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2. DETERMINATION OF THE EXTREME RAYS OF W

Inspired by [6] we consider the Green's function

$$G(x, t) = \begin{cases} (1-x)t & \text{for } 0 \leq t < x \leq 1, \\ (1-t)x & \text{for } 0 \leq x \leq t \leq 1. \end{cases}$$

If φ is a continuous function on $[0, 1]$ the unique solution $f \in C([0, 1]) \cap C^2(]0, 1[)$ to the equations

$$(2.1) \quad f'' = -\varphi \text{ in }]0, 1[, \quad f(0) = f(1) = 0$$

is

$$(2.2) \quad f(x) = \int_0^1 G(x, t) \varphi(t) dt.$$

The successive iterates of G are defined for $x, t \in [0, 1]$ by the equations

$$G_1(x, t) = G(x, t)$$

$$G_n(x, t) = \int_0^1 G(x, y) G_{n-1}(y, t) dy, \quad n \geq 2.$$

It is clear that $G_n(x, t) \geq 0$ for $x, t \in [0, 1]$.

We define recursively a sequence of polynomials $(A_n)_{n \geq 0}^1$ by the requirement

$$(2.3) \quad A_0(x) = x, \quad A_n'' = -A_{n-1} \quad \text{and} \quad A_n(0) = A_n(1) = 0$$

for $n \geq 1$.

The polynomial A_n is of degree $(2n + 1)$, and we clearly have

$$(2.4) \quad A_n(x) = \int_0^1 G(x, t) A_{n-1}(t) dt = \int_0^1 G_n(x, t) t dt \quad \text{for}$$

$n \geq 1, x \in [0, 1].$

It follows that $A_n \geq 0$ on $[0, 1]$ for all n , and since $(-1)^k A_n^{(2k)} = A_{n-k}$ for $k \leq n$ we see that $A_n \in W$.

We recall that a ray \mathbf{R}_+x of a cone C is called *extreme*, if an equation $x = f + g$ with $f, g \in C$ is possible only if $f, g \in \mathbf{R}_+x$, cf. [3].

¹) Our terminology is different from that of [6]; $(-1)^n A_n$ is equal to the n 'th Lidstone polynomial of [4] and [6].

PROPOSITION 2.1. *The polynomials $A_n, n \geq 0$, lie on extreme rays of W .*

Proof. If $A_0 = f + g$ with $f, g \in W$ we have $0 = f'' + g''$, but since f'' and g'' are both ≤ 0 , we conclude that f and g are affine. Furthermore, since $f(0) = g(0) = 0$, we conclude that f and g are proportional to A_0 .

Suppose now that $A_{n-1}, n \geq 1$, lies on an extreme ray of W , and assume that $A_n = f + g$ where $f, g \in W$. Then $A_{n-1} = -f'' + (-g'')$, and the induction hypothesis implies that $-f''$ and $-g''$ are proportional to A_{n-1} . Therefore we have $f = \lambda A_n(x) + ax + b$ for certain numbers $\lambda \geq 0, a, b$. Since $0 \leq f \leq A_n$, we have $f(0) = f(1) = 0$ which implies that $a = b = 0$. This proves that f (and similarly g) are proportional to A_n which then lies on an extreme ray of W .

Since $f \mapsto f^*$ is an affine isomorphism of W the polynomials A_n^* also lie on extreme rays of W . The following result is a special case of [6], Theorem 1.1.

PROPOSITION 2.2. *Every function $f \in W$ can for $n \geq 1$ be written as*

$$f(x) = \sum_{k=0}^{n-1} ((-1)^k f^{(2k)}(0) A_k^*(x) + (-1)^k f^{(2k)}(1) A_k(x)) + R_n(x),$$

where

$$R_n(x) = \int_0^1 G_n(x, t) (-1)^n f^{(2n)}(t) dt \in W.$$

Proof. For $n = 1$ the formula is equivalent with

$$(2.5) \quad f(x) - f(0)(1-x) - f(1)x = R_1(x) = - \int_0^1 G(x, t) f''(t) dt,$$

which follows directly from (2.2), and it is clear that $R_1 \in W$.

Suppose now the formula holds for some $n \geq 1$. Applying (2.5) to $(-1)^n f^{(2n)} \in W$ we get

$$\begin{aligned} (-1)^n f^{(2n)}(x) &= (-1)^n f^{(2n)}(0) A_0^*(x) + (-1)^n f^{(2n)}(1) A_0(x) \\ &+ \int_0^1 G(x, t) (-1)^{n+1} f^{(2n+2)}(t) dt, \end{aligned}$$

which substituted in the expression for R_n yields the formula for $n+1$ because of (2.4).

To see that $R_n \in W$ we notice that

$$(-1)^k R_n^{(2k)}(x) = \begin{cases} \int_0^1 G_{n-k}(x, t) (-1)^n f^{(2n)}(t) dt & \text{for } 0 \leq k \leq n-1 \\ (-1)^k f^{(2k)}(x) & \text{for } k \geq n. \end{cases}$$

The following lemma is easy to establish, but instead of giving the proof here we refer to [6].

LEMMA 2.3. *There exists a constant $M > 0$ such that*

$$0 \leq \int_0^1 G_n(x, t) dt \leq \frac{M}{\pi^{2n}} \quad \text{for} \quad 0 \leq x \leq 1, \quad n \geq 1.$$

PROPOSITION 2.4. *The only functions $f \in W$ satisfying $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \geq 0$ are $f(x) = a \sin(\pi x)$ with $a \geq 0$.*

Proof. Suppose $f \in W$ satisfies $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \geq 0$. Defining $a = \sup \{ \alpha \geq 0 \mid f - \alpha \sin(\pi x) \in W \}$, $g = f - a \sin(\pi x)$ belongs to W because W is closed in \mathbf{R}^I . Furthermore

$$g^{(2k)}(0) = g^{(2k)}(1) = 0 \quad \text{for all } k \geq 0.$$

Let $\varepsilon > 0$ be given. Since $\varphi = g - \varepsilon \sin(\pi x) \notin W$, there exist $k \geq 0$ and $x_0 \in]0, 1[$ such that $(-1)^k \varphi^{(2k)}(x_0) < 0$, hence

$$(-1)^k g^{(2k)}(x_0) < \varepsilon \pi^{2k} \sin(\pi x_0).$$

By Lemma 1.2 (iii) applied to $(-1)^k g^{(2k)}$ we get

$$(-1)^k g^{(2k)}(t) \leq \varepsilon \pi^{2k+1} \quad \text{for} \quad 0 < t < 1,$$

and therefore by Proposition 2.2 and Lemma 2.3 for $0 < x < 1$

$$\begin{aligned} g(x) &= \int_0^1 G_k(x, t) (-1)^k g^{(2k)}(t) dt \leq \varepsilon \pi^{2k+1} \int_0^1 G_k(x, t) dt \\ &\leq \varepsilon M \pi. \end{aligned}$$

This proves that g is identically zero.

PROPOSITION 2.5. *The extreme rays of W are precisely the rays generated by Λ_n and Λ_n^* , where $n \geq 0$, and $\sin(\pi x)$.*

Proof. We first show that $\sin(\pi x)$ lies on an extreme ray. If $\sin(\pi x) = f + g$ where $f, g \in W$, we have $f(0) = f(1) = g(0) = g(1) = 0$. Differentiating $2k$ times we similarly get $f^{(2k)}(0) = f^{(2k)}(1) = g^{(2k)}(0) = g^{(2k)}(1) = 0$, and it follows by Proposition 2.4 that f and g are proportional to $\sin(\pi x)$.

We finally have to show that an arbitrary extreme ray is generated by one of the above functions.

Assume that $f \in W$ generates an extreme ray. If $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \geq 0$ we already know by Proposition 2.4 that f is proportional to $\sin(\pi x)$. Otherwise let n be the smallest number ≥ 0 for which $f^{(2n)}(0) \neq 0$ or $f^{(2n)}(1) \neq 0$. By Proposition 2.2 we then have

$$f(x) = (-1)^n f^{(2n)}(0) A_n^*(x) + (-1)^n f^{(2n)}(1) A_n(x) + R_{n+1}(x),$$

but since f lies on an extreme ray all three terms on the right-hand side lie on this ray.

If $f^{(2n)}(0) \neq 0$ this shows that $(-1)^n f^{(2n)}(1) A_n$ and R_{n+1} are proportional to A_n^* . Therefore $f^{(2n)}(1) = 0$ and $R_{n+1}^{(2n+2)} = f^{(2n+2)}$ is proportional to $(A_n^*)^{(2n+2)} = 0$, so that $f^{(2n+2)} = 0$ and hence $R_{n+1} = 0$ (cf. Proposition 2.2).

If $f^{(2n)}(1) \neq 0$ we similarly get $f^{(2n)}(0) = 0$ and $R_{n+1} = 0$. This shows that f lies on the ray generated by either A_n^* or A_n .

3. DETERMINATION OF A BASE FOR W

There are several ways of determining a base for W . We choose the following set

$$B = \left\{ f \in W \mid \int_0^1 f(x) \sin(\pi x) dx = 1 \right\}.$$

By Lemma 1.2 (ii) we get for $f \in B$ and $x_0 \in]0, 1[$ that

$$1 \geq \frac{1}{\pi} f(x_0) \int_0^1 \sin^2(\pi x) dx = \frac{1}{2\pi} f(x_0),$$

so the functions in B are uniformly bounded by 2π .

It is therefore clear that B is a compact convex base for W .

The extreme points of B are exactly the intersections between B and the extreme rays of W . We see that $2 \sin(\pi x) \in B$.

We claim that the following formulas hold, cf. [4]:

$$(3.1) \quad A_n^*(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geq 0, \quad x \in]0, 1[,$$

$$(3.2) \quad A_n(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geq 0, \quad x \in]0, 1[.$$