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The Whitehead product theory for ex-spaces has been worked out by Eggar [4]. His definition is such that if A, B, Y are as in §2 and $\alpha \in \pi_G(\Sigma A, Y), \beta \in \pi_G(\Sigma B, Y)$ then

$$(4.1) \quad [P_{\#}\alpha, P_{\#}\beta] = P_{\#}[\alpha, \beta]$$

in $\pi_X(\Sigma(P_{\#}A \wedge P_{\#}B), P_{\#}Y)$. Since we shall only be concerned with elements in the image of $P_{\#}$ we can introduce (4.1) as a piece of notation, without going into the details of Eggar's theory.

5. THE REGISTER THEOREM

In this section we suppose that X is a finite simply-connected CW -complex, although the results obtained can no doubt be generalized. We define the *register* $\text{reg}(X)$ of X to be the number of positive integers r such that, for some abelian group A , the cohomology group $H^r(X; A)$ is non-trivial. If X is a sphere, for example, then $\text{reg}(X) = 1$.

Let $p: M \rightarrow X$ be a fibration with fibre N . If a cross-section $s: X \rightarrow M$ exists then $sp: M \rightarrow M$ is a fibre-preserving map which is constant on the fibre. Conversely if $k: M \rightarrow M$ is a fibre-preserving map which is nulhomotopic on the fibre then M admits a cross-section as shown by Noakes [11]. We use similar arguments to prove

THEOREM (5.1). *Let $k: M \rightarrow M$ be a fibre-preserving map such that $l: N \rightarrow N$ is nulhomotopic, where $l = k|_N$, and let $s, t: X \rightarrow M$ be cross-sections. Then $k^r s$ and $k^r t$ are vertically homotopic, where $r = \text{reg}(X)$.*

The n -section ($n=0, 1, \dots$) of the complex X is denoted by X^n . Since X is connected we have a vertical homotopy of s into t over X^0 . This starts an induction. Suppose that for some $n \geq 1$ and some $q = q(n) \geq 1$ we have a vertical homotopy of $k^q s$ into $k^q t$ over X^{n-1} , so that the separation class

$$d = d(k^q s, k^q t) \in H^n(X; \pi_n(N))$$

is defined. If the cohomology group vanishes then $d = 0$ and $k^q s \simeq k^q t$ over X^n . But in any case the induced endomorphism l_* of $\pi_n(N)$ is trivial, by hypothesis, and so d lies in the kernel of the coefficient endomorphism $l_{\#}$ determined by l_* . Therefore

$$d(k^{q+1} s, k^{q+1} t) = l_{\#} d = 0,$$

and so $k^{q+1}s \simeq k^{q+1}t$ over X^n . Hence, by induction, we obtain (5.1). Of course the value of r can often be improved in particular cases.

COROLLARY (5.2). *Let E be a fibre bundle over X with locally compact fibre F , which admits a cross-section. Choose a cross-section and so regard E as an ex-space. Let $f: E \rightarrow E$ be an ex-map such that $g: F \rightarrow F$ is null-homotopic, where $g = f|F$. Then $f^{r+1} \simeq c$, the trivial ex-map, where $r = \text{reg}(X)$.*

To see this, take $M = M_X(E, E)$, in (5.1), and define $k: M \rightarrow M$ by post-composition with f . We take s, t to be the cross-sections $f', e': X \rightarrow M$ determined by f, e , and obtain (5.2).

Now let $\alpha, \beta \in \pi_X(E, E)$ be elements such that

- (i) $\alpha^2 = \beta^2$ and $\alpha\beta = \beta\alpha$,
- (ii) $\Phi_*\alpha = \Phi_*\beta$,

where $\Phi_*: \pi_X(E, E) \rightarrow \pi(F, F)$ is given by restriction. Suppose that $E = \Sigma E'$, for some ex-space E' , and that $\alpha = \Sigma_*\alpha', \beta = \Sigma_*\beta'$, for some $\alpha', \beta' \in \pi_X(E', E')$. Take f in (5.2) to be a representative of $\alpha - \beta$. Then f^{r+1} is a representative of $(\alpha - \beta)^{r+1} = 2^r(\alpha - \beta)\alpha^r$, and so (5.2) shows that

$$(5.3) \quad 2^r\alpha = 2^r\beta.$$

Applications will be given in §8 below.

6. THE EXACT SEQUENCE

Let X be a CW-complex with basepoint x_0 a 0-cell. Let $p: M \rightarrow X$ be a fibration with fibre $N = p^{-1}(x_0)$, and let Γ denote the function-space of cross-sections. By evaluating at x_0 we obtain a fibration $q: \Gamma \rightarrow N$. It may be noted that, under fairly general conditions, this fibration admits a cross-section if and only if the original fibration is trivial, in the sense of fibre homotopy type.

Now choose a basepoint $y_0 \in N$ so that $q^{-1}(y_0) = \Gamma_0$, the space of pointed cross-sections. Choose such a cross-section s as basepoint in $\Gamma_0 \subset \Gamma$, and consider the homotopy exact sequence of the fibration as follows:

$$\dots \rightarrow \pi_{r+1}(N) \xrightarrow{\Delta} \pi_r(\Gamma_0) \xrightarrow{u_*} \pi_r(\Gamma) \xrightarrow{q_*} \pi_r(N) \rightarrow \dots$$

Note that Γ_0 is a deformation retract of $\tilde{\Gamma}_0$, the space of pointed maps $t: X \rightarrow M$ such that $pt \simeq 1$.