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The Whitehead product theory for ex-spaces has been worked out by Eggar [4]. His definition is such that if $A$, $B$, $Y$ are as in §2 and $\alpha \in \pi_G(\Sigma A, Y)$, $\beta \in \pi_G(\Sigma B, Y)$ then

$$[P_\# \alpha, P_\# \beta] = P_\# [\alpha, \beta] \tag{4.1}$$

in $\pi_X(\Sigma(P_\# A \wedge P_\# B), P_\# Y)$. Since we shall only be concerned with elements in the image of $P_\#$ we can introduce (4.1) as a piece of notation, without going into the details of Eggar's theory.

## 5. The register theorem

In this section we suppose that $X$ is a finite simply-connected $CW$-complex, although the results obtained can no doubt be generalized. We define the *register* $\operatorname{reg}(X)$ of $X$ to be the number of positive integers $r$ such that, for some abelian group $A$, the cohomology group $H^r(X; A)$ is non-trivial. If $X$ is a sphere, for example, then $\operatorname{reg}(X) = 1$.

Let $p: M \to X$ be a fibration with fibre $N$. If a cross-section $s: X \to M$ exists then $sp: M \to M$ is a fibre-preserving map which is constant on the fibre. Conversely if $k: M \to M$ is a fibre-preserving map which is nulhomotopic on the fibre then $M$ admits a cross-section as shown by Noakes [11]. We use similar arguments to prove

Theorem (5.1). *Let $k: M \to M$ be a fibre-preserving map such that $l: N \to N$ is nulhomotopic, where $l = k|N$, and let $s, t: X \to M$ be cross-sections. Then $k^r s$ and $k^r t$ are vertically homotopic, where $r = \operatorname{reg}(X)$.*

The $n$-section ($n = 0, 1, \ldots$) of the complex $X$ is denoted by $X^n$. Since $X$ is connected we have a vertical homotopy of $s$ into $t$ over $X^0$. This starts an induction. Suppose that for some $n \geqslant 1$ and some $q = q(n) \geqslant 1$ we have a vertical homotopy of $k^q s$ into $k^q t$ over $X^{n-1}$, so that the separation class

$$d = d(k^q s, k^q t) \in H^n(X; \pi_n(N))$$

is defined. If the cohomology group vanishes then $d = 0$ and $k^q s \simeq k^q t$ over $X^n$. But in any case the induced endomorphism $l_*$ of $\pi_n(N)$ is trivial, by hypothesis, and so $d$ lies in the kernel of the coefficient endomorphism $l_\#$ determined by $l_*$. Therefore

$$d(k^{q+1} s, k^{q+1} t) = l_\# d = 0,$$

and so $k^{q+1}s \simeq k^{q+1}t$ over $X^n$. Hence, by induction, we obtain (5.1). Of course the value of $r$ can often be improved in particular cases.

COROLLARY (5.2). *Let $E$ be a fibre bundle over $X$ with locally compact fibre $F$, which admits a cross-section. Choose a cross-section and so regard $E$ as an ex-space. Let $f : E \to E$ be an ex-map such that $g : F \to F$ is nul-homotopic, where $g = f \mid F$. Then $f^{r+1} \simeq c$, the trivial ex-map, where $r = reg\,(X)$.*

To see this, take $M = M_X(E, E)$, in (5.1), and define $k : M \to M$ by post-composition with $f$. We take $s$, $t$ to be the cross-sections $f', e' : X \to M$ determined by $f$, $e$, and obtain (5.2).

Now let $\alpha, \beta \in \pi_X(E, E)$ be elements such that

(i) $\alpha^2 = \beta^2$ and $\alpha\beta = \beta\alpha$,

(ii) $\Phi_*\alpha = \Phi_*\beta$,

where $\Phi_* : \pi_X(E, E) \to \pi(F, F)$ is given by restriction. Suppose that $E = \Sigma E'$, for some ex-space $E'$, and that $\alpha = \Sigma_*\alpha'$, $\beta = \Sigma_*\beta'$, for some $\alpha', \beta' \in \pi_X(E', E')$. Take $f$ in (5.2) to be a representative of $\alpha - \beta$. Then $f^{r+1}$ is a representative of $(\alpha-\beta)^{r+1} = 2^r(\alpha-\beta)\alpha^r$, and so (5.2) shows that

$$2^r\alpha = 2^r\beta\,. \tag{5.3}$$

Applications will be given in §8 below.

## 6. THE EXACT SEQUENCE

Let $X$ be a CW-complex with basepoint $x_0$ a 0-cell. Let $p : M \to X$ be a fibration with fibre $N = p^{-1}(x_0)$, and let $\Gamma$ denote the function-space of cross-sections. By evaluating at $x_0$ we obtain a fibration $q : \Gamma \to N$. It may be noted that, under fairly general conditions, this fibration admits a cross-section if and only if the original fibration is trivial, in the sense of fibre homotopy type.

Now choose a basepoint $y_0 \in N$ so that $q^{-1}(y_0) = \Gamma_0$, the space of pointed cross-sections. Choose such a cross-section $s$ as basepoint in $\Gamma_0 \subset \Gamma$, and consider the homotopy exact sequence of the fibration as follows:

$$\ldots \to \pi_{r+1}(N) \xrightarrow{\Delta} \pi_r(\Gamma_0) \xrightarrow{u_*} \pi_r(\Gamma) \xrightarrow{q_*} \pi_r(N) \to \ldots\,.$$

Note that $\Gamma_0$ is a deformation retract of $\tilde{\Gamma}_0$, the space of pointed maps $t : X \to M$ such that $pt \simeq 1$.