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REPRESENTATION OF COMPLETELY CONVEX FUNCTIONS BY THE EXTREME-POINT METHOD

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0. INTRODUCTION

A function $f:]0, 1[\to \mathbf{R}$ is called completely convex, if it is C^∞ and $(-1)^k f^{(2k)} \geqq 0$ for all $k \geqq 0$. A completely convex function f is called minimal if $f(x) - a \sin(\pi x)$ is not completely convex for any number $a > 0$. Widder showed (cf. [5]) that a completely convex function can be extended to an entire holomorphic function, and in the paper [6] he proved that a minimal completely convex function can be expanded in a Lidstone series. This indicates that the Lidstone polynomials lie on the extreme rays of the cone W of completely convex functions.

The purpose of the present paper is to treat the completely convex functions by the extreme-point method and to obtain the expansion in Lidstone series as a special case of the Choquet representation theorem.

We will proceed as follows: In the topology of point-wise convergence the set W of completely convex functions is a closed, metrizable convex cone. We prove directly that the extreme rays of W are generated by certain polynomials — essentially the Lidstone polynomials — and the function $\sin(\pi x)$. The occurrence of the extreme ray generated by $\sin(\pi x)$ is related to the fact that only minimal completely convex functions can be expanded in Lidstone series.

The cone W has a compact convex base B, and the extreme points of B are determined. It turns out that B is a Bauer simplex, i.e. B is a simplex and the extreme points form a closed set.

The author wants to acknowledge Widder's paper [6] as a source of inspiration. The reason for writing this paper is that we felt it natural to investigate the cone W by the extreme-point method.

Recently Mugler [2] showed that real part of the holomorphic extension of $f \in W$ to the strip $\mathrm{Re}\, z \in]0, 1[$ is completely convex on each segment $\{x + iy \mid 0 < x < 1\}$. We give a very short proof of this result.

1. Completely convex functions

Let I denote an open interval. A function $f : I \to \mathbf{R}$ is called *completely convex*, if it is C^∞ and $(-1)^k f^{(2k)} \geqq 0$ on I for $k \geqq 0$.

The set of completely convex functions is a convex cone denoted $W = W(I)$. We always equip W with the topology of pointwise convergence, i.e., with the topology induced by the product space $\mathbf{R}^I$.

Lemma 1.1. *If I is unbounded $W(I)$ consists of the non-negative affine functions, and $W(\mathbf{R})$ consists of the non-negative constants.*

Proof. Assume first that $\inf I = -\infty$. Then every $f \in W$ is decreasing since it is non-negative and concave. For $k \geqq 0$ and $f \in W$ we have $(-1)^k f^{(2k)} \in W$ and consequently $(-1)^k f^{(2k+1)} \leqq 0$. This shows that also $-f' \in W$ and then $-f'' \leqq 0$, but by definition $f'' \leqq 0$ and therefore f is affine.

The case $\sup I = \infty$ is treated in a similar manner. Finally, every non-negative concave function on $\mathbf{R}$ is constant.

Remark. Completely convex sequences are non-negative and affine.

For a sequence $a = (a_0, a_1, \ldots)$ of real numbers we define Δa to be the sequence $(\Delta a)_n = a_{n+1} - a_n$, $n \geqq 0$, and $\Delta^k a$ is defined as $\Delta(\Delta^{k-1} a)$ for $k \geqq 1$, where $\Delta^0 a = a$. A sequence a is called *completely convex* if $(-1)^k \Delta^{2k} a \geqq 0$ for $k \geqq 0$. The same method as in Lemma 1.1 leads to the conclusion that every completely convex sequence satisfies $\Delta a \geqq 0$ and $\Delta^2 a = 0$. The completely convex sequences are therefore exactly the sequences $a_n = \alpha n + \beta$, where $\alpha, \beta \geqq 0$.

This is an answer to a remark by Boas [1]: "Nothing seems to be known about completely convex sequences".

In the following we will always assume that I is bounded, and for the sake of convenience we choose I to be $I =]0, 1[$. We simply write W for $W(]0, 1[)$. For $f \in W$ we have $-f'' \in W$ and $f^* \in W$, where f^* is defined by $f^*(x) = f(1-x)$. The mapping $f \mapsto f^*$ is an affine isomorphism of W onto itself.

Lemma 1.2. *Let $f :]0, 1[\to \mathbf{R}$ be non-negative and concave. Then the following holds:*

(i) $f(x) \leqq 2f(\tfrac{1}{2})$ *for* $x \in]0, 1[$.

(ii) $f(x) \geqq \dfrac{1}{\pi} f(x_0) \sin(\pi x)$ *for* $x, x_0 \in]0, 1[$.

(iii) ([6], Lemma 7.1) *If there exists* $x_0 \in]0, 1[$ *and* $a > 0$ *such that* $f(x_0) < a \sin(\pi x_0)$ *then* $f(x) \leqq a\pi$ *for* $x \in]0, 1[$.

Proof. (i). For $x \in]0, \frac{1}{2}]$ we have that $f(x)$ lies below the line through $(\frac{1}{2}, f(\frac{1}{2}))$ and $(1, 0)$ and (i) follows for $x \in]0, \frac{1}{2}]$. The interval $[\frac{1}{2}, 1[$ is treated similarly.

(ii). Let $x_0 \in]0, 1[$. For $x \in]0, x_0]$ we have

$$f(x) \geqq \frac{f(x_0)}{x_0} x \geqq f(x_0)\, x \geqq \frac{f(x_0)}{\pi} \sin(\pi x),$$

and for $x \in [x_0, 1[$ we have

$$f(x) \geqq \frac{f(x_0)(1-x)}{1-x_0} \geqq f(x_0)(1-x) \geqq \frac{f(x_0)}{\pi} \sin \pi(1-x) = \frac{f(x_0)}{\pi} \sin(\pi x).$$

(iii). If $f(x_0) > a\pi$ the inequality (ii) implies that $f(x) > a \sin(\pi x)$ for $x \in]0, 1[$.

Since every $f \in W$ can be extended to an entire holomorphic function all derivatives of f have finite limits at 0 and 1. This can also be established in an elementary way from the property $(-1)^k f^{(2k)} \geqq 0$ for $k \geqq 0$. We will therefore freely use $f^{(k)}(x)$ for $x = 0, 1$ as the limit of $f^{(k)}(x)$ at these points.

LEMMA 1.3. *The cone* W *is a closed and metrizable subset of* $\mathbf{R}^I$.

Proof. The set of concave functions $f : I \to \mathbf{R}$ is a closed and metrizable subset of $\mathbf{R}^I$, and therefore it suffices to prove that the pointwise limit f of a sequence (f_n) from W belongs to W.

It follows by Lemma 1.2 (i) that there exists a constant A such that $f_n \leqq A$ for all n [1]). The dominated convergence theorem then shows that

$$\lim_{n \to \infty} \int_0^1 f_n(x)\, \varphi(x)\, dx = \int_0^1 f(x)\, \varphi(x)\, dx$$

for all $\varphi \in \mathcal{D}(]0, 1[)$, so (f_n) converges to f weakly in the distribution sense. Therefore $(-1)^k f^{(2k)} \geqq 0$ for all $k \geqq 0$ in the distribution sense, and this implies that f is C^∞ and hence $f \in W$.

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[1]) In fact, $A = 2 \sup f_n(\frac{1}{2})$ can be used. It is finite because $\lim_{n \to \infty} f_n(\frac{1}{2})$ exists.

2. Determination of the extreme rays of W

Inspired by [6] we consider the Green's function

$$G(x,t) = \begin{cases} (1-x)\,t & \text{for} \quad 0 \leqq t < x \leqq 1\,, \\ (1-t)\,x & \text{for} \quad 0 \leqq x \leqq t \leqq 1\,. \end{cases}$$

If φ is a continuous function on [0, 1] the unique solution $f \in C([0,1]) \cap C^2(]0,1[)$ to the equations

$$f'' = -\varphi \text{ in }]0,1[\,,\quad f(0) = f(1) = 0 \tag{2.1}$$

is

$$f(x) = \int_0^1 G(x,t)\,\varphi(t)\,dt\,. \tag{2.2}$$

The successive iterates of G are defined for $x, t \in [0,1]$ by the equations

$$G_1(x,t) = G(x,t)$$

$$G_n(x,t) = \int_0^1 G(x,y)\,G_{n-1}(y,t)]\,dy\,,\quad n \geqq 2\,.$$

It is clear that $G_n(x,t) \geqq 0$ for $x, t \in [0,1]$.

We define recursively a sequence of polynomials $(\Lambda_n)_{n \geqq 0}$ [1]) by the requirement

$$\Lambda_0(x) = x,\ \Lambda_n'' = -\Lambda_{n-1} \quad \text{and} \quad \Lambda_n(0) = \Lambda_n(1) = 0 \quad \text{for} \quad n \geqq 1\,. \tag{2.3}$$

The polynomial Λ_n is of degree $(2n+1)$, and we clearly have

$$\Lambda_n(x) = \int_0^1 G(x,t)\,\Lambda_{n-1}(t)\,dt = \int_0^1 G_n(x,t)\,t\,dt \quad \text{for} \quad n \geqq 1\,,\ x \in [0,1]\,. \tag{2.4}$$

It follows that $\Lambda_n \geqq 0$ on [0, 1] for all n, and since $(-1)^k \Lambda_n^{(2k)} = \Lambda_{n-k}$ for $k \leqq n$ we see that $\Lambda_n \in W$.

We recall that a ray $\mathbf{R}_+ x$ of a cone C is called *extreme*, if an equation $x = f + g$ with $f, g \in C$ is possible only if $f, g \in \mathbf{R}_+ x$, cf. [3].

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[1]) Our terminology is different from that of [6]; $(-1)^n \Lambda_n$ is equal to the n'th Lidstone polynomial of [4] and [6].

PROPOSITION 2.1. *The polynomials Λ_n, $n \geqq 0$, lie on extreme rays of W.*

Proof. If $\Lambda_0 = f + g$ with $f, g \in W$ we have $0 = f'' + g''$, but since f'' and g'' are both $\leqq 0$, we conclude that f and g are affine. Furthermore, since $f(0) = g(0) = 0$, we conclude that f and g are proportional to Λ_0.

Suppose now that Λ_{n-1}, $n \geqq 1$, lies on an extreme ray of W, and assume that $\Lambda_n = f + g$ where $f, g \in W$. Then $\Lambda_{n-1} = -f'' + (-g'')$, and the induction hypothesis implies that $-f''$ and $-g''$ are proportional to Λ_{n-1}. Therefore we have $f = \lambda \Lambda_n(x) + ax + b$ for certain numbers $\lambda \geqq 0$, a, b. Since $0 \leqq f \leqq \Lambda_n$, we have $f(0) = f(1) = 0$ which implies that $a = b = 0$. This proves that f (and similarly g) are proportional to Λ_n which then lies on an extreme ray of W.

Since $f \mapsto f^*$ is an affine isomorphism of W the polynomials Λ_n^* also lie on extreme rays of W. The following result is a special case of [6], Theorem 1.1.

PROPOSITION 2.2. *Every function $f \in W$ can for $n \geqq 1$ be written as*

$$f(x) = \sum_{k=0}^{n-1} ((-1)^k f^{(2k)}(0) \Lambda_k^*(x) + (-1)^k f^{(2k)}(1) \Lambda_k(x)) + R_n(x),$$

where

$$R_n(x) = \int_0^1 G_n(x,t)(-1)^n f^{(2n)}(t)\, dt \in W.$$

Proof. For $n = 1$ the formula is equivalent with

$$(2.5) \quad f(x) - f(0)(1-x) - f(1)x = R_1(x) = -\int_0^1 G(x,t) f''(t)\, dt,$$

which follows directly from (2.2), and it is clear that $R_1 \in W$.

Suppose now the formula holds for some $n \geqq 1$. Applying (2.5) to $(-1)^n f^{(2n)} \in W$ we get

$$(-1)^n f^{(2n)}(x) = (-1)^n f^{(2n)}(0) \Lambda_0^*(x) + (-1)^n f^{(2n)}(1) \Lambda_0(x)$$
$$+ \int_0^1 G(x,t)(-1)^{n+1} f^{(2n+2)}(t)\, dt,$$

which substituted in the expression for R_n yields the formula for $n+1$ because of (2.4).

To see that $R_n \in W$ we notice that

$$(-1)^k R_n^{(2k)}(x) = \begin{cases} \int_0^1 G_{n-k}(x,t)(-1)^n f^{(2n)}(t)\, dt & \text{for} \quad 0 \leqq k \leqq n-1 \\ (-1)^k f^{(2k)}(x) & \text{for} \quad k \geqq n. \end{cases}$$

The following lemma is easy to establish, but instead of giving the proof here we refer to [6].

LEMMA 2.3. *There exists a constant* $M > 0$ *such that*

$$0 \leqq \int_0^1 G_n(x,t)\,dt \leqq \frac{M}{\pi^{2n}} \quad \textit{for} \quad 0 \leqq x \leqq 1\,,\; n \geqq 1\,.$$

PROPOSITION 2.4. *The only functions* $f \in W$ *satisfying* $f^{(2k)}(0) = f^{(2k)}(1) = 0$ *for all* $k \geqq 0$ *are* $f(x) = a \sin(\pi x)$ *with* $a \geqq 0$.

Proof. Suppose $f \in W$ satisfies $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \geqq 0$. Defining $a = \sup \{ \alpha \geqq 0 \mid f - \alpha \sin(\pi x) \in W \}$, $g = f - a \sin(\pi x)$ belongs to W because W is closed in $\mathbf{R}^I$. Furthermore

$$g^{(2k)}(0) = g^{(2k)}(1) = 0 \text{ for all } k \geqq 0.$$

Let $\varepsilon > 0$ be given. Since $\varphi = g - \varepsilon \sin(\pi x) \notin W$, there exist $k \geqq 0$ and $x_0 \in]0, 1[$ such that $(-1)^k \varphi^{(2k)}(x_0) < 0$, hence

$$(-1)^k g^{(2k)}(x_0) < \varepsilon \pi^{2k} \sin(\pi x_0)\,.$$

By Lemma 1.2 (iii) applied to $(-1)^k g^{(2k)}$ we get

$$(-1)^k g^{(2k)}(t) \leqq \varepsilon \pi^{2k+1} \quad \text{for} \quad 0 < t < 1\,,$$

and therefore by Proposition 2.2 and Lemma 2.3 for $0 < x < 1$

$$g(x) = \int_0^1 G_k(x,t)(-1)^k g^{(2k)}(t)\,dt \leqq \varepsilon \pi^{2k+1} \int_0^1 G_k(x,t)\,dt$$
$$\leqq \varepsilon M \pi\,.$$

This proves that g is identically zero.

PROPOSITION 2.5. *The extreme rays of* W *are precisely the rays generated by* Λ_n *and* $\Lambda_{n'}^*$, *where* $n \geqq 0$, *and* $\sin(\pi x)$.

Proof. We first show that $\sin(\pi x)$ lies on an extreme ray. If $\sin(\pi x) = f + g$ where $f, g \in W$, we have $f(0) = f(1) = g(0) = g(1) = 0$. Differentiating $2k$ times we similarly get $f^{(2k)}(0) = f^{(2k)}(1) = g^{(2k)}(0) = g^{(2k)}(1) = 0$, and it follows by Proposition 2.4 that f and g are proportional to $\sin(\pi x)$.

We finally have to show that an arbitrary extreme ray is generated by one of the above functions.

Assume that $f \in W$ generates an extreme ray. If $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \geqq 0$ we already know by Proposition 2.4 that f is proportional to $\sin(\pi x)$. Otherwise let n be the smallest number $\geqq 0$ for which $f^{(2n)}(0) \neq 0$ or $f^{(2n)}(1) \neq 0$. By Proposition 2.2 we then have

$$f(x) = (-1)^n f^{(2n)}(0)\, \Lambda_n^*(x) + (-1)^n f^{(2n)}(1)\, \Lambda_n(x) + R_{n+1}(x),$$

but since f lies on an extreme ray all three terms on the right-hand side lie on this ray.

If $f^{(2n)}(0) \neq 0$ this shows that $(-1)^n f^{(2n)}(1)\Lambda_n$ and R_{n+1} are proportional to Λ_n^*. Therefore $f^{(2n)}(1) = 0$ and $R_{n+1}^{(2n+2)} = f^{(2n+2)}$ is proportional to $(\Lambda_n^*)^{(2n+2)} = 0$, so that $f^{(2n+2)} = 0$ and hence $R_{n+1} = 0$ (cf. Proposition 2.2).

If $f^{(2n)}(1) \neq 0$ we similarly get $f^{(2n)}(0) = 0$ and $R_{n+1} = 0$. This shows that f lies on the ray generated by either Λ_n^* or Λ_n.

3. Determination of a base for W

There are several ways of determining a base for W. We choose the following set

$$B = \left\{ f \in W \mid \int_0^1 f(x) \sin(\pi x)\, dx = 1 \right\}.$$

By Lemma 1.2 (ii) we get for $f \in B$ and $x_0 \in]0, 1[$ that

$$1 \geqq \frac{1}{\pi} f(x_0) \int_0^1 \sin^2(\pi x)\, dx = \frac{1}{2\pi} f(x_0),$$

so the functions in B are uniformly bounded by 2π.

It is therefore clear that B is a compact convex base for W.

The extreme points of B are exactly the intersections between B and the extreme rays of W. We see that $2 \sin(\pi x) \in B$.

We claim that the following formulas hold, cf. [4]:

$$(3.1) \quad \Lambda_n^*(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geqq 0, \ x \in]0, 1[,$$

$$(3.2) \quad \Lambda_n(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geqq 0, \ x \in]0, 1[.$$

Formula (3.2) follows immediately from (3.1). For $n = 0$ (3.1) is the familiar formula

$$\frac{\pi}{2}(1-x) = \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k}, \quad 0 < x < 1.$$

Suppose that (3.1) holds for n replaced by $n-1$ for some $n \geqq 1$. Denoting the right-hand side of (3.1) by f_n, we have $f_n(0) = f_n(1) = 0$ and

$$f_n''(x) = -\frac{2}{\pi^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k^{2n-1}}.$$

which is equal to $-\Lambda_{n-1}^*$ by the induction hypothesis. It follows by (2.3) that $f_n = \Lambda_n^*$, and (3.1) is proved. From (3.1) and (3.2) it follows that $\pi^{2n+1}\Lambda_n$ and $\pi^{2n+1}\Lambda_n^* \in B$. We also get $\lim_{n\to\infty} \pi^{2n+1}\Lambda_n(x) = \lim_{n\to\infty} \pi^{2n+1}\Lambda_n^*(x)$ $= 2\sin(\pi x)$. We have now established the following result:

PROPOSITION 3.1. *The set B is a compact convex base for W and the extreme points of B are* $2\sin(\pi x)$, $\pi^{2n+1}\Lambda_n^*(x)$, $\pi^{2n+1}\Lambda_n(x)$, $n \geqq 0$, *which form a closed subset of B.*

By l_+^1 we denote the set of sequences $(\alpha_n)_{n \geqq 0}$ of non-negative numbers such that $\sum_0^{\infty} \alpha_n < \infty$.

By the Choquet representation theorem or just by the Krein-Milman theorem we get the following, cf. [3]:

THEOREM 3.2. *For every $f \in W$ there exist $a \geqq 0$ and sequences (α_n), $(\beta_n) \in l_+^1$ such that*

$$(3.1) \qquad f(x) = 2a\sin(\pi x) + \sum_{n=0}^{\infty} \alpha_n \pi^{2n+1}\Lambda_n^*(x) + \sum_{n=0}^{\infty} \beta_n \pi^{2n+1}\Lambda_n(x); \quad 0 < x < 1.$$

The functions in B are uniformly bounded by 2π, and therefore the series (3.1) is uniformly convergent.

If we differentiate the series in (3.1) two times and change sign we get the series

$$\pi^2\left(2a\sin(\pi x) + \sum_{n=0}^{\infty} \alpha_{n+1}\pi^{2n+1}\Lambda_n^*(x) + \sum_{n=0}^{\infty} \beta_{n+1}\pi^{2n+1}\Lambda_n(x)\right),$$

which also converges uniformly on]0, 1[because $\sum_{n=0}^{\infty} \alpha_{n+1} + \sum_{n=0}^{\infty} \beta_{n+1} < \infty$.

It follows that the following formula holds:

$$(3.2) \quad (-1)^k f^{(2k)}(x) = \pi^{2k}\Bigg(2a \sin(\pi x) + \sum_{n=0}^{\infty} \alpha_{n+k}\pi^{2n+1}\Lambda_n^*(x) + \sum_{n=0}^{\infty} \beta_{n+k}\pi^{2n+1}\Lambda_n(x)\Bigg)$$

for $0 < x < 1$, $k \geqq 0$ and furthermore

$$(3.3) \quad \alpha_k = \pi^{-2k-1}(-1)^k f^{(2k)}(0), \; \beta_k = \pi^{-2k-1}(-1)^k f^{(2k)}(1) \quad \text{for} \quad k \geqq 0.$$

This proves that the sequences (α_n), (β_n) and hence also a are uniquely determined by f. We have thus shown that B is a simplex. The extreme points of B form a closed subset of B as remarked in Proposition 3.1 so we can formulate the following

COROLLARY 3.3. *The base B for W is a Bauer simplex.*

Whittaker proved in [4] that the series in (3.1) in fact converges uniformly over arbitrary compact subsets of the complex plane. This also proves that f can be extended to an entire holomorphic function which we also call f. For $x \in]0, 1[$ and $y \in \mathbf{R}$ we then have

$$f(x+iy) = \sum_{k=0}^{\infty} f^{(k)}(x)\frac{(iy)^k}{k!},$$

hence

$$\operatorname{Re} f(x+iy) = \sum_{k=0}^{\infty} (-1)^k f^{(2k)}(x)\frac{y^{2k}}{(2k)!},$$

which shows that $x \mapsto \operatorname{Re} f(x+iy)$ belongs to W for all $y \in \mathbf{R}$, as sum of the functions

$$x \mapsto (-1)^k f^{2k}(x)\frac{y^{2k}}{(2k)!}$$

which all belong to the closed cone W.

This gives a short proof of the recent result of Mugler [2].

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