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The Whitehead product theory for ex-spaces has been worked out by Eggar [4]. His definition is such that if  $A, B, Y$  are as in §2 and  $\alpha \in \pi_G(\Sigma A, Y)$ ,  $\beta \in \pi_G(\Sigma B, Y)$  then

$$(4.1) \quad [P_{\#}\alpha, P_{\#}\beta] = P_{\#}[\alpha, \beta]$$

in  $\pi_X(\Sigma(P_{\#}A \wedge P_{\#}B), P_{\#}Y)$ . Since we shall only be concerned with elements in the image of  $P_{\#}$  we can introduce (4.1) as a piece of notation, without going into the details of Eggar's theory.

## 5. THE REGISTER THEOREM

In this section we suppose that  $X$  is a finite simply-connected  $CW$ -complex, although the results obtained can no doubt be generalized. We define the *register*  $\text{reg}(X)$  of  $X$  to be the number of positive integers  $r$  such that, for some abelian group  $A$ , the cohomology group  $H^r(X; A)$  is non-trivial. If  $X$  is a sphere, for example, then  $\text{reg}(X) = 1$ .

Let  $p: M \rightarrow X$  be a fibration with fibre  $N$ . If a cross-section  $s: X \rightarrow M$  exists then  $sp: M \rightarrow M$  is a fibre-preserving map which is constant on the fibre. Conversely if  $k: M \rightarrow M$  is a fibre-preserving map which is nulhomotopic on the fibre then  $M$  admits a cross-section as shown by Noakes [11]. We use similar arguments to prove

**THEOREM (5.1).** *Let  $k: M \rightarrow M$  be a fibre-preserving map such that  $l: N \rightarrow N$  is nulhomotopic, where  $l = k|_N$ , and let  $s, t: X \rightarrow M$  be cross-sections. Then  $k^r s$  and  $k^r t$  are vertically homotopic, where  $r = \text{reg}(X)$ .*

The  $n$ -section ( $n=0, 1, \dots$ ) of the complex  $X$  is denoted by  $X^n$ . Since  $X$  is connected we have a vertical homotopy of  $s$  into  $t$  over  $X^0$ . This starts an induction. Suppose that for some  $n \geq 1$  and some  $q = q(n) \geq 1$  we have a vertical homotopy of  $k^q s$  into  $k^q t$  over  $X^{n-1}$ , so that the separation class

$$d = d(k^q s, k^q t) \in H^n(X; \pi_n(N))$$

is defined. If the cohomology group vanishes then  $d = 0$  and  $k^q s \simeq k^q t$  over  $X^n$ . But in any case the induced endomorphism  $l_*$  of  $\pi_n(N)$  is trivial, by hypothesis, and so  $d$  lies in the kernel of the coefficient endomorphism  $l_{\#}$  determined by  $l_*$ . Therefore

$$d(k^{q+1} s, k^{q+1} t) = l_{\#} d = 0,$$

and so  $k^{q+1}s \simeq k^{q+1}t$  over  $X^n$ . Hence, by induction, we obtain (5.1). Of course the value of  $r$  can often be improved in particular cases.

**COROLLARY (5.2).** *Let  $E$  be a fibre bundle over  $X$  with locally compact fibre  $F$ , which admits a cross-section. Choose a cross-section and so regard  $E$  as an ex-space. Let  $f: E \rightarrow E$  be an ex-map such that  $g: F \rightarrow F$  is null-homotopic, where  $g = f|F$ . Then  $f^{r+1} \simeq c$ , the trivial ex-map, where  $r = \text{reg}(X)$ .*

To see this, take  $M = M_X(E, E)$ , in (5.1), and define  $k: M \rightarrow M$  by post-composition with  $f$ . We take  $s, t$  to be the cross-sections  $f', e': X \rightarrow M$  determined by  $f, e$ , and obtain (5.2).

Now let  $\alpha, \beta \in \pi_X(E, E)$  be elements such that

- (i)  $\alpha^2 = \beta^2$  and  $\alpha\beta = \beta\alpha$ ,
- (ii)  $\Phi_*\alpha = \Phi_*\beta$ ,

where  $\Phi_*: \pi_X(E, E) \rightarrow \pi(F, F)$  is given by restriction. Suppose that  $E = \Sigma E'$ , for some ex-space  $E'$ , and that  $\alpha = \Sigma_*\alpha', \beta = \Sigma_*\beta'$ , for some  $\alpha', \beta' \in \pi_X(E', E')$ . Take  $f$  in (5.2) to be a representative of  $\alpha - \beta$ . Then  $f^{r+1}$  is a representative of  $(\alpha - \beta)^{r+1} = 2^r(\alpha - \beta)\alpha^r$ , and so (5.2) shows that

$$(5.3) \quad 2^r\alpha = 2^r\beta.$$

Applications will be given in §8 below.

## 6. THE EXACT SEQUENCE

Let  $X$  be a CW-complex with basepoint  $x_0$  a 0-cell. Let  $p: M \rightarrow X$  be a fibration with fibre  $N = p^{-1}(x_0)$ , and let  $\Gamma$  denote the function-space of cross-sections. By evaluating at  $x_0$  we obtain a fibration  $q: \Gamma \rightarrow N$ . It may be noted that, under fairly general conditions, this fibration admits a cross-section if and only if the original fibration is trivial, in the sense of fibre homotopy type.

Now choose a basepoint  $y_0 \in N$  so that  $q^{-1}(y_0) = \Gamma_0$ , the space of pointed cross-sections. Choose such a cross-section  $s$  as basepoint in  $\Gamma_0 \subset \Gamma$ , and consider the homotopy exact sequence of the fibration as follows:

$$\dots \rightarrow \pi_{r+1}(N) \xrightarrow{\Delta} \pi_r(\Gamma_0) \xrightarrow{u_*} \pi_r(\Gamma) \xrightarrow{q_*} \pi_r(N) \rightarrow \dots$$

Note that  $\Gamma_0$  is a deformation retract of  $\tilde{\Gamma}_0$ , the space of pointed maps  $t: X \rightarrow M$  such that  $pt \simeq 1$ .