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Artikel: SOME CONTRIBUTIONS OF BENO ECKMANN TO THE DEVELOPMENT OF TOPOLOGY AND RELATED FIELDS

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5. Simple homotopy type

Our final example of Eckmann's work brings us very much into the modern era. It will be understood why I have eschewed the temptation to deal with his very extensive contributions between 1958 and 1973 in detail. However, in 1970, he published a paper with Serge Maumary [70], dedicated to Georges de Rham, to which it will surely repay us to give some attention.

In 1950 Henry Whitehead recast in algebraic terms the theory of simple homotopy types which he had introduced many years earlier. In this theory one considers the collection of finite simplicial complexes [1]) homotopically equivalent to a given complex X and introduces the finer classification of *simple* equivalence into this collection. Whitehead showed that a single invariant sufficed to classify homotopy equivalences modulo simple equivalences; this invariant is now known as *Whitehead torsion* and is an element of an abelian group constructed functorially out of $\pi_1 X$. The importance of this theory has come to be recognized more clearly in recent years with the rise of differential topology and algebraic K-theory; indeed, the Whitehead group Wh π, to which the Whitehead torsion belongs ($\pi = \pi_1 X$) is a quotient of $K_1 \pi$. Chapman has proved the topological invariance of Whitehead torsion using the techniques of infinite-dimensional topology.

In [70] Eckmann and Maumary give a purely geometric description of the Whitehead group Wh X. They start with the classical description of a simple equivalence $s\colon Y \to Y'$ between finite cell complexes, as a sequence of elementary expansions and contractions. For a given finite cell complex X, they then introduce an equivalence relation into the family of maps emanating from X with target a finite cell complex. Thus $f\colon X \to Y$ and $f'\colon X \to Y'$ are equivalent if there exists a simple equivalence $s\colon Y \to Y'$ such that $sf \simeq f'$. Let $A(X)$ be the set of equivalence classes thus defined. One may introduce a binary operation into $A(X)$ by means of the *homotopy push-out*—here we adopt a description equivalent to but not identical with that of [70]. Thus let $g\colon X \to Y$, $h\colon X \to Z$ be maps which we may assume cellular. Replace one or both of g, h by cofibrations; this may be done by the mapping cylinder construction. Thus, assume that g, h are cofibrations and construct the topological push-out of g, h—this will be the double mapping cylinder if g, h are embeddings in mapping cylinders:

[1]) There is no problem in replacing simplicial complexes by (finite) CW-complexes.

(5.1)
$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle u} \\ Z & \xrightarrow[v]{} & M \end{array}$$

Then if $< g >$ stands for the class of g, etc., we define, from (5.1),

(5.2)
$$< g > + < h > \; = \; < ug >$$

Next one makes A a functor to sets, also by means of the homotopy push-out: if $f: X \to X'$ we take a cofibration g in the class $< g >$ and define

(5.3)
$$f_* < g > \; = \; < \bar{g} >$$

from the push-out

(5.4)
$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ X' & \xrightarrow[\bar{g}]{} & Y' \end{array}$$

The fact that f_* is a homomorphism with respect to the addition (5.2) is proved by purely categorical arguments; so too is the fact that $A\,(X)$ is an abelian monoid under the addition (5.2); of course the zero of $A\,(X)$ is $< 1_X >$.

Now let $E(X)$ be the subset of $A\,(X)$ consisting of classes $< g >$ such that g is a homotopy equivalence. It is plain that $E\,(X)$ is a submonoid; however, it is actually an abelian *group*. First, E is a functor, that is, $f_* EX \subseteq EX'$ if $f: X \to X'$. Second, let $g: X \to Y$ be a homotopy equivalence with homotopy inverse $g': Y \to X$ and set $< h > \; = g'_* < g' >$, $h: X \to Z$, so that $< h > \in EX$. Then from (5.1)-(5.4) it follows that

$$< g > + < h > \; = \; < g' g > \; = \; < 1_X > \; = 0 \, .$$

We have thus defined a functor $E: P \to Ab$, where P is the category of finite cell-complexes and Ab is the category of abelian groups. Finally, Eckmann and Maumary prove that EX depends only on the 2-skeleton of X; a general argument then enables them to deduce that E factors through the fundamental group functor.

The gain in conceptual simplicity achieved by this geometric viewpoint is substantial; of course, the hard calculations remain to be done to compute the Whitehead group. One may compare the achievement of this paper with that of [34; 1953], in which Eckmann and Schopf produce a very significant simplification and clarification of the concept of injective hull of a module and a very easy, natural proof of its existence (first proved by Reinhold Baer); or with a very recent paper [81; 1976], in which Eckmann gave a remarkably simple proof of the Dyer-Vasquez theorem that the complement of a higher-dimensional knot $S^{n-2} \subseteq S^n$, $n \geqslant 4$, is never aspherical unless the knot group is infinite cyclic (thus, if $n \geqslant 5$, unless the knot is unknotted).

The story goes on. I have on my desk the latest manuscript, a joint paper by Eckmann and Bieri, completed in the spring of 1977, entitled "Relative Homology and Poincaré duality for group pairs". As I have said, Beno Eckmann remains active and effective—but more is true. The Eckmann touch remains as sure as ever!

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