'_{\#}; \lambda) \in \pi_n(G)$$

to extending this vertical homotopy over  $S^n$ . Since  $A|_{(B^n \times I)}$  is null-homotopic, however, it follows that  $\delta$  is also represented by  $d: \tilde{\Sigma} S^{n-1} \rightarrow G$ , where

$$d(y, t) = (ly) \cdot (\lambda t) \cdot (ly)^{-1}.$$

For example, take  $G = SO(m)$ , with  $m$  even. Take  $c = -e$  and

$$\lambda(t) = e \cos \pi t + b \sin \pi t \quad (0 \leq t \leq 1),$$

where  $b$  denotes the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (m/2 \text{ summands}).$$

Then  $\delta = F\theta$ , by definition, where

$$F: \pi_{n-1} SO(m) \rightarrow \pi_n SO(m)$$

denotes the Bott suspension, as in [6].

Now let  $A_i (i=1, 2)$  be a locally compact pointed  $G$ -space and write  $E_i = P_{\#}A_i$ . Recall that  $N = N(A_1, A_2)$  denotes the function-space of pointed maps  $A_1 \rightarrow A_2$ . Given a pointed  $G$ -map  $f: A_1 \rightarrow A_2$  we can construct an ex-map  $P_{\#}f: E_1 \rightarrow E_2$  and a pointed  $G$ -map  $\bar{f}: G \rightarrow N$ , where  $\bar{f}(g) = g_{\#} \circ f = f \circ g_{\#}$ . I assert

PROPOSITION (7.2). *The ex-maps*

$$P_{\#}f, P_{\#}f \circ P_{\#}c: E_1 \rightarrow E_2$$

*are ex-homotopic if and only if*

$$\bar{f}^* \delta \in D\pi_1(N) \subset \pi_n(N),$$

*where  $\delta$  is as above.*

Here  $D$  is the operator which occurs in the modified exact sequence of the evaluation fibration derived from the function-space bundle, as in §6. The proof of (7.2) is by naturality, as follows.

First observe that  $\bar{f}$  extends to a fibre-preserving map  $\hat{f}: Q \rightarrow M$ , where  $M = M_X(E_1, E_2)$  denotes the function-space bundle. To see this we note that  $f$  determines a pointed  $G$ -map  $F: A_1 \times G \rightarrow A_2$ , where

$$F(x, g) = f(xg) \quad (x \in A_1, g \in G).$$

Hence  $P_{\#}f: E_1 \times Q \rightarrow E_2$  is defined and we take  $\hat{f}$  to be the adjoint.

We have  $X = S^n$  so that the evaluation fibrations can be modified as in §6. Clearly

$$(7.3) \quad \Gamma_0(\hat{f}) \circ k \simeq l \circ \Omega^n(\bar{f})$$

as shown below, where  $k$  is defined by subtracting the cross-section  $e'_{\#}$  and  $l$  by subtracting  $\hat{f} \circ e'_{\#}$ .

$$\begin{array}{ccc} \Omega^n(G) & \xrightarrow{k} & \Gamma_0(Q) \\ \Omega^n(\bar{f}) \downarrow & & \downarrow \Gamma_0(\hat{f}) \\ \Omega^n(N) & \xrightarrow{l} & \Gamma_0(M) \end{array}$$

Hence we obtain a commutative diagram as follows, relating the modified exact sequences for  $Q$  and  $M$ .

$$\begin{array}{ccc} \pi_n(G) & \xrightarrow{u_*} & \pi(\Gamma(Q)) \\ \bar{f}_* \downarrow & & \downarrow (\Gamma(\hat{f}))_* \\ \pi_n(N) & \xrightarrow{v_*} & \pi(\Gamma(M)) \end{array}$$

Recall that  $\delta$  is the obstruction to extending  $\lambda$  to a vertical homotopy of  $e'_{\#}$  into  $c'_{\#}$ . Hence  $\bar{f}_*\delta$  is the obstruction to extending  $\bar{f} \circ \lambda$  to a vertical homotopy of  $\hat{f} \circ e'_{\#}$  into  $\hat{f} \circ c'_{\#}$ . Hence it follows, as explained in the previous section, that  $\hat{f} \circ e'_{\#}$  and  $\hat{f} \circ c'_{\#}$  are vertically homotopic if and only if  $\delta \in D\pi_1(N)$ . Finally we use the correspondence between ex-maps and cross-sections to obtain (7.2) as stated.

## 8. EXAMPLES

Let  $X$  be a finite simply-connected complex and let  $P$  be a principal  $SO(m)$ -bundle over  $X$ . Consider the antipodal self-map  $a$  of  $S^{m-1}$ . The unreduced suspension  $\hat{a}$  is a pointed  $SO(m)$ -map of  $S^m$  into itself. Hence  $P_{\#}\hat{a}$  is an ex-map of  $E = P_{\#}S^m$  into itself; let  $\sigma \in \pi_X(E, E)$  denote the ex-homotopy class. Since  $\hat{a}$  is of degree  $(-1)^m$  we can apply (5.3) and obtain that

$$(8.1) \quad 2^r \Sigma_* \sigma = 2^r \quad (m \text{ even}),$$

where  $r = \text{reg}(X)$ . It follows at once that

$$(8.2) \quad 2^{r+1} [l_{\Sigma E}, l_{\Sigma E}] = 0 \quad (m \text{ even}),$$

by (2.1) and (3.1), and hence from (3.3) that

$$(8.3) \quad [l_{\Sigma E}, [l_{\Sigma E}, l_{\Sigma E}]] = 0 \quad (m \text{ even}).$$

Here  $l_{\Sigma E}$  denotes the ex-homotopy class of the identity on  $\Sigma E$ . Similar results, but under more restrictive conditions, have been obtained by Eggar [4]. It can also be shown that the quadruple Whitehead products

$$[[l_{\Sigma E}, l_{\Sigma E}], [l_{\Sigma E}, l_{\Sigma E}]], [l_{\Sigma E}, [l_{\Sigma E}, [l_{\Sigma E}, l_{\Sigma E}]]]$$

are trivial, whether  $m$  is even or odd.