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Artikel: REPRESENTATION OF COMPLETELY CONVEX FUNCTIONS BY THE EXTREME-POINT METHOD

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Assume that $f \in W$ generates an extreme ray. If $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \geqq 0$ we already know by Proposition 2.4 that f is proportional to $\sin(\pi x)$. Otherwise let n be the smallest number $\geqq 0$ for which $f^{(2n)}(0) \neq 0$ or $f^{(2n)}(1) \neq 0$. By Proposition 2.2 we then have

$$f(x) = (-1)^n f^{(2n)}(0)\, \Lambda_n^*(x) + (-1)^n f^{(2n)}(1)\, \Lambda_n(x) + R_{n+1}(x),$$

but since f lies on an extreme ray all three terms on the right-hand side lie on this ray.

If $f^{(2n)}(0) \neq 0$ this shows that $(-1)^n f^{(2n)}(1) \Lambda_n$ and R_{n+1} are proportional to Λ_n^*. Therefore $f^{(2n)}(1) = 0$ and $R_{n+1}^{(2n+2)} = f^{(2n+2)}$ is proportional to $(\Lambda_n^*)^{(2n+2)} = 0$, so that $f^{(2n+2)} = 0$ and hence $R_{n+1} = 0$ (cf. Proposition 2.2).

If $f^{(2n)}(1) \neq 0$ we similarly get $f^{(2n)}(0) = 0$ and $R_{n+1} = 0$. This shows that f lies on the ray generated by either Λ_n^* or Λ_n.

3. Determination of a base for W

There are several ways of determining a base for W. We choose the following set

$$B = \left\{ f \in W \mid \int_0^1 f(x) \sin(\pi x)\, dx = 1 \right\}.$$

By Lemma 1.2 (ii) we get for $f \in B$ and $x_0 \in]0, 1[$ that

$$1 \geqq \frac{1}{\pi} f(x_0) \int_0^1 \sin^2(\pi x)\, dx = \frac{1}{2\pi} f(x_0),$$

so the functions in B are uniformly bounded by 2π.

It is therefore clear that B is a compact convex base for W.

The extreme points of B are exactly the intersections between B and the extreme rays of W. We see that $2 \sin(\pi x) \in B$.

We claim that the following formulas hold, cf. [4]:

$$(3.1) \quad \Lambda_n^*(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geqq 0, \ x \in]0, 1[,$$

$$(3.2) \quad \Lambda_n(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geqq 0, \ x \in]0, 1[.$$

Formula (3.2) follows immediately from (3.1). For $n = 0$ (3.1) is the familiar formula

$$\frac{\pi}{2}(1-x) = \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k}, \quad 0 < x < 1 .$$

Suppose that (3.1) holds for n replaced by $n - 1$ for some $n \geqq 1$. Denoting the right-hand side of (3.1) by f_n, we have $f_n(0) = f_n(1) = 0$ and

$$f_n''(x) = -\frac{2}{\pi^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k^{2n-1}} .$$

which is equal to $-\Lambda_{n-1}^*$ by the induction hypothesis. It follows by (2.3) that $f_n = \Lambda_n^*$, and (3.1) is proved. From (3.1) and (3.2) it follows that $\pi^{2n+1}\Lambda_n$ and $\pi^{2n+1}\Lambda_n^* \in B$. We also get $\lim_{n\to\infty} \pi^{2n+1}\Lambda_n(x) = \lim_{n\to\infty} \pi^{2n+1}\Lambda_n^*(x)$ $= 2\sin(\pi x)$. We have now established the following result:

PROPOSITION 3.1. *The set* B *is a compact convex base for* W *and the extreme points of* B *are* $2\sin(\pi x)$, $\pi^{2n+1}\Lambda_n^*(x)$, $\pi^{2n+1}\Lambda_n(x)$, $n \geqq 0$, *which form a closed subset of* B.

By l_+^1 we denote the set of sequences $(\alpha_n)_{n\geqq 0}$ of non-negative numbers such that $\sum_0^{\infty} \alpha_n < \infty$.

By the Choquet representation theorem or just by the Krein-Milman theorem we get the following, cf. [3]:

THEOREM 3.2. *For every* $f \in W$ *there exist* $a \geqq 0$ *and sequences* $(\alpha_n), (\beta_n) \in l_+^1$ *such that*

$$f(x) = 2a\sin(\pi x) + \sum_{n=0}^{\infty} \alpha_n \pi^{2n+1}\Lambda_n^*(x) + \sum_{n=0}^{\infty} \beta_n \pi^{2n+1}\Lambda_n(x); \quad 0 < x < 1 . \tag{3.1}$$

The functions in B are uniformly bounded by 2π, and therefore the series (3.1) is uniformly convergent.

If we differentiate the series in (3.1) two times and change sign we get the series

$$\pi^2\left(2a\sin(\pi x) + \sum_{n=0}^{\infty} \alpha_{n+1}\pi^{2n+1}\Lambda_n^*(x) + \sum_{n=0}^{\infty} \beta_{n+1}\pi^{2n+1}\Lambda_n(x)\right),$$

which also converges uniformly on]0, 1[because $\sum_{n=0}^{\infty} \alpha_{n+1} + \sum_{n=0}^{\infty} \beta_{n+1} < \infty$.

It follows that the following formula holds:

$$(3.2) \quad (-1)^k f^{(2k)}(x) = \pi^{2k} \Big(2a \sin(\pi x) + \sum_{n=0}^{\infty} \alpha_{n+k} \pi^{2n+1} \Lambda_n^*(x) + \sum_{n=0}^{\infty} \beta_{n+k} \pi^{2n+1} \Lambda_n(x) \Big)$$

for $0 < x < 1$, $k \geqq 0$ and furthermore

$$(3.3) \quad \alpha_k = \pi^{-2k-1}(-1)^k f^{(2k)}(0), \quad \beta_k = \pi^{-2k-1}(-1)^k f^{(2k)}(1) \quad \text{for} \quad k \geqq 0.$$

This proves that the sequences (α_n), (β_n) and hence also a are uniquely determined by f. We have thus shown that B is a simplex. The extreme points of B form a closed subset of B as remarked in Proposition 3.1 so we can formulate the following

COROLLARY 3.3. *The base B for W is a Bauer simplex.*

Whittaker proved in [4] that the series in (3.1) in fact converges uniformly over arbitrary compact subsets of the complex plane. This also proves that f can be extended to an entire holomorphic function which we also call f. For $x \in]0, 1[$ and $y \in \mathbf{R}$ we then have

$$f(x+iy) = \sum_{k=0}^{\infty} f^{(k)}(x) \frac{(iy)^k}{k!},$$

hence

$$\operatorname{Re} f(x+iy) = \sum_{k=0}^{\infty} (-1)^k f^{(2k)}(x) \frac{y^{2k}}{(2k)!},$$

which shows that $x \mapsto \operatorname{Re} f(x+iy)$ belongs to W for all $y \in \mathbf{R}$, as sum of the functions

$$x \mapsto (-1)^k f^{2k}(x) \frac{y^{2k}}{(2k)!}$$

which all belong to the closed cone W.

This gives a short proof of the recent result of Mugler [2].