

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 23 (1977)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** REPRESENTATION OF COMPLETELY CONVEX FUNCTIONS BY  
THE EXTREME-POINT METHOD  
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**DOI:** <https://doi.org/10.5169/seals-48925>

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# REPRESENTATION OF COMPLETELY CONVEX FUNCTIONS BY THE EXTREME-POINT METHOD

by CHRISTIAN BERG

## 0. INTRODUCTION

A function  $f : ]0, 1[ \rightarrow \mathbf{R}$  is called completely convex, if it is  $C^\infty$  and  $(-1)^k f^{(2k)} \geq 0$  for all  $k \geq 0$ . A completely convex function  $f$  is called minimal if  $f(x) - a \sin(\pi x)$  is not completely convex for any number  $a > 0$ . Widder showed (cf. [5]) that a completely convex function can be extended to an entire holomorphic function, and in the paper [6] he proved that a minimal completely convex function can be expanded in a Lidstone series. This indicates that the Lidstone polynomials lie on the extreme rays of the cone  $W$  of completely convex functions.

The purpose of the present paper is to treat the completely convex functions by the extreme-point method and to obtain the expansion in Lidstone series as a special case of the Choquet representation theorem.

We will proceed as follows: In the topology of point-wise convergence the set  $W$  of completely convex functions is a closed, metrizable convex cone. We prove directly that the extreme rays of  $W$  are generated by certain polynomials — essentially the Lidstone polynomials — and the function  $\sin(\pi x)$ . The occurrence of the extreme ray generated by  $\sin(\pi x)$  is related to the fact that only minimal completely convex functions can be expanded in Lidstone series.

The cone  $W$  has a compact convex base  $B$ , and the extreme points of  $B$  are determined. It turns out that  $B$  is a Bauer simplex, i.e.  $B$  is a simplex and the extreme points form a closed set.

The author wants to acknowledge Widder's paper [6] as a source of inspiration. The reason for writing this paper is that we felt it natural to investigate the cone  $W$  by the extreme-point method.

Recently Mugler [2] showed that real part of the holomorphic extension of  $f \in W$  to the strip  $\operatorname{Re} z \in ]0, 1[$  is completely convex on each segment  $\{x + iy \mid 0 < x < 1\}$ . We give a very short proof of this result.

### 1. COMPLETELY CONVEX FUNCTIONS

Let  $I$  denote an open interval. A function  $f : I \rightarrow \mathbf{R}$  is called *completely convex*, if it is  $C^\infty$  and  $(-1)^k f^{(2k)} \geq 0$  on  $I$  for  $k \geq 0$ .

The set of completely convex functions is a convex cone denoted  $W = W(I)$ . We always equip  $W$  with the topology of pointwise convergence, i.e., with the topology induced by the product space  $\mathbf{R}^I$ .

LEMMA 1.1. *If  $I$  is unbounded  $W(I)$  consists of the non-negative affine functions, and  $W(\mathbf{R})$  consists of the non-negative constants.*

*Proof.* Assume first that  $\inf I = -\infty$ . Then every  $f \in W$  is decreasing since it is non-negative and concave. For  $k \geq 0$  and  $f \in W$  we have  $(-1)^k f^{(2k)} \in W$  and consequently  $(-1)^k f^{(2k+1)} \leq 0$ . This shows that also  $-f' \in W$  and then  $-f'' \leq 0$ , but by definition  $f'' \leq 0$  and therefore  $f$  is affine.

The case  $\sup I = \infty$  is treated in a similar manner. Finally, every non-negative concave function on  $\mathbf{R}$  is constant.

*Remark.* Completely convex sequences are non-negative and affine.

For a sequence  $a = (a_0, a_1, \dots)$  of real numbers we define  $\Delta a$  to be the sequence  $(\Delta a)_n = a_{n+1} - a_n, n \geq 0$ , and  $\Delta^k a$  is defined as  $\Delta(\Delta^{k-1} a)$  for  $k \geq 1$ , where  $\Delta^0 a = a$ . A sequence  $a$  is called *completely convex* if  $(-1)^k \Delta^{2k} a \geq 0$  for  $k \geq 0$ . The same method as in Lemma 1.1 leads to the conclusion that every completely convex sequence satisfies  $\Delta a \geq 0$  and  $\Delta^2 a = 0$ . The completely convex sequences are therefore exactly the sequences  $a_n = \alpha n + \beta$ , where  $\alpha, \beta \geq 0$ .

This is an answer to a remark by Boas [1]: "Nothing seems to be known about completely convex sequences".

In the following we will always assume that  $I$  is bounded, and for the sake of convenience we choose  $I$  to be  $I = ]0, 1[$ . We simply write  $W$  for  $W(]0, 1[)$ . For  $f \in W$  we have  $-f'' \in W$  and  $f^* \in W$ , where  $f^*$  is defined by  $f^*(x) = f(1-x)$ . The mapping  $f \mapsto f^*$  is an affine isomorphism of  $W$  onto itself.

LEMMA 1.2. *Let  $f : ]0, 1[ \rightarrow \mathbf{R}$  be non-negative and concave. Then the following holds :*

$$(i) \quad f(x) \leq 2f(1/2) \quad \text{for} \quad x \in ]0, 1[.$$

$$(ii) \quad f(x) \geq \frac{1}{\pi} f(x_0) \sin(\pi x) \quad \text{for} \quad x, x_0 \in ]0, 1[.$$

(iii) ([6], Lemma 7.1) *If there exists  $x_0 \in ]0, 1[$  and  $a > 0$  such that  $f(x_0) < a \sin(\pi x_0)$  then  $f(x) \leq a\pi$  for  $x \in ]0, 1[$ .*

*Proof.* (i). For  $x \in ]0, 1/2]$  we have that  $f(x)$  lies below the line through  $(1/2, f(1/2))$  and  $(1, 0)$  and (i) follows for  $x \in ]0, 1/2]$ . The interval  $]1/2, 1[$  is treated similarly.

(ii). Let  $x_0 \in ]0, 1[$ . For  $x \in ]0, x_0]$  we have

$$f(x) \geq \frac{f(x_0)}{x_0} x \geq f(x_0) x \geq \frac{f(x_0)}{\pi} \sin(\pi x),$$

and for  $x \in [x_0, 1[$  we have

$$f(x) \geq \frac{f(x_0)(1-x)}{1-x_0} \geq f(x_0)(1-x) \geq \frac{f(x_0)}{\pi} \sin \pi(1-x) = \frac{f(x_0)}{\pi} \sin(\pi x).$$

(iii). If  $f(x_0) > a\pi$  the inequality (ii) implies that  $f(x) > a \sin(\pi x)$  for  $x \in ]0, 1[$ .

Since every  $f \in W$  can be extended to an entire holomorphic function all derivatives of  $f$  have finite limits at 0 and 1. This can also be established in an elementary way from the property  $(-1)^k f^{(2k)} \geq 0$  for  $k \geq 0$ . We will therefore freely use  $f^{(k)}(x)$  for  $x = 0, 1$  as the limit of  $f^{(k)}(x)$  at these points.

LEMMA 1.3. *The cone  $W$  is a closed and metrizable subset of  $\mathbf{R}^I$ .*

*Proof.* The set of concave functions  $f : I \rightarrow \mathbf{R}$  is a closed and metrizable subset of  $\mathbf{R}^I$ , and therefore it suffices to prove that the pointwise limit  $f$  of a sequence  $(f_n)$  from  $W$  belongs to  $W$ .

It follows by Lemma 1.2 (i) that there exists a constant  $A$  such that  $f_n \leq A$  for all  $n$ <sup>1)</sup>. The dominated convergence theorem then shows that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) \varphi(x) dx = \int_0^1 f(x) \varphi(x) dx$$

for all  $\varphi \in \mathcal{D}(]0, 1[)$ , so  $(f_n)$  converges to  $f$  weakly in the distribution sense. Therefore  $(-1)^k f^{(2k)} \geq 0$  for all  $k \geq 0$  in the distribution sense, and this implies that  $f$  is  $C^\infty$  and hence  $f \in W$ .

<sup>1)</sup> In fact,  $A = 2 \sup f_n(1/2)$  can be used. It is finite because  $\lim_{n \rightarrow \infty} f_n(1/2)$  exists.

2. DETERMINATION OF THE EXTREME RAYS OF  $W$

Inspired by [6] we consider the Green's function

$$G(x, t) = \begin{cases} (1-x)t & \text{for } 0 \leq t < x \leq 1, \\ (1-t)x & \text{for } 0 \leq x \leq t \leq 1. \end{cases}$$

If  $\varphi$  is a continuous function on  $[0, 1]$  the unique solution  $f \in C([0, 1]) \cap C^2(]0, 1[)$  to the equations

$$(2.1) \quad f'' = -\varphi \text{ in } ]0, 1[, \quad f(0) = f(1) = 0$$

is

$$(2.2) \quad f(x) = \int_0^1 G(x, t) \varphi(t) dt.$$

The successive iterates of  $G$  are defined for  $x, t \in [0, 1]$  by the equations

$$G_1(x, t) = G(x, t)$$

$$G_n(x, t) = \int_0^1 G(x, y) G_{n-1}(y, t) dy, \quad n \geq 2.$$

It is clear that  $G_n(x, t) \geq 0$  for  $x, t \in [0, 1]$ .

We define recursively a sequence of polynomials  $(A_n)_{n \geq 0}^1$  by the requirement

$$(2.3) \quad A_0(x) = x, \quad A_n'' = -A_{n-1} \quad \text{and} \quad A_n(0) = A_n(1) = 0$$

for  $n \geq 1$ .

The polynomial  $A_n$  is of degree  $(2n + 1)$ , and we clearly have

$$(2.4) \quad A_n(x) = \int_0^1 G(x, t) A_{n-1}(t) dt = \int_0^1 G_n(x, t) t dt \quad \text{for}$$

$n \geq 1, x \in [0, 1].$

It follows that  $A_n \geq 0$  on  $[0, 1]$  for all  $n$ , and since  $(-1)^k A_n^{(2k)} = A_{n-k}$  for  $k \leq n$  we see that  $A_n \in W$ .

We recall that a ray  $\mathbf{R}_+x$  of a cone  $C$  is called *extreme*, if an equation  $x = f + g$  with  $f, g \in C$  is possible only if  $f, g \in \mathbf{R}_+x$ , cf. [3].

<sup>1)</sup> Our terminology is different from that of [6];  $(-1)^n A_n$  is equal to the  $n$ 'th Lidstone polynomial of [4] and [6].

PROPOSITION 2.1. *The polynomials  $A_n, n \geq 0$ , lie on extreme rays of  $W$ .*

*Proof.* If  $A_0 = f + g$  with  $f, g \in W$  we have  $0 = f'' + g''$ , but since  $f''$  and  $g''$  are both  $\leq 0$ , we conclude that  $f$  and  $g$  are affine. Furthermore, since  $f(0) = g(0) = 0$ , we conclude that  $f$  and  $g$  are proportional to  $A_0$ .

Suppose now that  $A_{n-1}, n \geq 1$ , lies on an extreme ray of  $W$ , and assume that  $A_n = f + g$  where  $f, g \in W$ . Then  $A_{n-1} = -f'' + (-g'')$ , and the induction hypothesis implies that  $-f''$  and  $-g''$  are proportional to  $A_{n-1}$ . Therefore we have  $f = \lambda A_n(x) + ax + b$  for certain numbers  $\lambda \geq 0, a, b$ . Since  $0 \leq f \leq A_n$ , we have  $f(0) = f(1) = 0$  which implies that  $a = b = 0$ . This proves that  $f$  (and similarly  $g$ ) are proportional to  $A_n$  which then lies on an extreme ray of  $W$ .

Since  $f \mapsto f^*$  is an affine isomorphism of  $W$  the polynomials  $A_n^*$  also lie on extreme rays of  $W$ . The following result is a special case of [6], Theorem 1.1.

PROPOSITION 2.2. *Every function  $f \in W$  can for  $n \geq 1$  be written as*

$$f(x) = \sum_{k=0}^{n-1} ((-1)^k f^{(2k)}(0) A_k^*(x) + (-1)^k f^{(2k)}(1) A_k(x)) + R_n(x),$$

where

$$R_n(x) = \int_0^1 G_n(x, t) (-1)^n f^{(2n)}(t) dt \in W.$$

*Proof.* For  $n = 1$  the formula is equivalent with

$$(2.5) \quad f(x) - f(0)(1-x) - f(1)x = R_1(x) = - \int_0^1 G(x, t) f''(t) dt,$$

which follows directly from (2.2), and it is clear that  $R_1 \in W$ .

Suppose now the formula holds for some  $n \geq 1$ . Applying (2.5) to  $(-1)^n f^{(2n)} \in W$  we get

$$\begin{aligned} (-1)^n f^{(2n)}(x) &= (-1)^n f^{(2n)}(0) A_0^*(x) + (-1)^n f^{(2n)}(1) A_0(x) \\ &+ \int_0^1 G(x, t) (-1)^{n+1} f^{(2n+2)}(t) dt, \end{aligned}$$

which substituted in the expression for  $R_n$  yields the formula for  $n+1$  because of (2.4).

To see that  $R_n \in W$  we notice that

$$(-1)^k R_n^{(2k)}(x) = \begin{cases} \int_0^1 G_{n-k}(x, t) (-1)^n f^{(2n)}(t) dt & \text{for } 0 \leq k \leq n-1 \\ (-1)^k f^{(2k)}(x) & \text{for } k \geq n. \end{cases}$$

The following lemma is easy to establish, but instead of giving the proof here we refer to [6].

LEMMA 2.3. *There exists a constant  $M > 0$  such that*

$$0 \leq \int_0^1 G_n(x, t) dt \leq \frac{M}{\pi^{2n}} \quad \text{for} \quad 0 \leq x \leq 1, \quad n \geq 1.$$

PROPOSITION 2.4. *The only functions  $f \in W$  satisfying  $f^{(2k)}(0) = f^{(2k)}(1) = 0$  for all  $k \geq 0$  are  $f(x) = a \sin(\pi x)$  with  $a \geq 0$ .*

*Proof.* Suppose  $f \in W$  satisfies  $f^{(2k)}(0) = f^{(2k)}(1) = 0$  for all  $k \geq 0$ . Defining  $a = \sup \{ \alpha \geq 0 \mid f - \alpha \sin(\pi x) \in W \}$ ,  $g = f - a \sin(\pi x)$  belongs to  $W$  because  $W$  is closed in  $\mathbf{R}^I$ . Furthermore

$$g^{(2k)}(0) = g^{(2k)}(1) = 0 \quad \text{for all } k \geq 0.$$

Let  $\varepsilon > 0$  be given. Since  $\varphi = g - \varepsilon \sin(\pi x) \notin W$ , there exist  $k \geq 0$  and  $x_0 \in ]0, 1[$  such that  $(-1)^k \varphi^{(2k)}(x_0) < 0$ , hence

$$(-1)^k g^{(2k)}(x_0) < \varepsilon \pi^{2k} \sin(\pi x_0).$$

By Lemma 1.2 (iii) applied to  $(-1)^k g^{(2k)}$  we get

$$(-1)^k g^{(2k)}(t) \leq \varepsilon \pi^{2k+1} \quad \text{for} \quad 0 < t < 1,$$

and therefore by Proposition 2.2 and Lemma 2.3 for  $0 < x < 1$

$$\begin{aligned} g(x) &= \int_0^1 G_k(x, t) (-1)^k g^{(2k)}(t) dt \leq \varepsilon \pi^{2k+1} \int_0^1 G_k(x, t) dt \\ &\leq \varepsilon M \pi. \end{aligned}$$

This proves that  $g$  is identically zero.

PROPOSITION 2.5. *The extreme rays of  $W$  are precisely the rays generated by  $\Lambda_n$  and  $\Lambda_n^*$ , where  $n \geq 0$ , and  $\sin(\pi x)$ .*

*Proof.* We first show that  $\sin(\pi x)$  lies on an extreme ray. If  $\sin(\pi x) = f + g$  where  $f, g \in W$ , we have  $f(0) = f(1) = g(0) = g(1) = 0$ . Differentiating  $2k$  times we similarly get  $f^{(2k)}(0) = f^{(2k)}(1) = g^{(2k)}(0) = g^{(2k)}(1) = 0$ , and it follows by Proposition 2.4 that  $f$  and  $g$  are proportional to  $\sin(\pi x)$ .

We finally have to show that an arbitrary extreme ray is generated by one of the above functions.

Assume that  $f \in W$  generates an extreme ray. If  $f^{(2k)}(0) = f^{(2k)}(1) = 0$  for all  $k \geq 0$  we already know by Proposition 2.4 that  $f$  is proportional to  $\sin(\pi x)$ . Otherwise let  $n$  be the smallest number  $\geq 0$  for which  $f^{(2n)}(0) \neq 0$  or  $f^{(2n)}(1) \neq 0$ . By Proposition 2.2 we then have

$$f(x) = (-1)^n f^{(2n)}(0) A_n^*(x) + (-1)^n f^{(2n)}(1) A_n(x) + R_{n+1}(x),$$

but since  $f$  lies on an extreme ray all three terms on the right-hand side lie on this ray.

If  $f^{(2n)}(0) \neq 0$  this shows that  $(-1)^n f^{(2n)}(1) A_n$  and  $R_{n+1}$  are proportional to  $A_n^*$ . Therefore  $f^{(2n)}(1) = 0$  and  $R_{n+1}^{(2n+2)} = f^{(2n+2)}$  is proportional to  $(A_n^*)^{(2n+2)} = 0$ , so that  $f^{(2n+2)} = 0$  and hence  $R_{n+1} = 0$  (cf. Proposition 2.2).

If  $f^{(2n)}(1) \neq 0$  we similarly get  $f^{(2n)}(0) = 0$  and  $R_{n+1} = 0$ . This shows that  $f$  lies on the ray generated by either  $A_n^*$  or  $A_n$ .

### 3. DETERMINATION OF A BASE FOR $W$

There are several ways of determining a base for  $W$ . We choose the following set

$$B = \left\{ f \in W \mid \int_0^1 f(x) \sin(\pi x) dx = 1 \right\}.$$

By Lemma 1.2 (ii) we get for  $f \in B$  and  $x_0 \in ]0, 1[$  that

$$1 \geq \frac{1}{\pi} f(x_0) \int_0^1 \sin^2(\pi x) dx = \frac{1}{2\pi} f(x_0),$$

so the functions in  $B$  are uniformly bounded by  $2\pi$ .

It is therefore clear that  $B$  is a compact convex base for  $W$ .

The extreme points of  $B$  are exactly the intersections between  $B$  and the extreme rays of  $W$ . We see that  $2 \sin(\pi x) \in B$ .

We claim that the following formulas hold, cf. [4]:

$$(3.1) \quad A_n^*(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geq 0, \quad x \in ]0, 1[ ,$$

$$(3.2) \quad A_n(x) = \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin(k\pi x)}{k^{2n+1}}, \quad n \geq 0, \quad x \in ]0, 1[ .$$

Formula (3.2) follows immediately from (3.1). For  $n = 0$  (3.1) is the familiar formula

$$\frac{\pi}{2} (1-x) = \sum_{k=1}^{\infty} \frac{\sin (k\pi x)}{k}, \quad 0 < x < 1.$$

Suppose that (3.1) holds for  $n$  replaced by  $n - 1$  for some  $n \geq 1$ . Denoting the right-hand side of (3.1) by  $f_n$ , we have  $f_n(0) = f_n(1) = 0$  and

$$f_n''(x) = -\frac{2}{\pi^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin (k\pi x)}{k^{2n-1}}.$$

which is equal to  $-A_{n-1}^*$  by the induction hypothesis. It follows by (2.3) that  $f_n = A_n^*$ , and (3.1) is proved. From (3.1) and (3.2) it follows that  $\pi^{2n+1}A_n$  and  $\pi^{2n+1}A_n^* \in B$ . We also get  $\lim_{n \rightarrow \infty} \pi^{2n+1}A_n(x) = \lim_{n \rightarrow \infty} \pi^{2n+1}A_n^*(x) = 2 \sin (\pi x)$ . We have now established the following result:

**PROPOSITION 3.1.** *The set  $B$  is a compact convex base for  $W$  and the extreme points of  $B$  are  $2 \sin (\pi x)$ ,  $\pi^{2n+1}A_n^*(x)$ ,  $\pi^{2n+1}A_n(x)$ ,  $n \geq 0$ , which form a closed subset of  $B$ .*

By  $l_+^1$  we denote the set of sequences  $(\alpha_n)_{n \geq 0}$  of non-negative numbers such that  $\sum_0^{\infty} \alpha_n < \infty$ .

By the Choquet representation theorem or just by the Krein-Milman theorem we get the following, cf. [3]:

**THEOREM 3.2.** *For every  $f \in W$  there exist  $a \geq 0$  and sequences  $(\alpha_n), (\beta_n) \in l_+^1$  such that*

$$(3.1) \quad f(x) = 2a \sin (\pi x) + \sum_{n=0}^{\infty} \alpha_n \pi^{2n+1} A_n^*(x) + \sum_{n=0}^{\infty} \beta_n \pi^{2n+1} A_n(x); \quad 0 < x < 1.$$

The functions in  $B$  are uniformly bounded by  $2\pi$ , and therefore the series (3.1) is uniformly convergent.

If we differentiate the series in (3.1) two times and change sign we get the series

$$\pi^2 \left( 2a \sin (\pi x) + \sum_{n=0}^{\infty} \alpha_{n+1} \pi^{2n+1} A_n^*(x) + \sum_{n=0}^{\infty} \beta_{n+1} \pi^{2n+1} A_n(x) \right),$$

which also converges uniformly on  $]0, 1[$  because  $\sum_{n=0}^{\infty} \alpha_{n+1} + \sum_{n=0}^{\infty} \beta_{n+1} < \infty$ .

It follows that the following formula holds:

$$(3.2) \quad (-1)^k f^{(2k)}(x) = \pi^{2k} \left( 2a \sin(\pi x) + \sum_{n=0}^{\infty} \alpha_{n+k} \pi^{2n+1} A_n^*(x) + \sum_{n=0}^{\infty} \beta_{n+k} \pi^{2n+1} A_n(x) \right)$$

for  $0 < x < 1$ ,  $k \geq 0$  and furthermore

$$(3.3) \quad \alpha_k = \pi^{-2k-1} (-1)^k f^{(2k)}(0), \quad \beta_k = \pi^{-2k-1} (-1)^k f^{(2k)}(1)$$

for  $k \geq 0$ .

This proves that the sequences  $(\alpha_n)$ ,  $(\beta_n)$  and hence also  $a$  are uniquely determined by  $f$ . We have thus shown that  $B$  is a simplex. The extreme points of  $B$  form a closed subset of  $B$  as remarked in Proposition 3.1 so we can formulate the following

**COROLLARY 3.3.** *The base  $B$  for  $W$  is a Bauer simplex.*

Whittaker proved in [4] that the series in (3.1) in fact converges uniformly over arbitrary compact subsets of the complex plane. This also proves that  $f$  can be extended to an entire holomorphic function which we also call  $f$ . For  $x \in ]0, 1[$  and  $y \in \mathbf{R}$  we then have

$$f(x + iy) = \sum_{k=0}^{\infty} f^{(k)}(x) \frac{(iy)^k}{k!},$$

hence

$$\operatorname{Re} f(x + iy) = \sum_{k=0}^{\infty} (-1)^k f^{(2k)}(x) \frac{y^{2k}}{(2k)!},$$

which shows that  $x \mapsto \operatorname{Re} f(x + iy)$  belongs to  $W$  for all  $y \in \mathbf{R}$ , as sum of the functions

$$x \mapsto (-1)^k f^{(2k)}(x) \frac{y^{2k}}{(2k)!}$$

which all belong to the closed cone  $W$ .

This gives a short proof of the recent result of Mugler [2].

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(Reçu le 26 avril 1977)

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