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REPRESENTATION OF COMPLETELY CONVEX FUNCTIONS BY THE EXTREME-POINT METHOD

by Christian Berg

0. INTRODUCTION

A function $f: [0, 1[\rightarrow \mathbb{R}$ is called completely convex, if it is C^{∞} and $(-1)^{k} f^{(2k)} \geq 0$ for all $k \geq 0$. A completely convex function f is called minimal if $f(x) - a \sin(\pi x)$ is not completely convex for any number a > 0. Widder showed (cf. [5]) that a completely convex function can be extended to an entire holomorphic function, and in the paper [6] he proved that a minimal completely convex function can be expanded in a Lidstone series. This indicates that the Lidstone polynomials lie on the extreme rays of the cone W of completely convex functions.

The purpose of the present paper is to treat the completely convex functions by the extreme-point method and to obtain the expansion in Lidstone series as a special case of the Choquet representation theorem.

We will proceed as follows: In the topology of point-wise convergence the set W of completely convex functions is a closed, metrizable convex cone. We prove directly that the extreme rays of W are generated by certain polynomials — essentially the Lidstone polynomials — and the function sin (πx) . The occurrence of the extreme ray generated by sin (πx) is related to the fact that only minimal completely convex functions can be expanded in Lidstone series.

The cone W has a compact convex base B, and the extreme points of B are determined. It turns out that B is a Bauer simplex, i.e. B is a simplex and the extreme points form a closed set.

The author wants to acknowledge Widder's paper [6] as a source of inspiration. The reason for writing this paper is that we felt it natural to investigate the cone W by the extreme-point method.

Recently Mugler [2] showed that real part of the holomorphic extension of $f \in W$ to the strip Re $z \in [0, 1[$ is completely convex on each segment $\{x + iy \mid 0 < x < 1\}$. We give a very short proof of this result.

1. Completely convex functions

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Let *I* denote an open interval. A function $f: I \to \mathbb{R}$ is called *completely* convex, if it is C^{∞} and $(-1)^k f^{(2k)} \ge 0$ on *I* for $k \ge 0$.

The set of completely convex functions is a convex cone denoted W = W(I). We always equip W with the topology of pointwise convergence, i.e., with the topology induced by the product space \mathbb{R}^{I} .

LEMMA 1.1. If I is unbounded W(I) consists of the non-negative affine functions, and $W(\mathbf{R})$ consists of the non-negative constants.

Proof. Assume first that $\inf I = -\infty$. Then every $f \in W$ is decreasing since it is non-negative and concave. For $k \ge 0$ and $f \in W$ we have $(-1)^k f^{(2k)} \in W$ and consequently $(-1)^k f^{(2k+1)} \le 0$. This shows that also $-f' \in W$ and then $-f'' \le 0$, but by definition $f'' \le 0$ and therefore f is affine.

The case sup $I = \infty$ is treated in a similar manner. Finally, every non-negative concave function on **R** is constant.

Remark. Completely convex sequences are non-negative and affine. For a sequence $a = (a_0, a_1, ...)$ of real numbers we define Δa to be the sequence $(\Delta a)_n = a_{n+1} - a_n, n \ge 0$, and $\Delta^k a$ is defined as $\Delta (\Delta^{k-1}a)$ for $k \ge 1$, where $\Delta^0 a = a$. A sequence a is called *completely convex* if $(-1)^k \Delta^{2k} a \ge 0$ for $k \ge 0$. The same method as in Lemma 1.1 leads to the conclusion that every completely convex sequence satisfies $\Delta a \ge 0$ and $\Delta^2 a = 0$. The completely convex sequences are therefore exactly the sequences $a_n = \alpha n + \beta$, where $\alpha, \beta \ge 0$.

This is an answer to a remark by Boas [1]: "Nothing seems to be known about completely convex sequences".

In the following we will always assume that I is bounded, and for the sake of convenience we choose I to be I = [0, 1[. We simply write W for W ([0, 1[). For $f \in W$ we have $-f'' \in W$ and $f^* \in W$, where f^* is defined by $f^*(x) = f(1-x)$. The mapping $f \mapsto f^*$ is an affine isomorphism of W onto itself.

LEMMA 1.2. Let $f:]0, 1[\rightarrow \mathbb{R}$ be non-negative and concave. Then the following holds:

(i)
$$f(x) \leq 2f(\frac{1}{2})$$
 for $x \in]0, 1[$.

(ii) $f(x) \ge \frac{1}{\pi} f(x_0) \sin(\pi x)$ for $x, x_0 \in]0, 1[$.

(iii) ([6], Lemma 7.1) If there exists $x_0 \in [0, 1[$ and a > 0 such that $f(x_0) < a \sin(\pi x_0)$ then $f(x) \leq a\pi$ for $x \in [0, 1[$.

Proof. (i). For $x \in [0, \frac{1}{2}]$ we have that f(x) lies below the line through $(\frac{1}{2}, f(\frac{1}{2}))$ and (1, 0) and (i) follows for $x \in [0, \frac{1}{2}]$. The interval $[\frac{1}{2}, 1]$ is treated similarly.

(ii). Let $x_0 \in [0, 1[$. For $x \in [0, x_0]$ we have

$$f(x) \ge \frac{f(x_0)}{x_0} x \ge f(x_0) x \ge \frac{f(x_0)}{\pi} \sin (\pi x),$$

and for $x \in [x_0, 1]$ we have

$$f(x) \ge \frac{f(x_0)(1-x)}{1-x_0} \ge f(x_0)(1-x) \ge \frac{f(x_0)}{\pi} \sin \pi (1-x) = \frac{f(x_0)}{\pi} \sin (\pi x) .$$

(iii). If $f(x_0) > a\pi$ the inequality (ii) implies that $f(x) > a \sin(\pi x)$ for $x \in [0, 1[.$

Since every $f \in W$ can be extended to an entire holomorphic function all derivatives of f have finite limits at 0 and 1. This can also be established in an elementary way from the property $(-1)^k f^{(2k)} \ge 0$ for $k \ge 0$. We will therefore freely use $f^{(k)}(x)$ for x = 0, 1 as the limit of $f^{(k)}(x)$ at these points.

LEMMA 1.3. The cone W is a closed and metrizable subset of \mathbf{R}^{I} .

Proof. The set of concave functions $f: I \rightarrow \mathbf{R}$ is a closed and metrizable subset of \mathbf{R}^{I} , and therefore it suffices to prove that the pointwise limit f of a sequence (f_n) from W belongs to W.

It follows by Lemma 1.2 (i) that there exists a constant A such that $f_n \leq A$ for all n^1). The dominated convergence theorem then shows that

$$\lim_{n \to \infty} \int_0^1 f_n(x) \varphi(x) dx = \int_0^1 f(x) \varphi(x) dx$$

for all $\varphi \in \mathcal{D}([0, 1[), so (f_n) \text{ converges to } f \text{ weakly in the distribution})$ sense. Therefore $(-1)^k f^{(2k)} \ge 0$ for all $k \ge 0$ in the distribution sense, and this implies that f is C^{∞} and hence $f \in W$.

2. Determination of the extreme rays of W

Inspired by [6] we consider the Green's function

$$G(x,t) = \begin{cases} (1-x)t & \text{for} \quad 0 \leq t < x \leq 1, \\ (1-t)x & \text{for} \quad 0 \leq x \leq t \leq 1. \end{cases}$$

If φ is a continuous function on [0, 1] the unique solution $f \in C([0, 1])$ $\cap C^2([0, 1[)$ to the equations

(2.1)
$$f'' = -\varphi \text{ in }]0, 1[, f(0) = f(1) = 0$$

is

(2.2)
$$f(x) = \int_0^1 G(x, t) \varphi(t) dt.$$

The successive iterates of G are defined for $x, t \in [0,1]$ by the equations

$$G_{1}(x,t) = G(x,t)$$

$$G_{n}(x,t) = \int_{0}^{1} G(x,y) G_{n-1}(y,t) dy, n \ge 2.$$

It is clear that $G_n(x, t) \ge 0$ for $x, t \in [0, 1]$.

We define recursively a sequence of polynomials $(\Lambda_n)_{n \ge 0}$ ¹) by the requirement

(2.3) $\Lambda_0(x) = x, \Lambda_n'' = -\Lambda_{n-1}$ and $\Lambda_n(0) = \Lambda_n(1) = 0$ for $n \ge 1$.

The polynomial Λ_n is of degree (2n + 1), and we clearly have

(2.4)
$$\Lambda_n(x) = \int_0^1 G(x,t) \Lambda_{n-1}(t) dt = \int_0^1 G_n(x,t) t dt$$
 for $n \ge 1, x \in [0,1].$

It follows that $\Lambda_n \ge 0$ on [0, 1] for all *n*, and since $(-1)^k \Lambda_n^{(2k)} = \Lambda_{n-k}$ for $k \le n$ we see that $\Lambda_n \in W$.

We recall that a ray $\mathbf{R}_+ x$ of a cone C is called *extreme*, if an equation x = f + g with $f, g \in C$ is possible only if $f, g \in \mathbf{R}_+ x$, cf. [3].

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¹) Our terminology is different from that of [6]; $(-1)^n \Lambda_n$ is equal to the n'th Lidstone polynomial of [4] and [6].

PROPOSITION 2.1. The polynomials A_n , $n \ge 0$, lie on extreme rays of W.

Proof. If $\Lambda_0 = f + g$ with $f, g \in W$ we have 0 = f'' + g'', but since f'' and g'' are both ≤ 0 , we conclude that f and g are affine. Furthermore, since f(0) = g(0) = 0, we conclude that f and g are proportional to Λ_0 .

Suppose now that Λ_{n-1} , $n \ge 1$, lies on an extreme ray of W, and assume that $\Lambda_n = f + g$ where $f, g \in W$. Then $\Lambda_{n-1} = -f'' + (-g'')$, and the induction hypothesis implies that -f'' and -g'' are proportional to Λ_{n-1} . Therefore we have $f = \lambda \Lambda_n(x) + ax + b$ for certain numbers $\lambda \ge 0, a, b$. Since $0 \le f \le \Lambda_n$, we have f(0) = f(1) = 0 which implies that a = b = 0. This proves that f (and similarly g) are proportional to Λ_n which then lies on an extreme ray of W.

Since $f \mapsto f^*$ is an affine isomorphism of W the polynomials Λ_n^* also lie on extreme rays of W. The following result is a special case of [6], Theorem 1.1.

PROPOSITION 2.2. Every function $f \in W$ can for $n \ge 1$ be written as

$$f(x) = \sum_{k=0}^{n-1} \left((-1)^k f^{(2k)}(0) \Lambda_k^*(x) + (-1)^k f^{(2k)}(1) \Lambda_k(x) \right) + R_n(x),$$

where

$$R_n(x) = \int_0^1 G_n(x,t) (-1)^n f^{(2n)}(t) dt \in W.$$

Proof. For n = 1 the formula is equivalent with

(2.5)
$$f(x) - f(0)(1-x) - f(1)x = R_1(x) = -\int_0^1 G(x,t)f''(t) dt$$
,

which follows directly from (2.2), and it is clear that $R_1 \in W$.

Suppose now the formula holds for some $n \ge 1$. Applying (2.5) to $(-1)^n f^{(2n)} \in W$ we get

$$(-1)^{n} f^{(2n)}(x) = (-1)^{n} f^{(2n)}(0) \Lambda_{0}^{*}(x) + (-1)^{n} f^{(2n)}(1) \Lambda_{0}(x) + \int_{0}^{1} G(x,t) (-1)^{n+1} f^{(2n+2)}(t) dt,$$

which substituted in the expression for R_n yields the formula for n+1 because of (2.4).

To see that $R_n \in W$ we notice that

$$(-1)^{k} R_{n}^{(2k)}(x) = \begin{cases} \int_{0}^{1} G_{n-k}(x,t) (-1)^{n} f^{(2n)}(t) dt & \text{for } 0 \leq k \leq n-1 \\ (-1)^{k} f^{(2k)}(x) & \text{for } k \geq n \end{cases}$$

L'Enseignement mathém., t. XXIII, fasc. 3-4.

The following lemma is easy to establish, but instead of giving the proof here we refer to [6].

LEMMA 2.3. There exists a constant M > 0 such that

$$0 \leq \int_0^1 G_n(x,t) dt \leq \frac{M}{\pi^{2n}} \quad \text{for} \quad 0 \leq x \leq 1, \ n \geq 1.$$

PROPOSITION 2.4. The only functions $f \in W$ satisfying $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \ge 0$ are $f(x) = a \sin(\pi x)$ with $a \ge 0$.

Proof. Suppose $f \in W$ satisfies $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \ge 0$. Defining $a = \sup \{ \alpha \ge 0 \mid f - \alpha \sin(\pi x) \in W \}$, $g = f - a \sin(\pi x)$ belongs to W because W is closed in \mathbb{R}^{I} . Furthermore

$$g^{(2k)}(0) = g^{(2k)}(1) = 0$$
 for all $k \ge 0$.

Let $\varepsilon > 0$ be given. Since $\varphi = g - \varepsilon \sin(\pi x) \notin W$, there exist $k \ge 0$ and $x_0 \in]0, 1[$ such that $(-1)^k \varphi^{(2k)}(x_0) < 0$, hence

$$(-1)^k g^{(2k)}(x_0) < \varepsilon \pi^{2k} \sin (\pi x_0).$$

By Lemma 1.2 (iii) applied to $(-1)^k g^{(2k)}$ we get

 $(-1)^k g^{(2k)}(t) \leq \varepsilon \pi^{2k+1} \quad \text{for} \quad 0 < t < 1,$

and therefore by Proposition 2.2 and Lemma 2.3 for 0 < x < 1

$$g(x) = \int_{0}^{1} G_{k}(x,t) (-1)^{k} g^{(2k)}(t) dt \leq \varepsilon \pi^{2k+1} \int_{0}^{1} G_{k}(x,t) dt$$
$$\leq \varepsilon M \pi.$$

This proves that g is identically zero.

PROPOSITION 2.5. The extreme rays of W are precisely the rays generated by Λ_n and $\Lambda_{n'}^*$, where $n \ge 0$, and $\sin(\pi x)$.

Proof. We first show that $\sin(\pi x)$ lies on an extreme ray. If $\sin(\pi x) = f + g$ where $f, g \in W$, we have f(0) = f(1) = g(0) = g(1) = 0. Differentiating 2k times we similarly get $f^{(2k)}(0) = f^{(2k)}(1) = g^{(2k)}(0) = g^{(2k)}(1) = 0$, and it follows by Proposition 2.4 that f and g are proportional to $\sin(\pi x)$.

We finally have to show that an arbitrary extreme ray is generated by one of the above functions. Assume that $f \in W$ generates an extreme ray. If $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k \ge 0$ we already know by Proposition 2.4 that f is proportional to sin (πx) . Otherwise let n be the smallest number ≥ 0 for which $f^{(2n)}(0) \ne 0$ or $f^{(2n)}(1) \ne 0$. By Proposition 2.2 we then have

$$f(x) = (-1)^n f^{(2n)}(0) \Lambda_n^*(x) + (-1)^n f^{(2n)}(1) \Lambda_n(x) + R_{n+1}(x),$$

but since f lies on an extreme ray all three terms on the right-hand side lie on this ray.

If $f^{(2n)}(0) \neq 0$ this shows that $(-1)^n f^{(2n)}(1) \Lambda_n$ and R_{n+1} are proportional to Λ_n^* . Therefore $f^{(2n)}(1) = 0$ and $R_{n+1}^{(2n+2)} = f^{(2n+2)}$ is proportional to $(\Lambda_n^*)^{(2n+2)} = 0$, so that $f^{(2n+2)} = 0$ and hence $R_{n+1} = 0$ (cf. Proposition 2.2).

If $f^{(2n)}(1) \neq 0$ we similarly get $f^{(2n)}(0) = 0$ and $R_{n+1} = 0$. This shows that f lies on the ray generated by either Λ_n^* or Λ_n .

3. Determination of a base for W

There are several ways of determining a base for W. We choose the following set

$$B = \left\{ f \in W \mid \int_{0}^{1} f(x) \sin(\pi x) \, dx = 1 \right\}.$$

By Lemma 1.2 (ii) we get for $f \in B$ and $x_0 \in [0, 1[$ that

$$1 \ge \frac{1}{\pi} f(x_0) \int_0^1 \sin^2 (\pi x) \, dx = \frac{1}{2\pi} f(x_0) \, ,$$

so the functions in B are uniformly bounded by 2π .

It is therefore clear that B is a compact convex base for W.

The extreme points of B are exactly the intersections between B and the extreme rays of W. We see that $2 \sin(\pi x) \in B$.

We claim that the following formulas hold, cf. [4]:

(3.1)
$$\Lambda_n^*(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^\infty \frac{\sin(k\pi x)}{k^{2n+1}}, \ n \ge 0, \ x \in]0, 1[,$$

(3.2)
$$A_n(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin(k\pi x)}{k^{2n+1}}, \ n \ge 0, x \in]0, 1[.$$

Formula (3.2) follows immediately from (3.1). For n = 0 (3.1) is the familiar formula

$$\frac{\pi}{2}(1-x) = \sum_{k=1}^{\infty} \frac{\sin (k\pi x)}{k}, \ 0 < x < 1.$$

Suppose that (3.1) holds for *n* replaced by n - 1 for some $n \ge 1$. Denoting the right-hand side of (3.1) by f_n , we have $f_n(0) = f_n(1) = 0$ and

$$f_n''(x) = -\frac{2}{\pi^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin (k\pi x)}{k^{2n-1}}.$$

which is equal to $-\Lambda_{n-1}^*$ by the induction hypothesis. It follows by (2.3) that $f_n = \Lambda_n^*$, and (3.1) is proved. From (3.1) and (3.2) it follows that $\pi^{2n+1}\Lambda_n$ and $\pi^{2n+1}\Lambda_n^* \in B$. We also get $\lim_{n \to \infty} \pi^{2n+1}\Lambda_n(x) = \lim_{n \to \infty} \pi^{2n+1}\Lambda_n^*(x)$ = 2 sin (πx). We have now established the following result:

PROPOSITION 3.1. The set B is a compact convex base for W and the extreme points of B are $2 \sin(\pi x)$, $\pi^{2n+1} \Lambda_n^*(x)$, $\pi^{2n+1} \Lambda_n(x)$, $n \ge 0$, which form a closed subset of B.

By l_{+}^{1} we denote the set of sequences $(\alpha_{n})_{n \ge 0}$ of non-negative numbers such that $\sum_{n=0}^{\infty} \alpha_{n} < \infty$.

By the Choquet representation theorem or just by the Krein-Milman theorem we get the following, cf. [3]:

THEOREM 3.2. For every $f \in W$ there exist $a \ge 0$ and sequences $(\alpha_n), (\beta_n) \in l_+^1$ such that

(3.1)
$$f(x) = 2a \sin (\pi x) + \sum_{n=0}^{\infty} \alpha_n \pi^{2n+1} \Lambda_n^*(x) + \sum_{n=0}^{\infty} \beta_n \pi^{2n+1} \Lambda_n(x); \quad 0 < x < 1.$$

The functions in *B* are uniformly bounded by 2π , and therefore the series (3.1) is uniformly convergent.

If we differentiate the series in (3.1) two times and change sign we get the series

$$\pi^{2}\left(2a \sin(\pi x) + \sum_{n=0}^{\infty} \alpha_{n+1}\pi^{2n+1}\Lambda_{n}^{*}(x) + \sum_{n=0}^{\infty} \beta_{n+1}\pi^{2n+1}\Lambda_{n}(x)\right),$$

which also converges uniformly on]0, 1[because $\sum_{n=0}^{\infty} \alpha_{n+1} + \sum_{n=0}^{\infty} \beta_{n+1} < \infty$.

It follows that the following formula holds:

$$(3.2) \quad (-1)^k f^{(2k)}(x) = \pi^{2k} \left(2a \sin(\pi x) + \sum_{n=0}^{\infty} \alpha_{n+k} \pi^{2n+1} \Lambda_n^*(x) + \sum_{n=0}^{\infty} \beta_{n+k} \pi^{2n+1} \Lambda_n(x) \right)$$

for 0 < x < 1, $k \ge 0$ and furthermore

(3.3)
$$\alpha_k = \pi^{-2k-1} (-1)^k f^{(2k)}(0), \ \beta_k = \pi^{-2k-1} (-1)^k f^{(2k)}(1)$$

for $k \ge 0$.

This proves that the sequences (α_n) , (β_n) and hence also a are uniquely determined by f. We have thus shown that B is a simplex. The extreme points of B form a closed subset of B as remarked in Proposition 3.1 so we can formulate the following

COROLLARY 3.3. The base B for W is a Bauer simplex.

Whittaker proved in [4] that the series in (3.1) in fact converges uniformly over arbitrary compact subsets of the complex plane. This also proves that fcan be extended to an entire holomorphic function which we also call f. For $x \in [0, 1[$ and $y \in \mathbf{R}$ we then have

$$f(x+iy) = \sum_{k=0}^{\infty} f^{(k)}(x) \frac{(iy)^k}{k!},$$

hence

Re
$$f(x+iy) = \sum_{k=0}^{\infty} (-1)^k f^{(2k)}(x) \frac{y^{2k}}{(2k)!}$$
,

which shows that $x \mapsto \text{Ref}(x+iy)$ belongs to W for all $y \in \mathbf{R}$, as sum of the functions

$$x \mapsto (-1)^k f^{2k}(x) \frac{y^{2k}}{(2k)!}$$

which all belong to the closed cone W.

This gives a short proof of the recent result of Mugler [2].

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