Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 23 (1977)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ALTERNATIVE HOMOTOPY THEORIES
Autor: James, I. M.
Kapitel: 6. The exact sequence
DOI: https://doi.org/10.5169/seals-48928

and so $k^{q+1}s \simeq k^{q+1}t$ over X^n. Hence, by induction, we obtain (5.1). Of course the value of r can often be improved in particular cases.

COROLLARY (5.2). *Let E be a fibre bundle over X with locally compact fibre F, which admits a cross-section. Choose a cross-section and so regard E as an ex-space. Let $f : E \to E$ be an ex-map such that $g: F \to F$ is nul-homotopic, where $g = f \mid F$. Then $f^{r+1} \simeq c$, the trivial ex-map, where $r = reg\,(X)$.*

To see this, take $M = M_X(E, E)$, in (5.1), and define $k: M \to M$ by post-composition with f. We take s, t to be the cross-sections $f', e': X \to M$ determined by f, e, and obtain (5.2).

Now let $\alpha, \beta \in \pi_X(E, E)$ be elements such that

(i) $\alpha^2 = \beta^2$ and $\alpha\beta = \beta\alpha$,

(ii) $\Phi_*\alpha = \Phi_*\beta$,

where $\Phi_* : \pi_X(E, E) \to \pi(F, F)$ is given by restriction. Suppose that $E = \Sigma E'$, for some ex-space E', and that $\alpha = \Sigma_*\alpha'$, $\beta = \Sigma_*\beta'$, for some $\alpha', \beta' \in \pi_X(E', E')$. Take f in (5.2) to be a representative of $\alpha - \beta$. Then f^{r+1} is a representative of $(\alpha - \beta)^{r+1} = 2^r(\alpha - \beta)\,\alpha^r$, and so (5.2) shows that

(5.3) $$2^r\alpha = 2^r\beta\,.$$

Applications will be given in §8 below.

6. THE EXACT SEQUENCE

Let X be a CW-complex with basepoint x_0 a 0-cell. Let $p: M \to X$ be a fibration with fibre $N = p^{-1}(x_0)$, and let Γ denote the function-space of cross-sections. By evaluating at x_0 we obtain a fibration $q: \Gamma \to N$. It may be noted that, under fairly general conditions, this fibration admits a cross-section if and only if the original fibration is trivial, in the sense of fibre homotopy type.

Now choose a basepoint $y_0 \in N$ so that $q^{-1}(y_0) = \Gamma_0$, the space of pointed cross-sections. Choose such a cross-section s as basepoint in $\Gamma_0 \subset \Gamma$, and consider the homotopy exact sequence of the fibration as follows:

$$\dots \to \pi_{r+1}(N) \xrightarrow{\Delta} \pi_r(\Gamma_0) \xrightarrow{u_*} \pi_r(\Gamma) \xrightarrow{q_*} \pi_r(N) \to \dots .$$

Note that Γ_0 is a deformation retract of $\tilde{\Gamma}_0$, the space of pointed maps $t: X \to M$ such that $pt \simeq 1$.

In particular, suppose that $X = S^n$ $(n \geqslant 1)$. Then s determines a homotopy equivalence $k: \Omega^n(N) \to \Gamma_0$ as follows. Consider the map $l: \Omega^n(N) \to \tilde{\Gamma}_0$ which transforms each pointed map $f: S^n \to N$ into the track sum $s + jf$, where $j: N \subset M$. Then l is a homotopy equivalence and k is obtained by composing l with a deformation retraction $\tilde{\Gamma}_0 \to \Gamma$. In this case, therefore, we can transform our exact sequence into

$$\ldots \to \pi_{r+1}(N) \xrightarrow{D} \pi_{r+n}(N) \xrightarrow{v_*} \pi_r(\Gamma) \xrightarrow{q_*} \pi_r(N) \to \ldots,$$

where $v_* = u_* k_*$ and $D = k_*^{-1} \Delta$. We refer to this as the *modified exact sequence* of the evaluation fibration. The operator D has been determined in §3 of [8]. Specifically, let $\sigma \in \pi_n(M)$ denote the class of s, and let $\alpha \in \pi_{r+1}(N)$. Then

$$j_* D\alpha = [j_*\alpha, \sigma], \tag{6.1}$$

the Whitehead product in $\pi_*(M)$.

We shall be particularly concerned with the tail end of this sequence, which reads

$$\pi_1(N) \xrightarrow{D} \pi_n(N) \xrightarrow{v_*} \pi(\Gamma) \xrightarrow{q_*} \pi(N).$$

Let $t: S^n \to M$ also be a cross-section. Suppose that we have a path λ in N from $s(x_0)$ to $t(x_0)$. We can regard λ as a vertical homotopy of s into t over $\{x_0\}$. The obstruction to extending this to a vertical homotopy over S^n is an element

$$\delta(s, t; \lambda) \in \pi_n(N).$$

If μ is another path in N from $s(x_0)$ to $t(x_0)$ the track difference $\lambda - \mu$ forms a loop in N and it is easy to check that the homotopy class $\alpha \in \pi_1(N)$ of this loop satisfies the relation

$$D\alpha = \delta(s, t; \lambda) - \delta(s, t; \mu).$$

Hence s and t are vertically homotopic if and only if the obstruction is contained in $D\pi_1(N)$.

Now let E_i $(i = 1, 2)$ be a sectioned bundle over X with locally compact fibre F_i. We can apply the above to the function-space bundle $M = M_X(E_1, E_2)$ with fibre $N = N(F_1, F_2)$, and obtain useful information about the ex-homotopy groups $\pi_X(\Sigma^r E_1, E_2)$ $(r = 1, 2, \ldots)$. Details are given in [9] where the operator D is calculated, as follows, in case E_1 and E_2 are sphere-bundles over $X = S^n$.

Given a representation $\phi: SO(m) \to SO(q)$ write

$$J_\phi = J \circ \phi_*: \pi_r SO(m) \to \pi_{r+q}(S^q),$$

where J denotes the usual Hopf-Whitehead homomorphism. For example, if $q > m$ and ϕ is the inclusion then

$$J_\phi = (-1)^{m-q} \Sigma_*^{m-q} J, \tag{6.2}$$

by (3.2) of [5] (cf. [8]). If $q = 2m$ and $\phi = 1 \oplus 1$ it is easily seen that

$$J_\phi = 2(-1)^m \Sigma_*^m J. \tag{6.3}$$

Consider the function-space $N = N(S^p, S^q)$ of pointed maps $S^p \to S^q$. We identify $\pi_i(N)$ $(i=0, 1, ...)$ with $\pi_{i+p}(S^q)$ in the standard way (see [15]). Let G be a topological group and let

$$\phi: G \to SO(p), \quad \psi: G \to SO(q)$$

be representations of G. We regard S^p, S^q as pointed G-spaces using ϕ, ψ, respectively. Choose a principal G-bundle P over S^n with classifying element $\theta \in \pi_{n-1}(G)$, and take $E_1 = P_\#(S^p)$, $E_2 = P_\#(S^q)$. Then the operator D in our exact sequence is given

$$D\alpha = \alpha \circ \Sigma_*^{r+p-q+1} J_\psi \theta - J_\phi \theta \circ \Sigma_*^{n+p-q-1} \alpha, \tag{6.4}$$

where $\alpha \in \pi_{r+p+1}(S^q)$. The case $r = 1$ of this result will be needed in §8 below.

7. The adjoint G-bundle

Let X be any space and let P be a principal G-bundle over X. We regard P as a (right) G-space in the usual way. By a *principal automorphism* we mean an equivariant fibre-preserving map of P into itself. By the *adjoint G-bundle* we mean the sectioned bundle $Q = P_\# G$, where G acts on itself by conjugation. Note that Q is a group ex-space since G is a group G-space. We can construct Q from $G \times P$ by identifying

$$(gag^{-1}, b) \sim (a, bg) \quad (a \in G, b \in P) \tag{7.1}$$

for all $g \in G$. The group ex-structure is given by

$$\{a_1, b\} \cdot \{a_2, b\} = \{a_1 \cdot a_2, b\} \quad (a_1, a_2 \in G),$$

where $\{\ ,\ \}$ denotes the equivalence class of $(\ ,\ )$. Every principal automorphism f of P determines a cross-section $f': X \to Q$ as follows.