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# ALTERNATIVE HOMOTOPY THEORIES [1]

by I. M. JAMES

*Dedicated to Beno Eckmann on the occasion of his 60th birthday*

## 1. INTRODUCTION

Recently there has been considerable interest in the theory of $G$-spaces, where $G$ is a topological group. The purpose of this lecture is to describe some of the work that has been done at Oxford in the past few years, particularly work concerned with equivariant homotopy theory and the associated homotopy theory of spaces over a given space. Little is known about these alternative homotopy theories outside the "stable range". Special emphasis will therefore be placed on non-stable questions, such as the existence of Hopf structures and the Whitehead product theory. Before embarking on this, however, I would like to make a few preliminary remarks.

Let us begin by considering the category of (right) $G$-spaces, where $G$ is a topological group. Both the product $\times$ and the join $*$ are defined in this category. Among the concepts which seem to belong here is that of group $G$-space. We say that a $G$-space $A$ is a *group $G$-space* if $G$ is a topological group with equivariant multiplication $A \times A \to A$. This implies, of course, that inversion is also equivariant and that the neutral element $e$ is a fixed point. Note that $G$ itself constitutes a group $G$-space under the action of conjugation.

Let $f : X \to Y$ be a $G$-map, where $X$ and $Y$ are $G$-spaces. Let $f^H : X^H \to Y^H$ denote the corresponding map of the fixed-point sets, for any subgroup $H \subset G$. Clearly $f^H$ is a homotopy equivalence if $f$ is a $G$-homotopy equivalence. Recently Segal [13] has proved that conversely $f$ is a $G$-homotopy equivalence provided (i) $X$ and $Y$ are $G$-ANR's, (ii) $G$ is a compact Lie group and (iii) $f^H$ is a homotopy equivalence for every closed subgroup $H$ of $G$. This important theorem enables many results of ordinary homotopy theory to be generalized to equivariant homotopy theory.

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1) Presented at the Colloquium on Topology and Algebra, April 1977, Zurich.

In general it is difficult to say very much about the space of equivariant maps. However, the following example, due to George Wilson [19], is instructive [1]). Take $G = SO(q)$, the rotation group, and $A = S^{q-1}$, where $q \geqslant 1$. Let $B$ and $C$ be trivial $G$-spaces. Choose a point $a_0 \in S^{q-1}$ and consider both the cone $TB = \{ a_0 \} * B \subset S^{q-1} * B$ and the suspension $SC = \{ \pm a_0 \} * C \subset S^{q-1} * C$. Every $G$-map $f: S^{q-1}*B \to S^{q-1}*C$ is determined by its values on $TB$. Moreover $f \mid TB$ determines a map $f'$: $(TB, B) \to (SC, C)$, from consideration of the fixed point sets of the isotropy subgroups, and conversely every such map $f'$ determines a $G$-map $f$ by $f(xg) = (f'x)\, g \; (x \in TB, g \in G)$.

Let $P$ be a principal $G$-bundle over a space $X$. To every $G$-space $A$ there is associated a $G$-bundle $P_{\#}A$ with fibre $A$, and similarly with $G$-maps. We refer to $P_{\#}$ as the *principal functor*. Note that trivial $G$-spaces transform into trivial bundles; thus the fixed point set $A^G$ of $A$ transforms into the trivial subbundle $P_{\#}A^G$ of $P_{\#}A$. In particular every fixed point $a \in A$ determines a cross-section $P_{\#}a$ of $P_{\#}A$.

Any self-functor $F$ on the category of $G$-spaces can be extended to the category of $G$-bundles by defining $FP_{\#}A = P_{\#}FA$. Binary functors are treated similarly. Thus the product $\times$ and the join $*$ in the category of $G$-spaces transform, under $P_{\#}$, into the (fibre) product $\times$ and the (fibre) join $*$ in the category of $G$-bundles.

Here is a less familiar example, currently being investigated by my research student Duncan Harvey. Let $A$ be a $G$-space with distinct fixed points $(a_1, ..., a_m)$. For $n = 1, 2, ...$ the configuration space $F_{n,m}A$ of $A$ is defined as the space of $n$-tuples $(x_1, ..., x_n)$ of distinct points in $A - \{ a_1, ..., a_m \}$. Regard $F_{n,m}A$ as a $G$-space with action as in $A^n$. Write $E_{n,m} = P_{\#}F_{n,m}A$, where $E = P_{\#}A$. Then $E_{n,m}$ can be described as the bundle whose fibre over the point $x \in X$ is the configuration space $F_{n,m}E_x$ relative to $(s_1x, ..., s_mx)$, where $s_1 = P_{\#}a_1, ..., s_m = P_{\#}a_m$.

It may happen that a $G$-map is homotopic to the identity but not $G$-homotopic. In that case it is interesting to try and determine the principal $G$-bundles $P$ with the property that the image of the $G$-map under $P_{\#}$ is fibre homotopic to the identity. This question has been studied in [8] for the antipodal map on the $SO(q)$-space $S^{q-1}$ ($q$ even), and we continue this investigation here from a rather different point of view.

In ordinary homotopy theory the advantages of introducing basepoints are well understood. In equivariant homotopy theory the advantages are

[1]) Actually [19] treats the case $q = 3$ when $B$ and $C$ are spheres.

similar. We work with the category of pointed $G$-spaces, i.e. $G$-spaces with fixed point. The suspension functor $\Sigma$ and the loop-space functor $\Omega$ are then defined, also the binary functors wedge $\vee$ and smash $\wedge$. This corresponds, of course, to the category of sectioned $G$-bundles, i.e. $G$-bundles with cross-section, where these functors are also defined and commute with the principal functor. We prefer, however, to enlarge this to the category of ex-spaces — see § 4.

## 2. Equivariant homotopy theory

Let $G$ be a topological group and let $A_i$ $(i=1, 2)$ be a pointed $G$-space. The space of pointed $G$-maps $f: A_1 \to A_2$ is denoted by $M_G(A_1, A_2)$, and the set of pointed $G$-homotopy classes of pointed $G$-maps by $\pi_G(A_1, A_2)$. The class of the constant map $e: A_1 \to A_2$ is denoted by 0. In this context we reserve the symbol $\simeq$ for the relation of pointed $G$-homotopy.

Let $A$ be a pointed $G$-space with base-point $a_0$, and let $p, q: A \to A \times A$ be given by

$$p(x) = (x, a_0), \quad q(x) = (a_0, x) \qquad (x \in A).$$

By a *Hopf G-structure* on $A$ we mean a pointed $G$-map $m: A \times A \to A$ such that

$$mp \simeq 1 \simeq mq: A \to A;$$

given such a structure we refer to $A$ as a *Hopf G-space*. For example, the reduced product space $A_\infty$ (see [5]) of any pointed $G$-space $A$ is an associative Hopf $G$-space [1]. If $A_2$ is a Hopf $G$-space then $\pi_G(A_1, A_2)$, for any pointed $G$-space $A_1$, obtains a natural binary operation with 0 as neutral element [2].

If $m: A \times A \to A$ satisfies the conditions for a topological group then, as before, we describe $A$ as a group $G$-space. If $m$ satisfies these conditions up to pointed $G$-homotopy then we describe $A$ as a *group-like G-space.* Note that $\pi_G(A_1, A_2)$ is a group when $A_2$ is group-like. This is so, in particular, when $A_2 = \Omega A_2'$ with standard Hopf $G$-structure for any pointed $G$-space $A_2'$. If $A_2'$ itself is a Hopf $G$-space then the group is abelian, by the usual argument.

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[1]) Under suitable conditions it can be shown, using Segal's theorem, that $A_\infty$ has the same pointed $G$-homotopy type as $\Omega\Sigma A$.

[2]) Another application of Segal's theorem is to show, following Sugawara [14], that this binary set forms a loop, under suitable conditions, and hence a group when the Hopf $G$-structure is pointed $G$-homotopy associative.

The notions of *coHopf G-space* and *cogroup-like G-space* are defined in the obvious way. If $A_1$ is a coHopf $G$-space then $\pi_G(A_1, A_2)$, for any pointed $G$-space $A_2$, obtains a binary operation with 0 as neutral element; moreover $\pi_G(A_1, A_2)$ is a group when $A_1$ is cogroup-like. In particular $\Sigma A_1$ is cogroup-like, for any pointed $G$-space $A_1$, and the adjoint functor determines an isomorphism

$$\xi\colon \pi_G(\Sigma A_1, A_2) \to \pi_G(A_1, \Omega A_2).$$

Using this we can define Whitehead products in equivariant homotopy theory as follows.

We say that a pointed $G$-space $X$ is *well-based* if there exists a neighbourhood $U$ of the basepoint $x_0$ in $X$ such that

(i) $x_0$ is an equivariant deformation retract, rel $x_0$, of $U$, and

(ii) there exists an invariant map $u\colon X \to I$ such that $ux_0 = 1$ and $ux = 0$ for $x \notin U$.

Let $A$, $B$ be well-based $G$-spaces. Then (cf. [12]) for any pointed $G$-space $Y$ the equivariant form of the Puppe sequence

$$0 \to \pi_G(A \wedge B, \Omega Y) \overset{p^*}{\to} \pi_G(A \times B, \Omega Y) \overset{q^*}{\to} \pi_G(A \vee B, \Omega Y) \to 0$$

is exact. Here $q$ denotes the inclusion and $p$ the collapsing map. From given elements $\alpha' \in \pi_G(A, \Omega Y)$, $\beta' \in \pi_G(B, \Omega Y)$ we can obtain $\alpha'', \beta'' \in \pi_G(A \times B, \Omega Y)$ by precomposition with the structural maps of the product. Since the commutator $\alpha''^{-1} . \beta''^{-1} . \alpha'' . \beta''$ lies in the kernel of $q^*$ there exists, by exactness, a (unique) element

$$<\alpha', \beta'> \in \pi_G(A \wedge B, \Omega Y)$$

with image this commutator. The *Samelson pairing*

$$\pi_G(A, \Omega Y) \times \pi_G(B, \Omega Y) \to \pi_G(A \wedge B, \Omega Y)$$

thus defined is bilinear, just as in [1], and has the property that

$$<\alpha', \beta'> \;=\; - T^* <\beta', \alpha'>,$$

where $T$ denotes the switching map. The *Whitehead product* $[\alpha, \beta]$ of elements $\alpha \in \pi_G(\Sigma A, Y)$, $\beta \in \pi_G(\Sigma B, Y)$ is defined by

$$\xi[\alpha, \beta] \;=\; <\xi\alpha, \xi\beta>,$$

where $\xi$ denotes the adjoint isomorphism. Clearly the Whitehead pairing

$$\pi_G(\Sigma A, Y) \times \pi_G(\Sigma B, Y) \to \pi_G(\Sigma(A \wedge B), Y)$$

thus defined is bilinear and has the property that

(2.1) $$[\alpha, \beta] = -(\Sigma T)^* [\beta, \alpha] .$$

It is a straightforward exercise, as in the ordinary theory, to show that the Whitehead pairing vanishes if $Y$ is a Hopf $G$-space, and hence vanishes under suspension. Moreover the suspension $\Sigma Z$ of a compact well-based $G$-space $Z$ is a Hopf $G$-space if and only if the Whitehead square

$$w(\Sigma Z) \in \pi_G(\Sigma(Z \wedge Z), \Sigma Z)$$

of the identity vanishes.

It should also be noted that the Jacobi identity holds for Samelson products and hence for Whitehead products, by an equivariant version of the argument given by G. W. Whitehead [16]. Specifically, consider the permutations

$$B \wedge C \wedge A \xrightarrow{\sigma} A \wedge B \wedge C \xrightarrow{\tau} C \wedge B \wedge A ,$$

where $A$, $B$, $C$ are suspensions of pointed $G$-spaces. Let

$$\alpha \in \pi_G(\Sigma A, Y) , \quad \beta \in \pi_G(\Sigma B, Y) , \quad \gamma \in \pi_G(\Sigma C, Y) ,$$

where $Y$ is a pointed $G$-space. Then the relation

(2.2) $$[\alpha, [\beta, \gamma]] + (\Sigma\sigma)^* [\beta, [\gamma, \alpha]] + (\Sigma\tau)^* [\gamma, [\alpha, \beta]] = 0$$

holds in the group $\pi_G(\Sigma(A \wedge B \wedge C), Y)$.

## 3. Some examples

We need to begin by discussing briefly some relations between the category of $G$-spaces and the category of pointed $G$-spaces, as follows. Given spaces $A$, $B$ we denote points of the join $A*B$ by triples $(a, b, t)$ where $a \in A$, $b \in B$, $t \in I$, so that $(a, b, t)$ is independent of $a$ when $t = 0$, of $b$ when $t = 1$. A basepoint $b_0 \in B$ determines a basepoint $(a, b_0, 0)$ in $A*B$. If $A$, $B$ are $G$-spaces we make $A*B$ a $G$-space with action

$$(a, b, t) g = (ag, bg, t) \qquad (g \in G) .$$

Note that $A*B$ is pointed if $B$ is. When $B = S^0$, with trivial action, then $A*B = \tilde{\Sigma} A$, the unreduced suspension [2]).

[1]) This differs by an automorphism from the normal definition.
[2]) We regard this as an identification space of the cylinder, in the usual way.

When $A$ is a pointed $G$-space the reduced suspension $\Sigma A$ is also defined and the natural projection $\tilde{\Sigma} A \to \Sigma A$ is a pointed $G$-homotopy equivalence if $A$ is well-based. Moreover, if $A$ and $B$ are well-based the natural projection $A^*B \to \Sigma(A \wedge B)$ is also a pointed $G$-homotopy equivalence. We need a variant of this result.

Suppose now that $A = \tilde{\Sigma} A'$, $B = \tilde{\Sigma} B'$, where $A'$, $B'$ are $G$-spaces. Consider the pointed $G$-map

$$k: \tilde{\Sigma}(A'^*B') \to (\tilde{\Sigma} A' \wedge \tilde{\Sigma} B') = A \wedge B$$

which is given by the formula

$$k((a', b', t), s) = \begin{cases} ((a', 2st), (b', s)) & (0 \leqslant t \leqslant \tfrac{1}{2}), \\ ((a', s), (b', s(2-2t))) & (\tfrac{1}{2} \leqslant t \leqslant 1). \end{cases}$$

It is easy to check that $k$ is a pointed $G$-homotopy equivalence, under similar conditions, and has the property that

(3.1) $$k \tilde{\Sigma} S = Tk,$$

where $S: A^*B \to B^*A$ denotes the switching map of the join and $T: A \wedge B \to B \wedge A$ the switching map of the smash product.

In particular, take $G$ to be the group $O(m)$ $(m \geqslant 2)$ of orthogonal transformations of the sphere $S^{m-1}$. The antipodal map $a: S^{m-1} \to S^{m-1}$ is an $O(m)$-map, hence $\hat{a} = \tilde{\Sigma} a: S^m \to S^m$ is a pointed $O(m)$-map. I assert that

(3.2) $$T \simeq \hat{a} \wedge 1: S^m \wedge S^m \to S^m \wedge S^m.$$

For the switching self-map $S$ of $S^{m-1} * S^{m-1}$ is $O(m)$-homotopic to $a^*1$, by elementary rotation in $R^{2m} = R^m \times R^m$. Hence $\tilde{\Sigma} S$ is pointed $O(m)$-homotopic to $\tilde{\Sigma}(a^*1)$. From (3.1), therefore, we obtain that

$$Tk = k \tilde{\Sigma} S \simeq k \tilde{\Sigma}(a^*1) = (\hat{a} \wedge 1) k.$$

Since $k$ is a pointed $O(m)$-homotopy equivalence this proves (3.2). In this case, therefore, $T$ can be replaced by $\tilde{a} \wedge 1$ in the commutation law (2.1).

We now turn to the permutations appearing in (2.2), the Jacobi identity. More generally let $A_i'$ $(i = 1, \ldots, n)$ be a $G$-space. Points of the multiple join $A_1' * \ldots * A_n'$ can be represented by $n$-tuples of the form

$$(t_1a_1', \ldots, t_na_n') \qquad (a_i' \in A_i', t_i \in I)$$

where $t_1^2 + \ldots + t_n^2 = 1$. The radius vector through $(t_1, \ldots, t_n)$ meets the boundary of the $n$-cube $I^n$ in a point $(x_1, \ldots, x_n)$, say, where at least one coordinate is equal to 1. Thus a pointed $G$-homotopy equivalence

$$l\colon \Sigma(A_1' * \ldots * A_n') \to \Sigma A_1' \wedge \ldots \wedge \Sigma A_n'$$

is given by

$$l\left((t_1a_1', \ldots, t_na_n'), s\right) = \left((sx_1, a_1'), \ldots, (sx_n, a_n')\right).$$

Clearly $l$ is equivariant with respect to the action of the symmetric group on the suspension of the multiple join and on the multiple smash product.

In particular, take $G = O(m)$ and $A_i' = S^{m-1}$, for all $i$. Let $u$ be a permutation of the multiple join and $v$ the corresponding permutation of the multiple smash product. We distinguish cases according as to whether the degree of the permutation is even or odd. In the even case $u$ is $G$-homotopic to the identity $1_n$ on the $n$-fold join, using elementary rotations as before, and hence $v$ is pointed $G$-homotopic to the identity $1_n$ on the $n$-fold smash product. In the odd case it follows similarly that $u$ is $G$-homotopic to $1_{n-1}*a$, hence $v$ is pointed $G$-homotopic to $1_{n-1} \wedge \hat{a}$. Taking $n = 3$, therefore, we see that the automorphisms which appear in (2.2) are trivial, in this example, and so

$$3\left[\Sigma_* \iota_m, [\Sigma_* \iota_m, \Sigma_* \iota_m]\right] = 0 \tag{3.3}$$

in $\pi_G(S^{3m+1}, S^{m+1})$, where $\iota_m$ denotes the pointed $O(m)$-homotopy class of the identity on $S^m$. It is easy to see, incidentally, that the Whitehead square $[\Sigma_*\iota_m, \Sigma_*\iota_m] \in \pi_G(S^{2m+1}, S^{m+1})$ is of infinite order, for all $m \geqslant 2$.

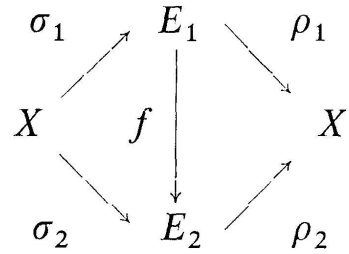
## 4. Ex-homotopy theory

For our second example of an alternative homotopy theory we take the category of ex-spaces (see [7] for details), which is an enlargement of the category of sectioned bundles mentioned earlier. We recall that, with regard to a given space $X$, an *ex-space* consists of a space $E$ together with maps

$$X \xrightarrow{\sigma} E \xrightarrow{\rho} X$$

such that $\rho\sigma = 1$. We refer to $\rho$ as the *projection*, to $\sigma$ as the *section*, and to $(\rho, \sigma)$ as the *ex-structure*. Let $E_i$ $(i = 1, 2)$ be an ex-space with ex-structure

$(\rho_i, \sigma_i)$. We describe a map $f: E_1 \to E_2$ as an *ex-map* if $f\sigma_1 = \sigma_2, \rho_2 f = \rho_1$, as shown in the following diagram.




In particular we refer to $c = \sigma_2 \rho_1$ as the *trivial ex-map*. We also describe a homotopy $h_t: E_1 \to E_2$ as an *ex-homotopy* if $h_t$ is an ex-map throughout. The set of ex-homotopy classes of ex-maps is denoted by $\pi_X(E_1, E_2)$ and the class of the trivial ex-map by 0.

In particular, suppose that $E_i$ is a sectioned bundle with locally compact fibre. For each point $x \in X$ the fibre $\rho_i^{-1}(x)$ is equipped with basepoint $\sigma_i(x)$. Consider the fibre bundle $M = M_X(E_1, E_2)$ which is formed, in the usual way (see [2]) from the function-spaces of pointed maps $\rho_1^{-1}(x) \to \rho_2^{-1}(x)$. To each ex-map $f: E_1 \to E_2$ there corresponds a cross-section $f': X \to M$, where $f'(x)$ is given by the restriction of $f$ to the fibre over $x$, and conversely every such cross-section determines an ex-map. We shall exploit this correspondence in the next section.

Now let $P$ be a principal $G$-bundle over $X$, where $G$ is a topological group. For any pointed $G$-space $A$ the pointed $G$-bundle $P_{\#}A$ can be regarded as an ex-space, and similarly with pointed $G$-maps. Thus $P_{\#}$ constitutes a functor from the category of pointed $G$-spaces to the category of ex-spaces, and determines a function

$$P_{\#}: \pi_G(A_1, A_2) \to \pi_X(E_1, E_2),$$

where $A_i$ $(i = 1, 2)$ is a pointed $G$-space and $E_i = P_{\#}A_i$. Of course, in general $P_{\#}$ is neither injective nor surjective.

As we have seen in § 1 a functor $F$ in the category of pointed $G$-spaces defines a functor $F$ in the category of sectioned $G$-bundles; in many cases such a functor can be extended to the category of ex-spaces. For example, the suspension functor $\Sigma$ and the loop-space functor $\Omega$ can be so extended, also the binary functors product $\times$, wedge $\vee$, and smash $\wedge$. Similarly the notions of Hopf ex-space, etc.; can be introduced, following the standard formal procedure, so that $P_{\#}$ transforms Hopf $G$-spaces into Hopf ex-spaces, and so forth. Note that $\Sigma E$ is cogroup-like and $\Omega E$ group-like, for any ex-space $E$.

The Whitehead product theory for ex-spaces has been worked out by Eggar [4]. His definition is such that if $A$, $B$, $Y$ are as in §2 and $\alpha \in \pi_G(\Sigma A, Y)$, $\beta \in \pi_G(\Sigma B, Y)$ then

$$[P_{\#}\alpha, P_{\#}\beta] = P_{\#}[\alpha, \beta] \tag{4.1}$$

in $\pi_X(\Sigma(P_{\#}A \wedge P_{\#}B), P_{\#}Y)$. Since we shall only be concerned with elements in the image of $P_{\#}$ we can introduce (4.1) as a piece of notation, without going into the details of Eggar's theory.

## 5. The register theorem

In this section we suppose that $X$ is a finite simply-connected $CW$-complex, although the results obtained can no doubt be generalized. We define the *register* $\operatorname{reg}(X)$ of $X$ to be the number of positive integers $r$ such that, for some abelian group $A$, the cohomology group $H^r(X; A)$ is non-trivial. If $X$ is a sphere, for example, then $\operatorname{reg}(X) = 1$.

Let $p: M \to X$ be a fibration with fibre $N$. If a cross-section $s: X \to M$ exists then $sp: M \to M$ is a fibre-preserving map which is constant on the fibre. Conversely if $k: M \to M$ is a fibre-preserving map which is nulhomotopic on the fibre then $M$ admits a cross-section as shown by Noakes [11]. We use similar arguments to prove

THEOREM (5.1). *Let $k: M \to M$ be a fibre-preserving map such that $l: N \to N$ is nulhomotopic, where $l = k|N$, and let $s, t: X \to M$ be cross-sections. Then $k^r s$ and $k^r t$ are vertically homotopic, where $r = \operatorname{reg}(X)$.*

The $n$-section ($n=0, 1, \ldots$) of the complex $X$ is denoted by $X^n$. Since $X$ is connected we have a vertical homotopy of $s$ into $t$ over $X^0$. This starts an induction. Suppose that for some $n \geqslant 1$ and some $q = q(n) \geqslant 1$ we have a vertical homotopy of $k^q s$ into $k^q t$ over $X^{n-1}$, so that the separation class

$$d = d(k^q s, k^q t) \in H^n(X; \pi_n(N))$$

is defined. If the cohomology group vanishes then $d = 0$ and $k^q s \simeq k^q t$ over $X^n$. But in any case the induced endomorphism $l_*$ of $\pi_n(N)$ is trivial, by hypothesis, and so $d$ lies in the kernel of the coefficient endomorphism $l_{\#}$ determined by $l_*$. Therefore

$$d(k^{q+1}s, k^{q+1}t) = l_{\#}d = 0,$$

and so $k^{q+1}s \simeq k^{q+1}t$ over $X^n$. Hence, by induction, we obtain (5.1). Of course the value of $r$ can often be improved in particular cases.

COROLLARY (5.2). *Let $E$ be a fibre bundle over $X$ with locally compact fibre $F$, which admits a cross-section. Choose a cross-section and so regard $E$ as an ex-space. Let $f: E \to E$ be an ex-map such that $g: F \to F$ is nulhomotopic, where $g = f \mid F$. Then $f^{r+1} \simeq c$, the trivial ex-map, where $r = reg(X)$.*

To see this, take $M = M_X(E, E)$, in (5.1), and define $k: M \to M$ by post-composition with $f$. We take $s$, $t$ to be the cross-sections $f', e': X \to M$ determined by $f$, $e$, and obtain (5.2).

Now let $\alpha, \beta \in \pi_X(E, E)$ be elements such that

(i) $\alpha^2 = \beta^2$ and $\alpha\beta = \beta\alpha$,

(ii) $\Phi_*\alpha = \Phi_*\beta$,

where $\Phi_*: \pi_X(E, E) \to \pi(F, F)$ is given by restriction. Suppose that $E = \Sigma E'$, for some ex-space $E'$, and that $\alpha = \Sigma_*\alpha'$, $\beta = \Sigma_*\beta'$, for some $\alpha', \beta' \in \pi_X(E', E')$. Take $f$ in (5.2) to be a representative of $\alpha - \beta$. Then $f^{r+1}$ is a representative of $(\alpha - \beta)^{r+1} = 2^r(\alpha - \beta)\alpha^r$, and so (5.2) shows that

$$2^r\alpha = 2^r\beta . \tag{5.3}$$

Applications will be given in §8 below.

## 6. THE EXACT SEQUENCE

Let $X$ be a CW-complex with basepoint $x_0$ a 0-cell. Let $p: M \to X$ be a fibration with fibre $N = p^{-1}(x_0)$, and let $\Gamma$ denote the function-space of cross-sections. By evaluating at $x_0$ we obtain a fibration $q: \Gamma \to N$. It may be noted that, under fairly general conditions, this fibration admits a cross-section if and only if the original fibration is trivial, in the sense of fibre homotopy type.

Now choose a basepoint $y_0 \in N$ so that $q^{-1}(y_0) = \Gamma_0$, the space of pointed cross-sections. Choose such a cross-section $s$ as basepoint in $\Gamma_0 \subset \Gamma$, and consider the homotopy exact sequence of the fibration as follows:

$$\ldots \to \pi_{r+1}(N) \xrightarrow{\Delta} \pi_r(\Gamma_0) \xrightarrow{u_*} \pi_r(\Gamma) \xrightarrow{q_*} \pi_r(N) \to \ldots .$$

Note that $\Gamma_0$ is a deformation retract of $\tilde{\Gamma}_0$, the space of pointed maps $t: X \to M$ such that $pt \simeq 1$.

In particular, suppose that $X = S^n$ $(n \geqslant 1)$. Then $s$ determines a homotopy equivalence $k: \Omega^n(N) \to \Gamma_0$ as follows. Consider the map $l: \Omega^n(N) \to \tilde{\Gamma}_0$ which transforms each pointed map $f: S^n \to N$ into the track sum $s + jf$, where $j: N \subset M$. Then $l$ is a homotopy equivalence and $k$ is obtained by composing $l$ with a deformation retraction $\tilde{\Gamma}_0 \to \Gamma$. In this case, therefore, we can transform our exact sequence into

$$\ldots \to \pi_{r+1}(N) \xrightarrow{D} \pi_{r+n}(N) \xrightarrow{v_*} \pi_r(\Gamma) \xrightarrow{q_*} \pi_r(N) \to \ldots,$$

where $v_* = u_* k_*$ and $D = k_*^{-1} \Delta$. We refer to this as the *modified exact sequence* of the evaluation fibration. The operator $D$ has been determined in §3 of [8]. Specifically, let $\sigma \in \pi_n(M)$ denote the class of $s$, and let $\alpha \in \pi_{r+1}(N)$. Then

$$(6.1) \qquad j_* D\alpha = [j_* \alpha, \sigma],$$

the Whitehead product in $\pi_*(M)$.

We shall be particularly concerned with the tail end of this sequence, which reads

$$\pi_1(N) \xrightarrow{D} \pi_n(N) \xrightarrow{v_*} \pi(\Gamma) \xrightarrow{q_*} \pi(N).$$

Let $t: S^n \to M$ also be a cross-section. Suppose that we have a path $\lambda$ in $N$ from $s(x_0)$ to $t(x_0)$. We can regard $\lambda$ as a vertical homotopy of $s$ into $t$ over $\{x_0\}$. The obstruction to extending this to a vertical homotopy over $S^n$ is an element

$$\delta(s, t; \lambda) \in \pi_n(N).$$

If $\mu$ is another path in $N$ from $s(x_0)$ to $t(x_0)$ the track difference $\lambda - \mu$ forms a loop in $N$ and it is easy to check that the homotopy class $\alpha \in \pi_1(N)$ of this loop satisfies the relation

$$D\alpha = \delta(s, t; \lambda) - \delta(s, t; \mu).$$

Hence $s$ and $t$ are vertically homotopic if and only if the obstruction is contained in $D\pi_1(N)$.

Now let $E_i$ $(i=1, 2)$ be a sectioned bundle over $X$ with locally compact fibre $F_i$. We can apply the above to the function-space bundle $M = M_X(E_1, E_2)$ with fibre $N = N(F_1, F_2)$, and obtain useful information about the ex-homotopy groups $\pi_X(\Sigma^r E_1, E_2)$ $(r=1, 2, \ldots)$. Details are given in [9] where the operator $D$ is calculated, as follows, in case $E_1$ and $E_2$ are sphere-bundles over $X = S^n$.

Given a representation $\phi: SO(m) \to SO(q)$ write

$$J_\phi = J \circ \phi_*: \pi_r SO(m) \to \pi_{r+q}(S^q),$$

where $J$ denotes the usual Hopf-Whitehead homomorphism. For example, if $q > m$ and $\phi$ is the inclusion then

$$J_\phi = (-1)^{m-q} \Sigma_*^{m-q} J, \tag{6.2}$$

by (3.2) of [5] (cf. [8]). If $q = 2m$ and $\phi = 1 \oplus 1$ it is easily seen that

$$J_\phi = 2(-1)^m \Sigma_*^m J. \tag{6.3}$$

Consider the function-space $N = N(S^p, S^q)$ of pointed maps $S^p \to S^q$. We identify $\pi_i(N)$ $(i=0, 1, ...)$ with $\pi_{i+p}(S^q)$ in the standard way (see [15]). Let $G$ be a topological group and let

$$\phi: G \to SO(p), \quad \psi: G \to SO(q)$$

be representations of $G$. We regard $S^p$, $S^q$ as pointed $G$-spaces using $\phi$, $\psi$, respectively. Choose a principal $G$-bundle $P$ over $S^n$ with classifying element $\theta \in \pi_{n-1}(G)$, and take $E_1 = P_\#(S^p)$, $E_2 = P_\#(S^q)$. Then the operator $D$ in our exact sequence is given

$$D\alpha = \alpha \circ \Sigma_*^{r+p-q+1} J_\psi \theta - J_\phi \theta \circ \Sigma_*^{n+p-q-1} \alpha, \tag{6.4}$$

where $\alpha \in \pi_{r+p+1}(S^q)$. The case $r = 1$ of this result will be needed in §8 below.

## 7. The adjoint $G$-bundle

Let $X$ be any space and let $P$ be a principal $G$-bundle over $X$. We regard $P$ as a (right) $G$-space in the usual way. By a *principal automorphism* we mean an equivariant fibre-preserving map of $P$ into itself. By the *adjoint G-bundle* we mean the sectioned bundle $Q = P_\# G$, where $G$ acts on itself by conjugation. Note that $Q$ is a group ex-space since $G$ is a group $G$-space. We can construct $Q$ from $G \times P$ by identifying

$$(gag^{-1}, b) \sim (a, bg) \quad (a \in G, b \in P) \tag{7.1}$$

for all $g \in G$. The group ex-structure is given by

$$\{a_1, b\} \cdot \{a_2, b\} = \{a_1 \cdot a_2, b\} \quad (a_1, a_2 \in G),$$

where $\{\ ,\ \}$ denotes the equivalence class of $(\ ,\ )$. Every principal automorphism $f$ of $P$ determines a cross-section $f': X \to Q$ as follows.

Given $x \in X$ choose any $b \in P_x$; then $fb = bg$, for some $g \in G$, and we define $f'x = \{ g, b \}$. This correspondence establishes an isomorphism between the group of principal automorphisms of $P$ and the group of cross-sections of $Q$.

Any element $c$ of the centre of $G$ determines a $G$-map $c_{\#}$ for any $G$-space $A$. Notice that $c_{\#}$ is a principal automorphism in the case of $P$ and that the corresponding cross-section $c'_{\#}$ of $Q$ is given by $c'_{\#} \{ b \} = \{ c, b \}$. When $X$ is a sphere these central cross-sections of $Q$ can be analysed as follows.

Take $X = S^n$ $(n \geqslant 2)$, so that $P$ is a principal $G$-bundle over $S^n$. Let $B^n$ denote the $n$-ball with boundary $S^{n-1}$. Choose a relative homeomorphism $(B^n, S^{n-1}) \to (S^n, x_0)$ and lift this to a map $k: (B^n, S^{n-1}) \to (P, G)$. The homotopy class $\theta \in \pi_{n-1}(G)$ of $l = k | S^{n-1}$ classifies the bundle according to clutching theory.

Let $c \in G$ be central and let $\lambda: I \to G$ be a path such that $\lambda(0) = e$, $\lambda(1) = c$. Consider the map $\Lambda: B^n \times I \to Q$ which is given by

$$\Lambda(y, t) = \{ \lambda(t), k(y) \} \qquad (y \in B^n, t \in I).$$

The boundary of $B^n \times I$ is the sphere

$$B^n \times 0 \cup S^{n-1} \times I \cup B^n \times 1,$$

and $\Lambda$ maps $S^{n-1} \times I$ into $G \subset Q$ by

$$\Lambda(y, t) = (ly) . (\lambda t) . (ly)^{-1},$$

using (7.1). Let us compare this with the map $\Lambda'$ of the boundary which agrees with $\Lambda$ on $B^n \times \dot{I}$ but is given on $S^{n-1} \times I$ by $\Lambda'(y, t) = \lambda t$. Now $\lambda$ can be regarded as a vertical homotopy of $e'_{\#}$ into $c'_{\#}$ over $\{ x_0 \}$ and $\Lambda$ represents the obstruction

$$\delta = \delta(e'_{\#}, c'_{\#}; \lambda) \in \pi_n(G)$$

to extending this vertical homotopy over $S^n$. Since $\Lambda \mid (B^n \times I)^{\cdot}$ is nul-homotopic, however, it follows that $\delta$ is also represented by $d: \tilde{\Sigma} S^{n-1} \to G$, where

$$d(y, t) = (ly) . (\lambda t) . (ly)^{-1}.$$

For example, take $G = SO(m)$, with $m$ even. Take $c = -e$ and

$$\lambda(t) = e \cos \pi t + b \sin \pi t \qquad (0 \leqslant t \leqslant 1),$$

where $b$ denotes the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (m/2 \quad \text{summands}).$$

Then $\delta = F\theta$, by definition, where

$$F : \pi_{n-1}\, SO(m) \to \pi_n SO(m)$$

denotes the Bott suspension, as in [6].

Now let $A_i$ $(i = 1, 2)$ be a locally compact pointed $G$-space and write $E_i = P_{\#} A_i$. Recall that $N = N(A_1, A_2)$ denotes the function-space of pointed maps $A_1 \to A_2$. Given a pointed $G$-map $f : A_1 \to A_2$ we can construct an ex-map $P_{\#} f : E_1 \to E_2$ and a pointed $G$-map $\bar{f} : G \to N$, where $\bar{f}(g) = g_{\#} \circ f = f \circ g_{\#}$. I assert

PROPOSITION (7.2). *The ex-maps*

$$P_{\#} f, P_{\#} f \circ P_{\#} c : E_1 \to E_2$$

*are ex-homotopic if and only if*

$$\bar{f}^* \delta \in D\pi_1(N) \subset \pi_n(N),$$

*where* $\delta$ *is as above.*

Here $D$ is the operator which occurs in the modified exact sequence of the evaluation fibration derived from the function-space bundle, as in §6. The proof of (7.2) is by naturality, as follows.

First observe that $\bar{f}$ extends to a fibre-preserving map $\hat{f} : Q \to M$, where $M = M_X(E_1, E_2)$ denotes the function-space bundle. To see this we note that $f$ determines a pointed $G$-map $F : A_1 \times G \to A_2$, where

$$F(x, g) = f(xg) \qquad (x \in A_1, g \in G).$$

Hence $P_{\#} f : E_1 \times Q \to E_2$ is defined and we take $\hat{f}$ to be the adjoint.

We have $X = S^n$ so that the evaluation fibrations can be modified as in §6. Clearly

$$\Gamma_0(\hat{f}) \circ k \simeq l \circ \Omega^n(\bar{f}) \tag{7.3}$$

as shown below, where $k$ is defined by subtracting the cross-section $e'_{\#}$ and $l$ by subtracting $\hat{f} \circ e'_{\#}$.

$$\begin{array}{ccc} \Omega^n(G) & \xrightarrow{\;k\;} & \Gamma_0(Q) \\ {\scriptstyle \Omega^n(\bar{f})}\downarrow & & \downarrow{\scriptstyle \Gamma_0(\hat{f})} \\ \Omega^n(N) & \xrightarrow[\;l\;]{} & \Gamma_0(M) \end{array}$$

Hence we obtain a commutative diagram as follows, relating the modified exact sequences for $Q$ and $M$.

$$\begin{array}{ccc} \pi_n(G) & \xrightarrow{u_*} & \pi(\Gamma(Q)) \\ \bar{f}_* \downarrow & & \downarrow (\Gamma(\hat{f}))_* \\ \pi_n(N) & \xrightarrow[v_*]{} & \pi(\Gamma(M)) \end{array}$$

Recall that $\delta$ is the obstruction to extending $\lambda$ to a vertical homotopy of $e'_{\#}$ into $c'_{\#}$. Hence $\bar{f}_*\delta$ is the obstruction to extending $\bar{f} \circ \lambda$ to a vertical homotopy of $\hat{f} \circ e'_{\#}$ into $\hat{f} \circ c'_{\#}$. Hence it follows, as explained in the previous section, that $\hat{f} \circ e'_{\#}$ and $\hat{f} \circ c'_{\#}$ are vertically homotopic if and only if $\delta \in D\pi_1(N)$. Finally we use the correspondence between ex-maps and cross-sections to obtain (7.2) as stated.

## 8. Examples

Let $X$ be a finite simply-connected complex and let $P$ be a principal $SO(m)$-bundle over $X$. Consider the antipodal self-map $a$ of $S^{m-1}$. The unreduced suspension $\hat{a}$ is a pointed $SO(m)$-map of $S^m$ into itself. Hence $P_{\#}\hat{a}$ is an ex-map of $E = P_{\#}S^m$ into itself; let $\sigma \in \pi_X(E, E)$ denote the ex-homotopy class. Since $\hat{a}$ is of degree $(-1)^m$ we can apply (5.3) and obtain that

(8.1) $$2^r \Sigma_* \sigma = 2^r \qquad (m \text{ even}),$$

where $r = \text{reg}(X)$. It follows at once that

(8.2) $$2^{r+1}[\iota_{\Sigma E}, \iota_{\Sigma E}] = 0 \qquad (m \text{ even}),$$

by (2.1) and (3.1), and hence from (3.3) that

(8.3) $$[\iota_{\Sigma E}, [\iota_{\Sigma E}, \iota_{\Sigma E}]] = 0 \qquad (m \text{ even}).$$

Here $\iota_{\Sigma E}$ denotes the ex-homotopy class of the identity on $\Sigma E$. Similar results, but under more restrictive conditions, have been obtained by Eggar [4]. It can also be shown that the quadruple Whitehead products

$$[[\iota_{\Sigma E}, \iota_{\Sigma E}], [\iota_{\Sigma E}, \iota_{\Sigma E}]], [\iota_{\Sigma E}, [\iota_{\Sigma E}, [\iota_{\Sigma E}, \iota_{\Sigma E}]]]$$

are trivial, whether $m$ is even or odd.

In particular, let $X$ be a sphere. For $m$ even (8.1) shows that $2\Sigma_*\sigma = 2$ and (8.2) that $4\,[\iota_{\Sigma E}, \iota_{\Sigma E}] = 0$. However, more precise results can be obtained by using the methods of §7, as follows. Take $X = S^n$ ($n \geqslant 2$) and let $\theta \in \pi_{n-1}\, SO\,(m)$ be the classifying element of $P$. We apply (7.2) with $f = 1$ and, using (6.4), obtain

THEOREM (8.4). *Let $m$ be even. Then $\Sigma_*\sigma = 1$ if and only if $\Sigma_* JF\theta$ is contained in the image of*

$$D: \pi_{m+2}(S^{m+1}) \to \pi_{n+m+1}(S^{m+1}),$$

*where*

$$D\alpha = \Sigma_* J\,\theta \circ \Sigma_*^{n-1}\alpha - \alpha \circ \Sigma_*^2 J\,\theta\,.$$

In the stable range, where $m > n$, the homomorphism $D$ is trivial and $F\theta = \theta \circ \eta$, as in §6 of [6], where $\eta$ generates the 1-stem. In this range it does not matter whether we deal with ex-maps or over-maps, and so (8.4) agrees with (4.5) of [8].

Now let $\iota_m$ denote the pointed $SO\,(m)$-homotopy class of the identity on $S^m$, so that $\iota_{\Sigma E} = P_\# \Sigma_* \iota_m$. Represent the Whitehead square

$$w\,(\Sigma S^m) = [\Sigma_* \iota_m, \Sigma_* \iota_m]$$

by a pointed $SO\,(m)$-map $f: \Sigma\,(S^m \wedge S^m) \to \Sigma S^m$. Then $P_\# f$ represents the Whitehead square

$$w\,(\Sigma E) = P_\# w\,(\Sigma S^m) = [\iota_{\Sigma E}, \iota_{\Sigma E}]\,.$$

We apply (7.2) again and, using (6.2)-(6.4), obtain

THEOREM (8.5). *Let $m$ be even. Then $2w\,(\Sigma E) = 0$ if and only if $w_{m+1} \circ \Sigma_*^{m+1} JF\theta$ lies in the image of*

$$D: \pi_{2m+2}(S^{m+1}) \to \pi_{n+2m+1}(S^{m+1}),$$

*where* $D\alpha = \alpha \circ \Sigma_*^{m+2} J\theta - 2\Sigma^* J\theta \circ \Sigma_*^{n-1}\alpha.$

Here $w_{m+1} \in \pi_{2m+1}(S^{m+1})$ denotes the ordinary Whitehead square of the generator of $\pi_{m+1}(S^{m+1})$. Unless $m = 2$ or $6$ we have $w_{m+1} \neq 0$ and (8.5) determines the order of $w\,(\Sigma E)$. When $m = 2$ or $6$ it would be interesting to know when $w\,(\Sigma E) = 0$, i.e. when $\Sigma E$ is a Hopf ex-space, but unfortunately our method does not apply.

For some examples where the order of $w\,(\Sigma E)$ is (precisely) 4, consider the transgression $\Delta: \pi_n(S^m) \to \pi_{n-1}\, SO\,(m)$ in the homotopy exact sequence of the standard fibration of $SO(m+1)$. Take $\theta = \Delta\phi$, where $\phi \in \Sigma_* \pi_{n-1}(S^{m-1})$. Then $\Sigma_* J\theta = 0$ and so $D$ is trivial. However it follows from (4.1) and (6.3) of [6] that

$$\Sigma_* JF\theta = [\Sigma_* \iota_m, \Sigma_* \phi] .$$

This Whitehead product is non-zero if, for example, $m = n$ and $\phi = \iota_m$ with $m \neq 2, 6$. Of course $E$ is trivial as a bundle, in these examples, although not as a sectioned bundle.

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