Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 23 (1977)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ALTERNATIVE HOMOTOPY THEORIES

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Kapitel: 5. The register theorem

DOI: https://doi.org/10.5169/seals-48928

The Whitehead product theory for ex-spaces has been worked out by Eggar [4]. His definition is such that if A, B, Y are as in §2 and $\alpha \in \pi_G(\Sigma A, Y)$, $\beta \in \pi_G(\Sigma B, Y)$ then

$$(4.1) \qquad [P_{\#}\alpha, P_{\#}\beta] = P_{\#}[\alpha, \beta]$$

in $\pi_X(\Sigma(P_{\#}A \wedge P_{\#}B), P_{\#}Y)$. Since we shall only be concerned with elements in the image of $P_{\#}$ we can introduce (4.1) as a piece of notation, without going into the details of Eggar's theory.

5. The register theorem

In this section we suppose that X is a finite simply-connected CW-complex, although the results obtained can no doubt be generalized. We define the *register* reg (X) of X to be the number of positive integers r such that, for some abelian group A, the cohomology group $H^r(X; A)$ is non-trivial. If X is a sphere, for example, then reg $(X) = 1$.

Let $p: M \to X$ be a fibration with fibre N. If a cross-section $s: X \to M$ exists then $sp: M \to M$ is a fibre-preserving map which is constant on the fibre. Conversely if $k: M \to M$ is a fibre-preserving map which is nulhomotopic on the fibre then M admits a cross-section as shown by Noakes [11]. We use similar arguments to prove

THEOREM (5.1). *Let* $k: M \to M$ *be a fibre-preserving map such that* $l: N \to N$ *is nulhomotopic, where* $l = k|N$, *and let* $s, t: X \to M$ *be cross-sections. Then* $k^r s$ *and* $k^r t$ *are vertically homotopic, where* $r = reg(X)$.

The n-section ($n = 0, 1, \ldots$) of the complex X is denoted by X^n. Since X is connected we have a vertical homotopy of s into t over X^0. This starts an induction. Suppose that for some $n \geqslant 1$ and some $q = q(n) \geqslant 1$ we have a vertical homotopy of $k^q s$ into $k^q t$ over X^{n-1}, so that the separation class

$$d = d(k^q s, k^q t) \in H^n(X; \pi_n(N))$$

is defined. If the cohomology group vanishes then $d = 0$ and $k^q s \simeq k^q t$ over X^n. But in any case the induced endomorphism l_* of $\pi_n(N)$ is trivial, by hypothesis, and so d lies in the kernel of the coefficient endomorphism $l_{\#}$ determined by l_*. Therefore

$$d(k^{q+1}s, k^{q+1}t) = l_{\#}d = 0,$$

and so $k^{q+1}s \simeq k^{q+1}t$ over X^n. Hence, by induction, we obtain (5.1). Of course the value of r can often be improved in particular cases.

COROLLARY (5.2). *Let E be a fibre bundle over X with locally compact fibre F, which admits a cross-section. Choose a cross-section and so regard E as an ex-space. Let $f : E \to E$ be an ex-map such that $g : F \to F$ is nul-homotopic, where $g = f \mid F$. Then $f^{r+1} \simeq c$, the trivial ex-map, where $r = reg(X)$.*

To see this, take $M = M_X(E, E)$, in (5.1), and define $k: M \to M$ by post-composition with f. We take s, t to be the cross-sections $f', e': X \to M$ determined by f, e, and obtain (5.2).

Now let $\alpha, \beta \in \pi_X(E, E)$ be elements such that

(i) $\alpha^2 = \beta^2$ and $\alpha\beta = \beta\alpha$,

(ii) $\Phi_* \alpha = \Phi_* \beta$,

where $\Phi_* : \pi_X(E, E) \to \pi(F, F)$ is given by restriction. Suppose that $E = \Sigma E'$, for some ex-space E', and that $\alpha = \Sigma_* \alpha'$, $\beta = \Sigma_* \beta'$, for some $\alpha', \beta' \in \pi_X(E', E')$. Take f in (5.2) to be a representative of $\alpha - \beta$. Then f^{r+1} is a representative of $(\alpha - \beta)^{r+1} = 2^r(\alpha - \beta)\alpha^r$, and so (5.2) shows that

$$2^r \alpha = 2^r \beta . \tag{5.3}$$

Applications will be given in §8 below.

6. THE EXACT SEQUENCE

Let X be a CW-complex with basepoint x_0 a 0-cell. Let $p: M \to X$ be a fibration with fibre $N = p^{-1}(x_0)$, and let Γ denote the function-space of cross-sections. By evaluating at x_0 we obtain a fibration $q: \Gamma \to N$. It may be noted that, under fairly general conditions, this fibration admits a cross-section if and only if the original fibration is trivial, in the sense of fibre homotopy type.

Now choose a basepoint $y_0 \in N$ so that $q^{-1}(y_0) = \Gamma_0$, the space of pointed cross-sections. Choose such a cross-section s as basepoint in $\Gamma_0 \subset \Gamma$, and consider the homotopy exact sequence of the fibration as follows:

$$\ldots \to \pi_{r+1}(N) \xrightarrow{\Delta} \pi_r(\Gamma_0) \xrightarrow{u_*} \pi_r(\Gamma) \xrightarrow{q_*} \pi_r(N) \to \ldots .$$

Note that Γ_0 is a deformation retract of $\tilde{\Gamma}_0$, the space of pointed maps $t: X \to M$ such that $pt \simeq 1$.