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$$(t_1 a'_1, \dots, t_n a'_n) \quad (a'_i \in A'_i, t_i \in I)$$

where  $t_1^2 + \dots + t_n^2 = 1$ . The radius vector through  $(t_1, \dots, t_n)$  meets the boundary of the  $n$ -cube  $I^n$  in a point  $(x_1, \dots, x_n)$ , say, where at least one coordinate is equal to 1. Thus a pointed  $G$ -homotopy equivalence

$$l: \Sigma(A'_1 * \dots * A'_n) \rightarrow \Sigma A'_1 \wedge \dots \wedge \Sigma A'_n$$

is given by

$$l((t_1 a'_1, \dots, t_n a'_n), s) = ((sx_1, a'_1), \dots, (sx_n, a'_n)).$$

Clearly  $l$  is equivariant with respect to the action of the symmetric group on the suspension of the multiple join and on the multiple smash product.

In particular, take  $G = O(m)$  and  $A'_i = S^{m-1}$ , for all  $i$ . Let  $u$  be a permutation of the multiple join and  $v$  the corresponding permutation of the multiple smash product. We distinguish cases according as to whether the degree of the permutation is even or odd. In the even case  $u$  is  $G$ -homotopic to the identity  $1_n$  on the  $n$ -fold join, using elementary rotations as before, and hence  $v$  is pointed  $G$ -homotopic to the identity  $1_n$  on the  $n$ -fold smash product. In the odd case it follows similarly that  $u$  is  $G$ -homotopic to  $1_{n-1} * a$ , hence  $v$  is pointed  $G$ -homotopic to  $1_{n-1} \wedge \hat{a}$ . Taking  $n = 3$ , therefore, we see that the automorphisms which appear in (2.2) are trivial, in this example, and so

$$(3.3) \quad 3[\Sigma * l_m, [\Sigma * l_m, \Sigma * l_m]] = 0$$

in  $\pi_G(S^{3m+1}, S^{m+1})$ , where  $l_m$  denotes the pointed  $O(m)$ -homotopy class of the identity on  $S^m$ . It is easy to see, incidentally, that the Whitehead square  $[\Sigma * l_m, \Sigma * l_m] \in \pi_G(S^{2m+1}, S^{m+1})$  is of infinite order, for all  $m \geq 2$ .

#### 4. EX-HOMOTOPY THEORY

For our second example of an alternative homotopy theory we take the category of ex-spaces (see [7] for details), which is an enlargement of the category of sectioned bundles mentioned earlier. We recall that, with regard to a given space  $X$ , an *ex-space* consists of a space  $E$  together with maps

$$X \xrightarrow{\sigma} E \xrightarrow{\rho} X$$

such that  $\rho\sigma = 1$ . We refer to  $\rho$  as the *projection*, to  $\sigma$  as the *section*, and to  $(\rho, \sigma)$  as the *ex-structure*. Let  $E_i$  ( $i = 1, 2$ ) be an ex-space with ex-structure

$(\rho_i, \sigma_i)$ . We describe a map  $f: E_1 \rightarrow E_2$  as an *ex-map* if  $f\sigma_1 = \sigma_2$ ,  $\rho_2 f = \rho_1$ , as shown in the following diagram.

$$\begin{array}{ccccc} & \sigma_1 & & E_1 & & \rho_1 \\ & \nearrow & & \downarrow f & & \searrow \\ X & & & & & X \\ & \searrow & & \uparrow & & \nearrow \\ & \sigma_2 & & E_2 & & \rho_2 \end{array}$$

In particular we refer to  $c = \sigma_2 \rho_1$  as the *trivial ex-map*. We also describe a homotopy  $h_t: E_1 \rightarrow E_2$  as an *ex-homotopy* if  $h_t$  is an ex-map throughout. The set of ex-homotopy classes of ex-maps is denoted by  $\pi_X(E_1, E_2)$  and the class of the trivial ex-map by 0.

In particular, suppose that  $E_i$  is a sectioned bundle with locally compact fibre. For each point  $x \in X$  the fibre  $\rho_i^{-1}(x)$  is equipped with basepoint  $\sigma_i(x)$ . Consider the fibre bundle  $M = M_X(E_1, E_2)$  which is formed, in the usual way (see [2]) from the function-spaces of pointed maps  $\rho_1^{-1}(x) \rightarrow \rho_2^{-1}(x)$ . To each ex-map  $f: E_1 \rightarrow E_2$  there corresponds a cross-section  $f': X \rightarrow M$ , where  $f'(x)$  is given by the restriction of  $f$  to the fibre over  $x$ , and conversely every such cross-section determines an ex-map. We shall exploit this correspondence in the next section.

Now let  $P$  be a principal  $G$ -bundle over  $X$ , where  $G$  is a topological group. For any pointed  $G$ -space  $A$  the pointed  $G$ -bundle  $P_{\#}A$  can be regarded as an ex-space, and similarly with pointed  $G$ -maps. Thus  $P_{\#}$  constitutes a functor from the category of pointed  $G$ -spaces to the category of ex-spaces, and determines a function

$$P_{\#}: \pi_G(A_1, A_2) \rightarrow \pi_X(E_1, E_2),$$

where  $A_i$  ( $i = 1, 2$ ) is a pointed  $G$ -space and  $E_i = P_{\#}A_i$ . Of course, in general  $P_{\#}$  is neither injective nor surjective.

As we have seen in § 1 a functor  $F$  in the category of pointed  $G$ -spaces defines a functor  $F$  in the category of sectioned  $G$ -bundles; in many cases such a functor can be extended to the category of ex-spaces. For example, the suspension functor  $\Sigma$  and the loop-space functor  $\Omega$  can be so extended, also the binary functors product  $\times$ , wedge  $\vee$ , and smash  $\wedge$ . Similarly the notions of Hopf ex-space, etc.; can be introduced, following the standard formal procedure, so that  $P_{\#}$  transforms Hopf  $G$ -spaces into Hopf ex-spaces, and so forth. Note that  $\Sigma E$  is cogroup-like and  $\Omega E$  group-like, for any ex-space  $E$ .

The Whitehead product theory for ex-spaces has been worked out by Eggar [4]. His definition is such that if  $A, B, Y$  are as in §2 and  $\alpha \in \pi_G(\Sigma A, Y)$ ,  $\beta \in \pi_G(\Sigma B, Y)$  then

$$(4.1) \quad [P_{\#}\alpha, P_{\#}\beta] = P_{\#}[\alpha, \beta]$$

in  $\pi_X(\Sigma(P_{\#}A \wedge P_{\#}B), P_{\#}Y)$ . Since we shall only be concerned with elements in the image of  $P_{\#}$  we can introduce (4.1) as a piece of notation, without going into the details of Eggar's theory.

## 5. THE REGISTER THEOREM

In this section we suppose that  $X$  is a finite simply-connected  $CW$ -complex, although the results obtained can no doubt be generalized. We define the *register*  $\text{reg}(X)$  of  $X$  to be the number of positive integers  $r$  such that, for some abelian group  $A$ , the cohomology group  $H^r(X; A)$  is non-trivial. If  $X$  is a sphere, for example, then  $\text{reg}(X) = 1$ .

Let  $p: M \rightarrow X$  be a fibration with fibre  $N$ . If a cross-section  $s: X \rightarrow M$  exists then  $sp: M \rightarrow M$  is a fibre-preserving map which is constant on the fibre. Conversely if  $k: M \rightarrow M$  is a fibre-preserving map which is nulhomotopic on the fibre then  $M$  admits a cross-section as shown by Noakes [11]. We use similar arguments to prove

**THEOREM (5.1).** *Let  $k: M \rightarrow M$  be a fibre-preserving map such that  $l: N \rightarrow N$  is nulhomotopic, where  $l = k|_N$ , and let  $s, t: X \rightarrow M$  be cross-sections. Then  $k^r s$  and  $k^r t$  are vertically homotopic, where  $r = \text{reg}(X)$ .*

The  $n$ -section ( $n=0, 1, \dots$ ) of the complex  $X$  is denoted by  $X^n$ . Since  $X$  is connected we have a vertical homotopy of  $s$  into  $t$  over  $X^0$ . This starts an induction. Suppose that for some  $n \geq 1$  and some  $q = q(n) \geq 1$  we have a vertical homotopy of  $k^q s$  into  $k^q t$  over  $X^{n-1}$ , so that the separation class

$$d = d(k^q s, k^q t) \in H^n(X; \pi_n(N))$$

is defined. If the cohomology group vanishes then  $d = 0$  and  $k^q s \simeq k^q t$  over  $X^n$ . But in any case the induced endomorphism  $l_*$  of  $\pi_n(N)$  is trivial, by hypothesis, and so  $d$  lies in the kernel of the coefficient endomorphism  $l_{\#}$  determined by  $l_*$ . Therefore

$$d(k^{q+1} s, k^{q+1} t) = l_{\#} d = 0,$$