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— 232 —

Given a representation $\phi: SO(m) \rightarrow SO(q)$ write

$$J_{\phi} = J \circ \phi_* : \pi_r SO(m) \to \pi_{r+q}(S^q) ,$$

where J denotes the usual Hopf-Whitehead homomorphism. For example, if q > m and ϕ is the inclusion then

(6.2)
$$J_{\phi} = (-1)^{m-q} \Sigma_{*}^{m-q} J$$

by (3.2) of [5] (cf. [8]). If q = 2m and $\phi = 1 \oplus 1$ it is easily seen that

(6.3)
$$J_{\phi} = 2(-1)^m \Sigma_*^m J.$$

Consider the function-space $N = N(S^p, S^q)$ of pointed maps $S^p \to S^q$. We identify $\pi_i(N)$ (i=0, 1, ...) with $\pi_{i+p}(S^q)$ in the standard way (see [15]). Let G be a topological group and let

$$\phi: G \to SO(p), \quad \psi: G \to SO(q)$$

be representations of G. We regard S^p , S^q as pointed G-spaces using ϕ , ψ , respectively. Choose a principal G-bundle P over S^n with classifying element $\theta \in \pi_{n-1}(G)$, and take $E_1 = P_{\#}(S^p)$, $E_2 = P_{\#}(S^q)$. Then the operator D in our exact sequence is given

(6.4)
$$D\alpha = \alpha \circ \Sigma_*^{r+p-q+1} J_{\psi} \theta - J_{\phi} \theta \circ \Sigma_*^{n+p-q-1} \alpha,$$

where $\alpha \in \pi_{r+p+1}(S^q)$. The case r = 1 of this result will be needed in §8 below.

7. The adjoint G-bundle

Let X be any space and let P be a principal G-bundle over X. We regard P as a (right) G-space in the usual way. By a principal automorphism we mean an equivariant fibre-preserving map of P into itself. By the adjoint G-bundle we mean the sectioned bundle $Q = P_{\#}G$, where G acts on itself by conjugation. Note that Q is a group ex-space since G is a group G-space. We can construct Q from $G \times P$ by identifying

(7.1)
$$(gag^{-1}, b) \sim (a, bg) \quad (a \in G, b \in P)$$

for all $g \in G$. The group ex-structure is given by

$$\{a_1, b\} \cdot \{a_2, b\} = \{a_1 \cdot a_2, b\} \quad (a_1, a_2 \in G),$$

where $\{,\}$ denotes the equivalence class of (,). Every principal automorphism f of P determines a cross-section $f': X \to Q$ as follows.

Given $x \in X$ choose any $b \in P_x$; then fb = bg, for some $g \in G$, and we define $f'x = \{g, b\}$. This correspondence establishes an isomorphism between the group of principal automorphisms of P and the group of cross-sections of Q.

Any element c of the centre of G determines a G-map $c_{\#}$ for any G-space A. Notice that $c_{\#}$ is a principal automorphism in the case of P and that the corresponding cross-section $c'_{\#}$ of Q is given by $c'_{\#} \{b\} = \{c, b\}$. When X is a sphere these central cross-sections of Q can be analysed as follows.

Take $X = S^n$ $(n \ge 2)$, so that P is a principal G-bundle over S^n . Let B^n denote the *n*-ball with boundary S^{n-1} . Choose a relative homeomorphism $(B^n, S^{n-1}) \to (S^n, x_0)$ and lift this to a map $k: (B^n, S^{n-1}) \to (P, G)$. The homotopy class $\theta \in \pi_{n-1}(G)$ of $l = k | S^{n-1}$ classifies the bundle according to clutching theory.

Let $c \in G$ be central and let $\lambda: I \to G$ be a path such that $\lambda(0) = e$, $\lambda(1) = c$. Consider the map $\Lambda: B^n \times I \to Q$ which is given by

 $\Lambda(y,t) = \{\lambda(t), k(y)\} \quad (y \in B^n, t \in I).$

The boundary of $B^n \times I$ is the sphere

$$B^n \times 0 \cup S^{n-1} \times I \cup B^n \times 1$$
,

and Λ maps $S^{n-1} \times I$ into $G \subset Q$ by

$$\Lambda(y,t) = (ly) \cdot (\lambda t) \cdot (ly)^{-1},$$

using (7.1). Let us compare this with the map Λ' of the boundary which agrees with Λ on $B^n \times \dot{I}$ but is given on $S^{n-1} \times I$ by $\Lambda'(y, t) = \lambda t$. Now λ can be regarded as a vertical homotopy of $e'_{\#}$ into $c'_{\#}$ over $\{x_0\}$ and Λ represents the obstruction

$$\delta = \delta(e'_{\#}, c'_{\#}; \lambda) \in \pi_n(G)$$

to extending this vertical homotopy over S^n . Since $\Lambda | (B^n \times I)^{\cdot}$ is nulhomotopic, however, it follows that δ is also represented by $d: \Sigma S^{n-1} \to G$, where

$$d(y, t) = (ly) . (\lambda t) . (ly)^{-1}$$
.

For example, take G = SO(m), with m even. Take c = -e and

$$\lambda(t) = e \cos \pi t + b \sin \pi t \quad (0 \le t \le 1),$$

where b denotes the matrix

L'Enseignement mathém., t. XXIII, fasc. 3-4.

— 234 —

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
 (*m*/2 summands).

Then $\delta = F\theta$, by definition, where

$$F: \pi_{n-1} SO(m) \to \pi_n SO(m)$$

denotes the Bott suspension, as in [6].

Now let A_i (i=1, 2) be a locally compact pointed G-space and write $E_i = P_{\#}A_i$. Recall that $N = N(A_1, A_2)$ denotes the function-space of pointed maps $A_1 \rightarrow A_2$. Given a pointed G-map $f: A_1 \rightarrow A_2$ we can construct an ex-map $P_{\#}f: E_1 \rightarrow E_2$ and a pointed G-map $\overline{f}: G \rightarrow N$, where $\overline{f}(g) = g_{\#} \circ f = f \circ g_{\#}$. I assert

PROPOSITION (7.2). The ex-maps

 $P_{\#}f, P_{\#}f \circ P_{\#}c \colon E_1 \to E_2$

are ex-homotopic if and only if

$$\bar{f}^*\delta \in D\pi_1(N) \subset \pi_n(N) ,$$

where δ is as above.

Here D is the operator which occurs in the modified exact sequence of the evaluation fibration derived from the function-space bundle, as in §6. The proof of (7.2) is by naturality, as follows.

First observe that \overline{f} extends to a fibre-preserving map $f: Q \to M$, where $M = M_X(E_1, E_2)$ denotes the function-space bundle. To see this we note that f determines a pointed G-map $F: A_1 \times G \to A_2$, where

$$F(x,g) = f(xg) \quad (x \in A_1, g \in G).$$

Hence $P_{\#}f: E_1 \times Q \to E_2$ is defined and we take f to be the adjoint.

We have $X = S^n$ so that the evaluation fibrations can be modified as in §6. Clearly

(7.3)
$$\Gamma_0(\widehat{f}) \circ k \simeq l \circ \Omega^n(\overline{f})$$

as shown below, where k is defined by subtracting the cross-section $e'_{\#}$ and l by subtracting $\hat{f} \circ e'_{\#}$.

Hence we obtain a commutative diagram as follows, relating the modified exact sequences for Q and M.

Recall that δ is the obstruction to extending λ to a vertical homotopy of $e'_{\#}$ into $c'_{\#}$. Hence $\overline{f}_*\delta$ is the obstruction to extending $\overline{f} \circ \lambda$ to a vertical homotopy of $\widehat{f} \circ e'_{\#}$ into $\widehat{f} \circ c'_{\#}$. Hence it follows, as explained in the previous section, that $\widehat{f} \circ e'_{\#}$ and $\widehat{f} \circ c'_{\#}$ are vertically homotopic if and only if $\delta \in D\pi_1(N)$. Finally we use the correspondence between ex-maps and crosssections to obtain (7.2) as stated.

8. EXAMPLES

Let X be a finite simply-connected complex and let P be a principal SO (m)-bundle over X. Consider the antipodal self-map a of S^{m-1} . The unreduced suspension \hat{a} is a pointed SO (m)-map of S^m into itself. Hence $\hat{P}_{\#}a$ is an ex-map of $E = P_{\#}S^m$ into itself; let $\sigma \in \pi_X(E, E)$ denote the exhomotopy class. Since \hat{a} is of degree $(-1)^m$ we can apply (5.3) and obtain that

(8.1) $2^r \Sigma_* \sigma = 2^r \qquad (m \text{ even}),$

where $r = \operatorname{reg}(X)$. It follows at once that

(8.2)
$$2^{r+1}[\iota_{\Sigma E}, \iota_{\Sigma E}] = 0$$
 (*m* even),

by (2.1) and (3.1), and hence from (3.3) that

(8.3)
$$\left[\iota_{\Sigma E}, \left[\iota_{\Sigma E}, \iota_{\Sigma E}\right]\right] = 0 \qquad (m \text{ even}).$$

Here $\iota_{\Sigma E}$ denotes the ex-homotopy class of the identity on ΣE . Similar results, but under more restrictive conditions, have been obtained by Eggar [4]. It can also be shown that the quadruple Whitehead products

 $\left[\left[\iota_{\Sigma E}, \iota_{\Sigma E}\right], \left[\iota_{\Sigma E}, \iota_{\Sigma E}\right]\right], \left[\iota_{\Sigma E}, \left[\iota_{\Sigma E}, \left[\iota_{\Sigma E}, \iota_{\Sigma E}\right]\right]\right]$

are trivial, whether *m* is even or odd.