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# REPRESENTATION OF COMPLETELY CONVEX FUNCTIONS BY THE EXTREME-POINT METHOD

# by Christian Berg

## 0. Introduction

A function  $f: ]0, 1[ \to \mathbb{R}$  is called completely convex, if it is  $C^{\infty}$  and  $(-1)^k f^{(2k)} \geq 0$  for all  $k \geq 0$ . A completely convex function f is called minimal if f(x) — a sin  $(\pi x)$  is not completely convex for any number a > 0. Widder showed (cf. [5]) that a completely convex function can be extended to an entire holomorphic function, and in the paper [6] he proved that a minimal completely convex function can be expanded in a Lidstone series. This indicates that the Lidstone polynomials lie on the extreme rays of the cone W of completely convex functions.

The purpose of the present paper is to treat the completely convex functions by the extreme-point method and to obtain the expansion in Lidstone series as a special case of the Choquet representation theorem.

We will proceed as follows: In the topology of point-wise convergence the set W of completely convex functions is a closed, metrizable convex cone. We prove directly that the extreme rays of W are generated by certain polynomials — essentially the Lidstone polynomials — and the function  $\sin (\pi x)$ . The occurrence of the extreme ray generated by  $\sin (\pi x)$  is related to the fact that only minimal completely convex functions can be expanded in Lidstone series.

The cone W has a compact convex base B, and the extreme points of B are determined. It turns out that B is a Bauer simplex, i.e. B is a simplex and the extreme points form a closed set.

The author wants to acknowledge Widder's paper [6] as a source of inspiration. The reason for writing this paper is that we felt it natural to investigate the cone W by the extreme-point method.

Recently Mugler [2] showed that real part of the holomorphic extension of  $f \in W$  to the strip Re  $z \in ]0$ , 1[ is completely convex on each segment  $\{x + iy \mid 0 < x < 1\}$ . We give a very short proof of this result.

#### 1. Completely convex functions

Let *I* denote an open interval. A function  $f: I \to \mathbb{R}$  is called *completely convex*, if it is  $C^{\infty}$  and  $(-1)^k f^{(2k)} \ge 0$  on *I* for  $k \ge 0$ .

The set of completely convex functions is a convex cone denoted W = W(I). We always equip W with the topology of pointwise convergence, i.e., with the topology induced by the product space  $\mathbb{R}^{I}$ .

LEMMA 1.1. If I is unbounded W(I) consists of the non-negative affine functions, and  $W(\mathbf{R})$  consists of the non-negative constants.

*Proof.* Assume first that  $\inf I = -\infty$ . Then every  $f \in W$  is decreasing since it is non-negative and concave. For  $k \ge 0$  and  $f \in W$  we have  $(-1)^k f^{(2k)} \in W$  and consequently  $(-1)^k f^{(2k+1)} \le 0$ . This shows that also  $-f' \in W$  and then  $-f'' \le 0$ , but by definition  $f'' \le 0$  and therefore f is affine.

The case sup  $I = \infty$  is treated in a similar manner. Finally, every nonnegative concave function on  $\mathbf{R}$  is constant.

Remark. Completely convex sequences are non-negative and affine. For a sequence  $a = (a_0, a_1, ...)$  of real numbers we define  $\Delta a$  to be the sequence  $(\Delta a)_n = a_{n+1} - a_n, n \ge 0$ , and  $\Delta^k a$  is defined as  $\Delta (\Delta^{k-1} a)$  for  $k \ge 1$ , where  $\Delta^0 a = a$ . A sequence a is called completely convex if  $(-1)^k \Delta^{2k} a \ge 0$  for  $k \ge 0$ . The same method as in Lemma 1.1 leads to the conclusion that every completely convex sequence satisfies  $\Delta a \ge 0$  and  $\Delta^2 a = 0$ . The completely convex sequences are therefore exactly the sequences  $a_n = \alpha n + \beta$ , where  $\alpha, \beta \ge 0$ .

This is an answer to a remark by Boas [1]: "Nothing seems to be known about completely convex sequences".

In the following we will always assume that I is bounded, and for the sake of convenience we choose I to be I = ]0, 1[. We simply write W for W (]0, 1[). For  $f \in W$  we have  $-f'' \in W$  and  $f^* \in W$ , where  $f^*$  is defined by  $f^*(x) = f(1-x)$ . The mapping  $f \mapsto f^*$  is an affine isomorphism of W onto itself.

Lemma 1.2. Let  $f:]0,1[\to \mathbb{R}$  be non-negative and concave. Then the following holds:

(i) 
$$f(x) \le 2f(\frac{1}{2})$$
 for  $x \in ]0, 1[$ .

(ii) 
$$f(x) \ge \frac{1}{\pi} f(x_0) \sin (\pi x)$$
 for  $x, x_0 \in ]0, 1[$ .

(iii) ([6], Lemma 7.1) If there exists  $x_0 \in ]0, 1[$  and a > 0 such that  $f(x_0) < a \sin(\pi x_0)$  then  $f(x) \leq a\pi$  for  $x \in ]0, 1[$ .

*Proof.* (i). For  $x \in ]0, \frac{1}{2}$  we have that f(x) lies below the line through  $(\frac{1}{2}, f(\frac{1}{2}))$  and (1, 0) and (i) follows for  $x \in ]0, \frac{1}{2}]$ . The interval  $[\frac{1}{2}, 1]$  is treated similarly.

(ii). Let  $x_0 \in [0, 1[$ . For  $x \in [0, x_0]$  we have

$$f(x) \ge \frac{f(x_0)}{x_0} x \ge f(x_0) x \ge \frac{f(x_0)}{\pi} \sin (\pi x),$$

and for  $x \in [x_0, 1[$  we have

$$f(x) \ge \frac{f(x_0)(1-x)}{1-x_0} \ge f(x_0)(1-x) \ge \frac{f(x_0)}{\pi} \sin \pi (1-x) = \frac{f(x_0)}{\pi} \sin (\pi x).$$

(iii). If  $f(x_0) > a\pi$  the inequality (ii) implies that  $f(x) > a \sin(\pi x)$  for  $x \in ]0, 1[.$ 

Since every  $f \in W$  can be extended to an entire holomorphic function all derivatives of f have finite limits at 0 and 1. This can also be established in an elementary way from the property  $(-1)^k f^{(2k)} \ge 0$  for  $k \ge 0$ . We will therefore freely use  $f^{(k)}(x)$  for x = 0, 1 as the limit of  $f^{(k)}(x)$  at these points.

LEMMA 1.3. The cone W is a closed and metrizable subset of  $\mathbb{R}^{I}$ .

*Proof.* The set of concave functions  $f: I \to \mathbf{R}$  is a closed and metrizable subset of  $\mathbf{R}^{I}$ , and therefore it suffices to prove that the pointwise limit f of a sequence  $(f_n)$  from W belongs to W.

It follows by Lemma 1.2 (i) that there exists a constant A such that  $f_n \leq A$  for all  $n^{-1}$ ). The dominated convergence theorem then shows that

$$\lim_{n\to\infty} \int_0^1 f_n(x) \varphi(x) dx = \int_0^1 f(x) \varphi(x) dx$$

for all  $\varphi \in \mathcal{D}(]0, 1[)$ , so  $(f_n)$  converges to f weakly in the distribution sense. Therefore  $(-1)^k f^{(2k)} \ge 0$  for all  $k \ge 0$  in the distribution sense, and this implies that f is  $C^{\infty}$  and hence  $f \in W$ .

<sup>1)</sup> In fact,  $A = 2 \sup_{n \to \infty} f_n(\frac{1}{2})$  can be used. It is finite because  $\lim_{n \to \infty} f_n(\frac{1}{2})$  exists.

# 2. Determination of the extreme rays of W

Inspired by [6] we consider the Green's function

$$G(x,t) = \begin{cases} (1-x)t & \text{for } 0 \le t < x \le 1, \\ (1-t)x & \text{for } 0 \le x \le t \le 1. \end{cases}$$

If  $\varphi$  is a continuous function on [0, 1] the unique solution  $f \in C([0, 1])$  $\cap C^2([0, 1])$  to the equations

is

(2.2) 
$$f(x) = \int_0^1 G(x, t) \varphi(t) dt.$$

The successive iterates of G are defined for  $x, t \in [0,1]$  by the equations

$$G_1(x,t) = G(x,t)$$

$$G_n(x, t) = \int_0^1 G(x, y) G_{n-1}(y, t) dy, n \ge 2.$$

It is clear that  $G_n(x, t) \ge 0$  for  $x, t \in [0, 1]$ .

We define recursively a sequence of polynomials  $(\Lambda_n)_{n \ge 0}$  by the requirement

(2.3) 
$$\Lambda_0(x) = x, \Lambda_n'' = -\Lambda_{n-1}$$
 and  $\Lambda_n(0) = \Lambda_n(1) = 0$  for  $n \ge 1$ .

The polynomial  $\Lambda_n$  is of degree (2n + 1), and we clearly have

(2.4) 
$$\Lambda_n(x) = \int_0^1 G(x,t) \Lambda_{n-1}(t) dt = \int_0^1 G_n(x,t) t dt$$
 for  $n \ge 1, x \in [0,1].$ 

It follows that  $\Lambda_n \ge 0$  on [0, 1] for all n, and since  $(-1)^k \Lambda_n^{(2k)} = \Lambda_{n-k}$  for  $k \le n$  we see that  $\Lambda_n \in W$ .

We recall that a ray  $\mathbf{R}_+ x$  of a cone C is called *extreme*, if an equation x = f + g with  $f, g \in C$  is possible only if  $f, g \in \mathbf{R}_+ x$ , cf. [3].

<sup>&</sup>lt;sup>1</sup>) Our terminology is different from that of [6];  $(-1)^n \Lambda_n$  is equal to the n'th Lidstone polynomial of [4] and [6].

Proposition 2.1. The polynomials  $\Lambda_n$ ,  $n \geq 0$ , lie on extreme rays of W.

*Proof.* If  $\Lambda_0 = f + g$  with  $f, g \in W$  we have 0 = f'' + g'', but since f'' and g'' are both  $\leq 0$ , we conclude that f and g are affine. Furthermore, since f(0) = g(0) = 0, we conclude that f and g are proportional to  $\Lambda_0$ .

Suppose now that  $\Lambda_{n-1}$ ,  $n \ge 1$ , lies on an extreme ray of W, and assume that  $\Lambda_n = f + g$  where  $f, g \in W$ . Then  $\Lambda_{n-1} = -f'' + (-g'')$ , and the induction hypothesis implies that -f'' and -g'' are proportional to  $\Lambda_{n-1}$ . Therefore we have  $f = \lambda \Lambda_n(x) + ax + b$  for certain numbers  $\lambda \ge 0$ , a, b. Since  $0 \le f \le \Lambda_n$ , we have f(0) = f(1) = 0 which implies that a = b = 0. This proves that f (and similarly g) are proportional to  $\Lambda_n$  which then lies on an extreme ray of W.

Since  $f \mapsto f^*$  is an affine isomorphism of W the polynomials  $\Lambda_n^*$  also lie on extreme rays of W. The following result is a special case of [6], Theorem 1.1.

Proposition 2.2. Every function  $f \in W$  can for  $n \ge 1$  be written as

$$f(x) = \sum_{k=0}^{n-1} \left( (-1)^k f^{(2k)}(0) \Lambda_k^*(x) + (-1)^k f^{(2k)}(1) \Lambda_k(x) \right) + R_n(x),$$
where

$$R_n(x) = \int_0^1 G_n(x,t) (-1)^n f^{(2n)}(t) dt \in W.$$

*Proof.* For n = 1 the formula is equivalent with

$$(2.5) \quad f(x) - f(0)(1-x) - f(1)x = R_1(x) = -\int_0^1 G(x,t)f''(t) dt,$$

which follows directly from (2.2), and it is clear that  $R_1 \in W$ .

Suppose now the formula holds for some  $n \ge 1$ . Applying (2.5) to  $(-1)^n f^{(2n)} \in W$  we get

$$(-1)^{n} f^{(2n)}(x) = (-1)^{n} f^{(2n)}(0) \Lambda_{0}^{*}(x) + (-1)^{n} f^{(2n)}(1) \Lambda_{0}(x) + \int_{0}^{1} G(x, t) (-1)^{n+1} f^{(2n+2)}(t) dt,$$

which substituted in the expression for  $R_n$  yields the formula for n+1 because of (2.4).

To see that  $R_n \in W$  we notice that

$$(-1)^k R_n^{(2k)}(x) = \begin{cases} \int_0^1 G_{n-k}(x,t) (-1)^n f^{(2n)}(t) dt & \text{for } 0 \le k \le n-1 \\ (-1)^k f^{(2k)}(x) & \text{for } k \ge n \end{cases}.$$

The following lemma is easy to establish, but instead of giving the proof here we refer to [6].

Lemma 2.3. There exists a constant M > 0 such that

$$0 \le \int_0^1 G_n(x,t) dt \le \frac{M}{\pi^{2n}}$$
 for  $0 \le x \le 1, n \ge 1$ .

PROPOSITION 2.4. The only functions  $f \in W$  satisfying  $f^{(2k)}(0) = f^{(2k)}(1) = 0$  for all  $k \ge 0$  are  $f(x) = a \sin(\pi x)$  with  $a \ge 0$ .

*Proof.* Suppose  $f \in W$  satisfies  $f^{(2k)}(0) = f^{(2k)}(1) = 0$  for all  $k \ge 0$ . Defining  $a = \sup \{ \alpha \ge 0 \mid f - \alpha \sin (\pi x) \in W \}, g = f - a \sin (\pi x)$  belongs to W because W is closed in  $\mathbb{R}^I$ . Furthermore

$$g^{(2k)}(0) = g^{(2k)}(1) = 0 \text{ for all } k \ge 0.$$

Let  $\varepsilon > 0$  be given. Since  $\varphi = g - \varepsilon \sin(\pi x) \notin W$ , there exist  $k \ge 0$  and  $x_0 \in ]0, 1[$  such that  $(-1)^k \varphi^{(2k)}(x_0) < 0$ , hence

$$(-1)^k g^{(2k)}(x_0) < \varepsilon \pi^{2k} \sin (\pi x_0)$$
.

By Lemma 1.2 (iii) applied to  $(-1)^k g^{(2k)}$  we get

$$(-1)^k g^{(2k)}(t) \le \varepsilon \pi^{2k+1}$$
 for  $0 < t < 1$ ,

and therefore by Proposition 2.2 and Lemma 2.3 for 0 < x < 1

$$g(x) = \int_0^1 G_k(x,t) (-1)^k g^{(2k)}(t) dt \le \varepsilon \pi^{2k+1} \int_0^1 G_k(x,t) dt$$
  
\$\le \varepsilon M\pi\$.

This proves that g is identically zero.

PROPOSITION 2.5. The extreme rays of W are precisely the rays generated by  $\Lambda_n$  and  $\Lambda_{n'}^*$ , where  $n \ge 0$ , and  $\sin(\pi x)$ .

*Proof.* We first show that  $\sin(\pi x)$  lies on an extreme ray. If  $\sin(\pi x) = f + g$  where  $f, g \in W$ , we have f(0) = f(1) = g(0) = g(1) = 0. Differentiating 2k times we similarly get  $f^{(2k)}(0) = f^{(2k)}(1) = g^{(2k)}(0) = g^{(2k)}(1) = 0$ , and it follows by Proposition 2.4 that f and g are proportional to  $\sin(\pi x)$ .

We finally have to show that an arbitrary extreme ray is generated by one of the above functions.

Assume that  $f \in W$  generates an extreme ray. If  $f^{(2k)}(0) = f^{(2k)}(1) = 0$  for all  $k \ge 0$  we already know by Proposition 2.4 that f is proportional to  $\sin(\pi x)$ . Otherwise let n be the smallest number  $\ge 0$  for which  $f^{(2n)}(0) \ne 0$  or  $f^{(2n)}(1) \ne 0$ . By Proposition 2.2 we then have

$$f(x) = (-1)^n f^{(2n)}(0) \Lambda_n^*(x) + (-1)^n f^{(2n)}(1) \Lambda_n(x) + R_{n+1}(x),$$

but since f lies on an extreme ray all three terms on the right-hand side lie on this ray.

If  $f^{(2n)}(0) \neq 0$  this shows that  $(-1)^n f^{(2n)}(1) \Lambda_n$  and  $R_{n+1}$  are proportional to  $\Lambda_n^*$ . Therefore  $f^{(2n)}(1) = 0$  and  $R_{n+1}^{(2n+2)} = f^{(2n+2)}$  is proportional to  $(\Lambda_n^*)^{(2n+2)} = 0$ , so that  $f^{(2n+2)} = 0$  and hence  $R_{n+1} = 0$  (cf. Proposition 2.2).

If  $f^{(2n)}(1) \neq 0$  we similarly get  $f^{(2n)}(0) = 0$  and  $R_{n+1} = 0$ . This shows that f lies on the ray generated by either  $\Lambda_n^*$  or  $\Lambda_n$ .

### 3. Determination of a base for W

There are several ways of determining a base for W. We choose the following set

$$B = \left\{ f \in W \mid \int_0^1 f(x) \sin (\pi x) dx = 1 \right\}.$$

By Lemma 1.2 (ii) we get for  $f \in B$  and  $x_0 \in ]0, 1[$  that

$$1 \ge \frac{1}{\pi} f(x_0) \int_0^1 \sin^2(\pi x) \, dx = \frac{1}{2\pi} f(x_0) \,,$$

so the functions in B are uniformly bounded by  $2\pi$ .

It is therefore clear that B is a compact convex base for W.

The extreme points of B are exactly the intersections between B and the extreme rays of W. We see that  $2 \sin (\pi x) \in B$ .

We claim that the following formulas hold, cf. [4]:

(3.1) 
$$\Lambda_n^*(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k^{2n+1}}, n \ge 0, x \in ]0,1[$$
,

(3.2) 
$$\Lambda_n(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin(k\pi x)}{k^{2n+1}}, n \ge 0, x \in ]0, 1[.$$

Formula (3.2) follows immediately from (3.1). For n = 0 (3.1) is the familiar formula

$$\frac{\pi}{2}(1-x) = \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k}, \ 0 < x < 1.$$

Suppose that (3.1) holds for n replaced by n-1 for some  $n \ge 1$ . Denoting the right-hand side of (3.1) by  $f_n$ , we have  $f_n(0) = f_n(1) = 0$  and

$$f_n''(x) = -\frac{2}{\pi^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k^{2n-1}}.$$

which is equal to  $-\Lambda_{n-1}^*$  by the induction hypothesis. It follows by (2.3) that  $f_n = \Lambda_n^*$ , and (3.1) is proved. From (3.1) and (3.2) it follows that  $\pi^{2n+1}\Lambda_n$  and  $\pi^{2n+1}\Lambda_n^* \in B$ . We also get  $\lim_{n \to \infty} \pi^{2n+1}\Lambda_n(x) = \lim_{n \to \infty} \pi^{2n+1}\Lambda_n^*(x) = 2 \sin(\pi x)$ . We have now established the following result:

PROPOSITION 3.1. The set B is a compact convex base for W and the extreme points of B are  $2 \sin (\pi x)$ ,  $\pi^{2n+1} \Lambda_n^*(x)$ ,  $\pi^{2n+1} \Lambda_n(x)$ ,  $n \ge 0$ , which form a closed subset of B.

By  $l_+^1$  we denote the set of sequences  $(\alpha_n)_{n\geq 0}$  of non-negative numbers such that  $\sum_{n=0}^{\infty} \alpha_n < \infty$ .

By the Choquet representation theorem or just by the Krein-Milman theorem we get the following, cf. [3]:

THEOREM 3.2. For every  $f \in W$  there exist  $a \ge 0$  and sequences  $(\alpha_n)$ ,  $(\beta_n) \in l^1_+$  such that

(3.1) 
$$f(x) = 2a \sin (\pi x) + \sum_{n=0}^{\infty} \alpha_n \pi^{2n+1} \Lambda_n^*(x) + \sum_{n=0}^{\infty} \beta_n \pi^{2n+1} \Lambda_n(x); \quad 0 < x < 1.$$

The functions in B are uniformly bounded by  $2\pi$ , and therefore the series (3.1) is uniformly convergent.

If we differentiate the series in (3.1) two times and change sign we get the series

$$\pi^{2}\left(2a \sin (\pi x) + \sum_{n=0}^{\infty} \alpha_{n+1}\pi^{2n+1}\Lambda_{n}^{*}(x) + \sum_{n=0}^{\infty} \beta_{n+1}\pi^{2n+1}\Lambda_{n}(x)\right),\,$$

which also converges uniformly on ]0, 1[ because  $\sum_{n=0}^{\infty} \alpha_{n+1} + \sum_{n=0}^{\infty} \beta_{n+1} < \infty$ .

It follows that the following formula holds:

$$(3.2) \quad (-1)^k f^{(2k)}(x) = \pi^{2k} \left( 2a \sin(\pi x) + \sum_{n=0}^{\infty} \alpha_{n+k} \pi^{2n+1} \Lambda_n^*(x) + \sum_{n=0}^{\infty} \beta_{n+k} \pi^{2n+1} \Lambda_n(x) \right)$$

for 0 < x < 1,  $k \ge 0$  and furthermore

(3.3) 
$$\alpha_k = \pi^{-2k-1} (-1)^k f^{(2k)}(0), \ \beta_k = \pi^{-2k-1} (-1)^k f^{(2k)}(1)$$
  
for  $k \ge 0$ .

This proves that the sequences  $(\alpha_n)$ ,  $(\beta_n)$  and hence also a are uniquely determined by f. We have thus shown that B is a simplex. The extreme points of B form a closed subset of B as remarked in Proposition 3.1 so we can formulate the following

COROLLARY 3.3. The base B for W is a Bauer simplex.

Whittaker proved in [4] that the series in (3.1) in fact converges uniformly over arbitrary compact subsets of the complex plane. This also proves that f can be extended to an entire holomorphic function which we also call f. For  $x \in ]0, 1[$  and  $y \in \mathbf{R}$  we then have

$$f(x+iy) = \sum_{k=0}^{\infty} f^{(k)}(x) \frac{(iy)^k}{k!}$$

hence

Re 
$$f(x+iy) = \sum_{k=0}^{\infty} (-1)^k f^{(2k)}(x) \frac{y^{2k}}{(2k)!}$$
,

which shows that  $x \mapsto \text{Ref}(x+iy)$  belongs to W for all  $y \in \mathbb{R}$ , as sum of the functions

$$x \mapsto (-1)^k f^{2k}(x) \frac{y^{2k}}{(2k)!}$$

which all belong to the closed cone W.

This gives a short proof of the recent result of Mugler [2].

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