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complex conjugation. By the Stone-Weierstrass theorem [29, p. 151] we conclude that $B = C_0(\hat{A})$ and hence that $x \rightarrow \hat{x}$ is onto. Thus the proof of the representation theorem for commutative B^* -algebras is complete.

The reader who is interested in an unconventional proof of the preceding theorem may consult Edward Nelson [38, p. 78]. Quite simple proofs of the Gelfand-Naimark theorem in the special case of function algebras have been given by Nelson Dunford and Jacob T. Schwartz [14, pp. 274-275] and Karl E. Aubert [5].

5. THE GELFAND-NAIMARK THEOREM FOR ARBITRARY B^* -ALGEBRAS

The proof of the representation theorem for an arbitrary B^* -algebra is much more involved than the commutative case and it will be divided into several steps. After having established that the involution is continuous we will introduce a new equivalent B^* -norm with isometric involution. An investigation of the unitary elements will show that the original norm on the algebra coincides with this new norm. The representation of B^* -algebras will then easily be effected by the well known Gelfand-Naimark-Segal construction. General references for material in this section are [13], [37] and [43].

Step. 1. *The involution in a B^* -algebra A is continuous.*

Proof [39, Lemma 1.3]. First we show that the set $H(A) = \{ h \in A : h^* = h \}$ of *hermitian elements* in A is closed. Let $\{ h_n \}$ be a convergent sequence in $H(A)$ whose limit is $h + ik$, with $h, k \in H(A)$. Since $h_n - h \rightarrow ik$ we may assume (by putting h_n for $h_n - h$) that h_n converges to ik . The spectral mapping theorem for polynomials [43, p. 32] gives $\sigma_A(h_n^2 - h_n^4) = \{ \lambda^2 - \lambda^4 : \lambda \in \sigma_A(h_n) \}$; since $\| h \| = | h |_\sigma$ and $\sigma_A(h)$ is real (see the first part of the proof of Theorem I, the Aren's-Fukamiya arguments and recall $\sigma_A(h) = \hat{h}(\hat{A}) \cup \{ 0 \}$) we have

$$\begin{aligned} \| h_n^2 - h_n^4 \| &= \sup \{ \lambda^2 - \lambda^4 : \lambda \in \sigma_A(h_n) \} \\ &\leq \sup \{ \lambda^2 : \lambda \in \sigma_A(h_n) \} = \| h_n^2 \|. \end{aligned}$$

Letting $n \rightarrow \infty$ we obtain $\| -k^2 - k^4 \| \leq \| k^2 \|. Hence$

$$\sup \{ \lambda^2 + \lambda^4 : \lambda \in \sigma_A(k) \} \leq \sup \{ \lambda^2 : \lambda \in \sigma_A(k) \}.$$

Choose $\mu \in \sigma_A(k)$ such that $\mu^2 = \sup \{ \lambda^2 : \lambda \in \sigma_A(k) \}$. Then $\mu^2 + \mu^4 \leq \mu^2$, so $\mu = 0$. It follows that $\| k \| = | k |_\sigma = 0$ and hence $k = 0$. This shows that $H(A)$ is closed.

Now it is easy to prove that the graph of the map $x \rightarrow x^*$ of A onto A is closed. For suppose $x_n \rightarrow x$ and $x_n^* \rightarrow y$. Then $x_n + x_n^* \rightarrow x + y$ and $(x_n - x_n^*)/i \rightarrow (x - y)/i$. Since $H(A)$ is closed, $x + y$ and $(x - y)/i$ are hermitian and so $x + y = x^* + y^*$ and $x - y = y^* - x^*$, whence $y = x^*$. Thus by the closed graph theorem, valid for conjugate linear maps, the involution in A is continuous.

Step 2. Let A be a B^* -algebra. Then $\|x\|_0 = \|x^*x\|^{1/2}$ is an equivalent B^* -norm on A such that $\|x^*\|_0 = \|x\|_0$ for all $x \in A$, and $\|h\|_0 = \|h\|$ for all hermitian $h \in A$.

Proof. [2], [53]. By Step 1 there exists $M \geq 1$ such that $\|x^*\| \leq M \|x\|$ for all $x \in A$. Then

$$M^{-1/2} \|x\| \leq \|x^*\|^{1/2} \|x\|^{1/2} = \|x\|_0 \leq M^{1/2} \|x\|$$

so that $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent. Clearly $\|\cdot\|_0$ is homogeneous and submultiplicative. To prove the triangle inequality, let $x, y \in A$. Then

$$\|x + y\|_0^2 = \|(x + y)^*(x + y)\| \leq \|x^*x\| + \|y^*y\| + \|x^*y + y^*x\|$$

so it is enough to prove that $\|x^*y + y^*x\| \leq 2 \|x\|_0 \|y\|_0$. For any positive integer n

$$\begin{aligned} & \| (x^*y)^{2^{n-1}} + (y^*x)^{2^{n-1}} \|^2 \\ &= \| (x^*y)^{2^n} + (y^*x)^{2^n} + (x^*y)^{2^{n-1}} (y^*x)^{2^{n-1}} + (y^*x)^{2^{n-1}} (x^*y)^{2^{n-1}} \| \\ &\leq \| (x^*y)^{2^n} + (y^*x)^{2^n} \| + 2 (\|x^*x\| \cdot \|y^*y\|)^{2^{n-1}}. \end{aligned}$$

For every $\varepsilon > 0$ there is an integer n such that

$$\| (x^*y)^{2^n} \| \leq (|x^*y|_\sigma^2 + \varepsilon)^{2^{n-1}} \quad \text{and} \quad \| (y^*x)^{2^n} \| \leq (|y^*x|_\sigma^2 + \varepsilon)^{2^{n-1}}.$$

Then

$$\begin{aligned} \| (x^*y)^{2^n} \| &\leq (|x^*y|_\sigma |y^*x|_\sigma + \varepsilon)^{2^{n-1}} \leq (\|x^*y\| \cdot \|y^*x\| + \varepsilon)^{2^{n-1}} \\ &\leq (\|x^*x\| \cdot \|y^*y\| + \varepsilon)^{2^{n-1}} \end{aligned}$$

and similarly

$$\| (y^*x)^{2^n} \| \leq (\|x^*x\| \cdot \|y^*y\| + \varepsilon)^{2^{n-1}}$$

so that

$$\| (x^*y)^{2^n} + (y^*x)^{2^n} \|^2 \leq 2 (\|x^*x\| \cdot \|y^*y\| + \varepsilon)^{2^{n-1}}$$

Combining these results we recursively obtain

$$\| (x^*y)^{2^{k-1}} + (y^*x)^{2^{k-1}} \|^2 \leq 4(\|x^*x\| \cdot \|y^*y\| + \varepsilon)^{2^{k-1}}.$$

for any k , $1 \leq k \leq n$. Thus

$$\|x^*y + y^*x\|^2 \leq 4(\|x^*x\| \cdot \|y^*y\| + \varepsilon)$$

for arbitrary $\varepsilon > 0$. Hence $\|x^*y + y^*x\| \leq 2\|x\|_0\|y\|_0$. So we have seen that $\|\cdot\|_0$ is an equivalent algebra norm on A . Further, $\|h\|_0 = \|h^*h\|^{1/2} = \|h\|$ for all hermitian $h \in A$ and so $\|x\|_0^2 = \|x^*x\| = \|x^*x\|_0$; i.e., $\|\cdot\|_0$ is a B^* -norm on A with $\|x^*\|_0 = \|x\|_0$ for all $x \in A$.

Step 3. Positive elements and symmetry. Let A be a B^* -algebra with identity e . Then every hermitian $h \in A$ lies in a maximal commutative B^* -algebra B with identity e . Observe that $\sigma_B(x) = \sigma_A(x)$ for all $x \in B$ [43, p. 35]. By the characterization of commutative B^* -algebras B is isometrically $*$ -isomorphic to $C(\hat{B})$. Hence every hermitian element $h \in A$ has real spectrum.

A hermitian element $x \in A$ is called *positive*, and we write $x \geq 0$, if the spectrum of x in A is a subset of the nonnegative reals.

Clearly $x = h^2$ is positive for every hermitian $h \in A$. The set $P = \{x \in A : x \geq 0\}$ of all positive elements in A is called the *positive cone*. Indeed, P is a cone. For $\lambda \geq 0$ and $x \geq 0$ then $\lambda x \geq 0$ since $\sigma_A(\lambda x) = \lambda\sigma_A(x)$. That $x \geq 0$ and $y \geq 0$ implies $x + y \geq 0$ may be seen by the following *Kelley-Vaught argument* [31]:

Set $\alpha = \|x\|$, $\beta = \|y\|$, $z = x + y$, and $\gamma = \alpha + \beta$. Since $|x|_\sigma = \|x\|$ the assumption $x \geq 0$ implies $\sigma_A(x) \subset [0, \alpha]$, so that $\sigma_A(\alpha e - x) \subset [0, \alpha]$ and therefore $\|\alpha e - x\| = |\alpha e - x|_\sigma \leq \alpha$. For the same reason $\|\beta e - y\| \leq \beta$. Hence

$$\|\gamma e - z\| = \|(\alpha e - x) + (\beta e - y)\| \leq \alpha + \beta = \gamma.$$

Since $z^* = z$, $\sigma_A(\gamma e - z)$ is real so that $\sigma_A(\gamma e - z) \subset [-\gamma, \gamma]$ which implies that $\sigma_A(z) \subset [0, 2\gamma]$. Thus $x + y = z \geq 0$.

The symmetry of the involution in A now follows readily by *Kaplansky's argument* [45]:

We intend to show $x^*x \geq 0$ for all $x \in A$. By observing that a real-valued continuous function is the difference of two nonnegative real-valued continuous functions whose product is zero, we can write the hermitian element x^*x in the form

$$x^*x = u - v, \quad u \geq 0, \quad v \geq 0, \quad uv = 0 = vu.$$

Now $(xv)^*(xv) = v^*x^*xv = vx^*xv = v(u-v)v = -v^3$ so that $(xv)^*(xv) \leq 0$. Since $(xv)^*(xv)$ and $(xv)(xv)^*$ have the same nonzero spectrum, also $(xv)(xv)^* \leq 0$. Write $xv = h + ik$ with h and k hermitian. Then

$$0 \geq (xv)^*(xv) + (xv)(xv)^* = 2(h^2 + k^2) \geq 0.$$

Thus $h = 0 = k$ or $xv = 0$. But then $0 = (xv)^*(xv) = -v^3$ and so $v = 0$. Hence $x^*x = u \geq 0$; in particular, $e + x^*x$ is invertible for all $x \in A$.

Step 4. Let A be a B^* -algebra with isometric involution. Then there exists a net $\{e_\alpha\}$ of hermitian elements in A , bounded by one, such that $\lim e_\alpha x = x = \lim xe_\alpha$ for all $x \in A$. The net $\{e_\alpha\}$ is called an approximate identity.

Proof. The following construction is due to Irving E. Segal [50]. If A has no identity, we may embed A in a B^* -algebra A_e with identity e (see the proof of Theorem I). Thus in any case we can use the preceding results about positive elements.

For any $\alpha = \{x_1, \dots, x_n\}$ in the class of all finite subsets of A , ordered by inclusion, set $h = x_1^*x_1 + \dots + x_n^*x_n$. Then $h \geq 0$ and so $e_\alpha = nh(e + nh)^{-1}$ is a well defined element in A . Viewing h as a non-negative function on the structure space of some maximal commutative B^* -subalgebra we see that $\|e_\alpha\| = |e_\alpha|_\sigma \leq 1$. It remains to show that $\lim e_\alpha x = x = \lim xe_\alpha$. Observe that

$$\begin{aligned} [x_i(e - e_\alpha)]^* [x_i(e - e_\alpha)] &\leq \sum_{j=1}^n [x_j(e - e_\alpha)]^* [x_j(e - e_\alpha)] \\ &\leq (e - e_\alpha) h (e - e_\alpha) \\ &\leq h(e + nh)^{-2} \leq e/4n \end{aligned}$$

where the last inequality follows from the fact that the real function $t \rightarrow t(1 + nt)^{-2}$ ($t \geq 0$) has maximum value $1/4n$. Thus

$$\|x_i(e - e_\alpha)\|^2 = \|[x_i(e - e_\alpha)]^* [x_i(e - e_\alpha)]\| \leq 1/4n.$$

Now for arbitrary $x \in A$ and $\varepsilon > 0$ choose a finite set α_0 of n elements in A such that $x \in \alpha_0$ and $n > \varepsilon^{-2}$. Then for all $\alpha \geq \alpha_0$ we have $\|x - xe_\alpha\| = \|x(e - e_\alpha)\| < \varepsilon$. Hence $\lim xe_\alpha = x$ for every $x \in A$; and by the continuity of the involution also $\lim e_\alpha x = (\lim x^*e_\alpha)^* = (x^*)^* = x$.

Step 5. Every B^* -algebra without identity can be isometrically embedded in a B^* -algebra with identity.

Proof. Let A be a B^* -algebra without identity. By Step 2, A is a B^* -algebra with isometric involution with respect to the equivalent norm $\|x\|_0 = \|x^*x\|^{1/2}$. Hence, by Step 4, A has an approximate identity $\{e_\alpha\}$ consisting of hermitian elements such that $\|e_\alpha\| = \|e_\alpha\|_0 \leq 1$. Now observe that for every $x \in A$,

$$\|x\| = \sup \{ \|xy\| : y \in A, \|y\| \leq 1 \} = \sup \{ \|yx\| : y \in A, \|y\| \leq 1 \}$$

and extend the norm on A to A_e by

$$\begin{aligned} \|x + \lambda e\| &= \sup \{ \|(x + \lambda e)y\| : y \in A, \|y\| \leq 1 \} \\ &= \sup \{ \|y(x + \lambda e)\| : y \in A, \|y\| \leq 1 \}. \end{aligned}$$

Then A_e is a Banach $*$ -algebra with identity in which A is isometrically embedded as a closed ideal of codimension one. To see that the B^* -condition holds in A_e we first prove that

$$\|x + \lambda e\| = \lim_\alpha \| (x + \lambda e) e_\alpha \| = \lim_\alpha \| e_\alpha (x + \lambda e) \|.$$

Given any $\varepsilon > 0$ there exists $y \in A$ with $\|y\| \leq 1$ such that

$$\| (x + \lambda e) y \| > \|x + \lambda e\| - \varepsilon.$$

Since $\lim_\alpha (x + \lambda e) e_\alpha y = (x + \lambda e) y$, there exists α_0 such that for all $\alpha \geq \alpha_0$, $\| (x + \lambda e) e_\alpha y \| > \|x + \lambda e\| - \varepsilon$. Since $\| (x + \lambda e) e_\alpha y \| \leq \| (x + \lambda e) e_\alpha \| \leq \|x + \lambda e\|$, it follows that $\lim_\alpha \| (x + \lambda e) e_\alpha \|$ exists and is equal to $\|x + \lambda e\|$. Similarly $\lim_\alpha \| e_\alpha (x + \lambda e) \| = \|x + \lambda e\|$. Thus

$$\begin{aligned} \| (x + \lambda e)^* \| \cdot \| (x + \lambda e) \| &= \lim_\alpha \| e_\alpha (x + \lambda e)^* \| \cdot \lim_\alpha \| (x + \lambda e) e_\alpha \| \\ &= \lim_\alpha \| e_\alpha (x + \lambda e)^* (x + \lambda e) e_\alpha \| \\ &= \| (x + \lambda e)^* (x + \lambda e) \|. \end{aligned}$$

Therefore $\| (x + \lambda e)^* (x + \lambda e) \| = \| (x + \lambda e)^* \| \cdot \|x + \lambda e\|$, and so A_e is a B^* -algebra.

Step 6. Let A be a B^* -algebra with identity e and isometric involution. Denote by $U = \{ u \in A : u^*u = e = uu^* \}$ the group of unitary elements in A . Then every element x in A is a linear combination of unitary elements and $\|x\| = \|x\|_u$, where

$$\|x\|_u = \inf \left\{ \sum_{n=1}^N |\lambda_n| : x = \sum_{n=1}^N \lambda_n u_n, \lambda_n \in \mathbb{C}, u_n \in U \right\}.$$

Proof. To prove that every $x \in A$ is a linear combination of unitary elements it clearly suffices to show that every hermitian $h \in A$ with $\|h\| < 1$

can be written as a linear combination of unitary elements. If $\|h\| < 1$, then $\|h^2\| \leq \|h\|^2 < 1$ and so

$$k = \sum_{n=0}^{\infty} \binom{1/2}{n} (-h^2)^n$$

is a well-defined element in A . Clearly, k is a hermitian element commuting with h such that $k^2 = e - h^2$. Thus $u = h + ik$ is unitary and $h = \frac{1}{2}u + \frac{1}{2}u^*$.

It now follows that $\|x\|_u$ (as given in Step 6) is well-defined for each $x \in A$; further, it is clear from the definition that $\|\cdot\|_u$ is a seminorm on A . We shall call it the *unitary seminorm*. Since the unitary elements form a group under multiplication $\|\cdot\|_u$ is submultiplicative.

Let us compare the unitary seminorm with the B^* -norm on A . Observe that $\|h\|_u \leq \|h\|$ for every hermitian $h \in A$. Indeed, if $\|h\| < 1$, then $h = \frac{1}{2}u + \frac{1}{2}u^*$ for some unitary $u \in A$ and so $\|h\|_u \leq 1$. Thus $\|h\|_u \leq \|h\|$ for every hermitian $h \in A$. Further $\|x\|_u \leq 2\|x\|$ for every $x \in A$. For if $x = h + ik$ with hermitian h and k , then $\|x\|_u \leq \|h\|_u + \|k\|_u \leq 2\|x\|$. On the other hand $\|x\| \leq \|x\|_u$ for all $x \in A$. Indeed, if $x = \sum_{n=1}^N \lambda_n u_n$, $\lambda_n \in \mathbb{C}$, $u_n \in U$, then

$$\|x\| = \left\| \sum_{n=1}^N \lambda_n u_n \right\| \leq \sum_{n=1}^N |\lambda_n| \cdot \|u_n\| = \sum_{n=1}^N |\lambda_n|$$

since $\|u\|^2 = \|u^*u\| = 1$ for every unitary $u \in A$. Thus $\|x\| \leq \|x\|_u$. Hence the unitary seminorm and the B^* -norm on A are equivalent norms with $\|x\| \leq \|x\|_u \leq 2\|x\|$ for all $x \in A$. To see that these two norms are actually equal we need the following result of Russo and Dye [44] about the closure of the convex hull of the unitary elements in A .

Russo-Dye Theorem. *Let A be a B^* -algebra with identity e and isometric involution. Then the open unit ball of A is contained in the closed convex hull of the unitary elements of A ; that is, for each x in A with $\|x\| < 1$ and each $\varepsilon > 0$ there exists a positive integer m and unitary elements u_k such that $\left\| x - \sum_{k=1}^m \frac{1}{m} u_k \right\| < \varepsilon$.*

The equality of the unitary seminorm and the B^* -norm on A is an immediate consequence of this result. Indeed, let $x \in A$ with $\|x\| < 1$.

Then for every $\varepsilon > 0$ there is a positive integer m and unitary elements u_k such that $\left\| x - \sum_{k=1}^m \frac{1}{m} u_k \right\| < \varepsilon$ and so

$$\begin{aligned} \|x\|_u &\leq \left\| \sum_{k=1}^m \frac{1}{m} u_k \right\|_u + \left\| x - \sum_{k=1}^m \frac{1}{m} u_k \right\|_u \\ &\leq \sum_{k=1}^m \frac{1}{m} \|u_k\|_u + 2 \left\| x - \sum_{k=1}^m \frac{1}{m} u_k \right\| \leq 1 + 2\varepsilon; \end{aligned}$$

since $\varepsilon > 0$ was arbitrary, $\|x\|_u \leq 1$. This proves $\|x\|_u \leq \|x\|$ and so $\|x\| = \|x\|_u$ for all $x \in A$.

For completeness we will now prove the Russo-Dye Theorem. The following elementary proof, valid for arbitrary Banach *-algebras with isometric involution, is based on ideas of Harris [28].

Proof of the Russo-Dye Theorem: Let $x \in A$ with $\|x\| < 1$. Then $\|xx^*\| \leq \|x\| \cdot \|x^*\| = \|x\|^2 < 1$. Hence the hermitian element $e - xx^*$ is invertible and has the invertible hermitian square root $(e - xx^*)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-xx^*)^n$. Similarly $e - x^*x$ has invertible hermitian square root $(e - x^*x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-x^*x)^n$. For complex λ with $|\lambda| = 1$ define

$$u_\lambda = (e - xx^*)^{-1/2} (x - \lambda e) (e - \lambda x^*)^{-1} (e - x^*x)^{1/2}.$$

We intend to show that u_λ is unitary. Since $\lambda\bar{\lambda} = 1$,

$$\begin{aligned} u_\lambda^* &= (e - x^*x)^{1/2} (e - \bar{\lambda}x)^{-1} (x^* - \bar{\lambda}e) (e - xx^*)^{-1/2} \\ &= (e - x^*x)^{1/2} (\lambda e - x)^{-1} (\lambda x^* - e) (e - xx^*)^{-1/2}. \end{aligned}$$

Observe that

$$\begin{aligned} (\lambda e - x)^{-1} (\lambda x^* - e) &= (\lambda e - x)^{-1} [(\lambda e - x)x^* - (e - xx^*)] \\ &= x^* - (\lambda e - x)^{-1} (e - xx^*), \\ (e - \lambda x^*) (x - \lambda e)^{-1} &= [x^* (\lambda e - x) - (e - x^*x)] (\lambda e - x)^{-1} \\ &= x^* - (e - x^*x) (\lambda e - x)^{-1}, \end{aligned}$$

and

$$\begin{aligned} x (e - x^*x)^{1/2} &= \sum_{n=0}^{\infty} \binom{1/2}{n} x (-x^*x)^n = \sum_{n=0}^{\infty} \binom{1/2}{n} (-xx^*)^n x \\ &= (e - xx^*)^{1/2} x \end{aligned}$$

which may be conjugated to give the related equality

$$(e - x^*x)^{1/2} x^* = x^* (e - xx^*)^{1/2}.$$

Utilizing these relations it follows easily that $u_\lambda^* = u_\lambda^{-1}$ so u_λ is unitary.

Let $u_{k/m}$ denote the unitary element u_λ with $\lambda = \exp\left(2\pi i \frac{k}{m}\right)$ where k, m are positive integers. We will show that $x = \lim_{m \rightarrow \infty} \sum_{k=1}^m (1/m) u_{k/m}$.

With λ as above, let $x_{k/m}$ denote the element

$$x_\lambda = (x - \lambda e)(e - \lambda x^*)^{-1}.$$

Then

$$\begin{aligned} x - \sum_{k=1}^m \frac{1}{m} u_{k/m} &= x - \frac{1}{m} \sum_{k=1}^m (e - x x^*)^{-1/2} x_{k/m} (e - x^* x)^{1/2} \\ &= (e - x x^*)^{-1/2} \left[x - \frac{1}{m} \sum_{k=1}^m x_{k/m} \right] (e - x^* x)^{1/2} \end{aligned}$$

and so

$$\begin{aligned} (1) \quad & \left\| x - \sum_{k=1}^m \frac{1}{m} u_{k/m} \right\| \\ & \leq \left\| (e - x x^*)^{-1/2} \right\| \cdot \left\| x - \frac{1}{m} \sum_{k=1}^m x_{k/m} \right\| \cdot \left\| (e - x^* x)^{1/2} \right\|. \end{aligned}$$

Observe that

$$x_\lambda = \sum_{n=0}^{\infty} (x - \lambda e)(\lambda x^*)^n = \sum_{n=0}^{\infty} \lambda^n x (x^*)^n - \sum_{n=0}^{\infty} \lambda^{n+1} (x^*)^n$$

and so

$$\begin{aligned} x - x_\lambda &= \sum_{n=0}^{\infty} \lambda^{n+1} (x^*)^n - \sum_{n=1}^{\infty} \lambda^n x (x^*)^n \\ &= \sum_{n=1}^{\infty} \lambda^n [(x^*)^{n-1} - x (x^*)^n] \\ &= (e - x x^*) \sum_{n=1}^{\infty} \lambda^n (x^*)^{n-1}. \end{aligned}$$

Summing over $k, 1 \leq k \leq m$, and dividing by m we have

$$\begin{aligned} x - \frac{1}{m} \sum_{k=1}^m x_{k/m} &= \frac{1}{m} \sum_{k=1}^m (x - x_{k/m}) \\ &= (e - x x^*) \sum_{n=1}^{\infty} \frac{1}{m} \sum_{k=1}^m \left[\exp\left(2\pi i \frac{k}{m}\right) \right]^n (x^*)^{n-1} \\ &= (e - x x^*) \sum_{n=1}^{\infty} \frac{1}{m} \sum_{k=1}^m \left[\exp\left(2\pi i \frac{n}{m}\right) \right]^k (x^*)^{n-1}. \end{aligned}$$

Now, if $1 \leq n < m$, then $\exp\left(2\pi i \frac{n}{m}\right) \neq 1$ and so by the sum formula for

a finite geometric sum

$$\sum_{k=1}^m \left[\exp \left(2\pi i \frac{n}{m} \right) \right]^k = \frac{\exp \left(2\pi i \frac{n}{m} \right) - \exp \left(2\pi i \frac{n(m+1)}{m} \right)}{1 - \exp \left(2\pi i \frac{n}{m} \right)} = 0;$$

hence we have

$$x - \frac{1}{m} \sum_{k=1}^m x_{k/m} = (e - xx^*) \sum_{n=m}^{\infty} \frac{1}{m} \sum_{k=1}^m \left[\exp \left(2\pi i \frac{n}{m} \right) \right]^k (x^*)^{n-1}.$$

Then

$$\begin{aligned} \left\| x - \frac{1}{m} \sum_{k=1}^m x_{k/m} \right\| &\leq \| e - xx^* \| \sum_{n=m-1}^{\infty} \| (x^*)^n \| \\ &\leq \| e - xx^* \| \sum_{n=m-1}^{\infty} \| x \|^n \\ &\leq \| e - xx^* \| \frac{\| x \|^{m-1}}{1 - \| x \|}. \end{aligned}$$

Since the right hand side converges to 0 as $m \rightarrow \infty$, the theorem now follows immediately from relation (1) above.

Step 7. The involution in a B-algebra A is isometric.*

Proof. Since every B*-algebra without identity can be isometrically embedded in a B*-algebra with identity we may assume A has an identity. By Step 2 $\| x \|_0 = \| x^*x \|^{1/2}$ is an equivalent B*-norm on A such that $\| x^* \|_0 = \| x \|_0$ for all $x \in A$. Hence, by Step 6, $\| x \|_0 = \| x \|_u$ where $\| \cdot \|_u$ is the unitary seminorm on A .

Observe that $\| u \| = 1$ for every unitary $u \in A$. Indeed, since u and u^* commute, by the argument given in the first step of the proof of Theorem I, we have $\| u^* \| = \| u \|$ and so $\| u \| = 1$.

Now, if $x = \sum_{n=1}^N \lambda_n u_n$, $\lambda_n \in \mathbb{C}$, $u_n \in U$, then

$$\| x \| = \left\| \sum_{n=1}^N \lambda_n u_n \right\| \leq \sum_{n=1}^N |\lambda_n| \cdot \| u_n \| = \sum_{n=1}^N |\lambda_n|.$$

Thus $\| x \| \leq \| x \|_u = \| x \|_0 = \| x^*x \|^{1/2}$ and so $\| x^* \| = \| x \|$.

Step 8. The Gelfand-Naimark-Segal Construction. We have seen that the involution in a B*-algebra A is isometric. Further, if A has no identity

we can embed A isometrically as a closed ideal of codimension one in the B^* -algebra A_e with identity e . Thus we can and will assume without loss of generality that A has an identity e .

The representation of such an algebra A as a norm-closed $*$ -subalgebra of bounded linear operators on a Hilbert space is effected by means of positive functionals on A and a construction due to Gelfand-Naimark [23] and Segal [49].

A *positive functional* on A is a linear functional p such that $p(x^*x) \geq 0$ for all $x \in A$. For $x, y \in A$ set $(x, y) = p(y^*x)$. This scalar product on A is linear in x , conjugate linear in y and (x, x) is nonnegative for all x . Thus in particular $p(y^*x) = \overline{p(x^*y)}$ and $|p(y^*x)|^2 \leq p(x^*x)p(y^*y)$ (Schwarz inequality). Setting $y = e$ we get $p(x^*) = \overline{p(x)}$ and $|p(x)|^2 \leq p(e)p(x^*x)$.

In general the scalar product on A is degenerate so that a reduction is necessary to obtain nondegeneracy. To this end we define the associated *null ideal* $I = \{x \in A : p(x^*x) = 0\}$. Since by the above properties of positive functionals

$$I = \{x \in A : p(y^*x) = 0 \text{ for all } y \in A\},$$

the null ideal is clearly a left ideal in A . Then the quotient space $X = A/I$ is a pre-Hilbert space with respect to the induced scalar product

$$(x+I, y+I) = p(y^*x)$$

and, further, for each $a \in A$ we can define a linear operator T_a on X by $T_a(x+I) = ax + I$. The map $a \rightarrow T_a$ has the following easily verified properties: $T_{a+b} = T_a + T_b$, $T_{\lambda a} = \lambda T_a$, $T_{ab} = T_a T_b$ and T_e is the identity operator; also

$$(T_a(x+I), y+I) = (x+I, T_a^*(y+I))$$

so that $a \rightarrow T_a$ is a **-representation* of A on the pre-Hilbert space X .

Let H be the Hilbert space completion of X . We want to show that every operator T_a on X can be extended to a bounded operator on H . We claim that $\|T_a\| \leq \|a\|$. Note that $\|T_a(x+I)\|^2 = (ax+I, ax+I) = p(x^*a^*ax)$. For any $\alpha > \|a^*a\| = \|a\|^2$ there exists a hermitian $h \in A$ such that $h^2 = \alpha e - a^*a$. Hence

$$\alpha p(x^*x) - p(x^*a^*ax) = p(x^*(\alpha e - a^*a)x) = p((hx)^*(hx)) \geq 0$$

and so $p(x^*a^*ax) \leq \|a\|^2 p(x^*x)$. Thus $\|T_a\| \leq \|a\|$. Denote the extended operator on H also by T_a .

The preceding discussion has shown that for every positive functional on A there is associated a $*$ -representation of A as a $*$ -subalgebra of bounded linear operators on a Hilbert space H such that $\|T_a\| \leq \|a\|$. In general this representation is neither injective nor norm-preserving. By constructing appropriate positive functionals in the next step we will, however, be able to build a representation with these properties.

Step 9. *Construction of positive functionals.* We will construct for every fixed $z \in A$ a positive functional p on A such that $p(e) = 1$ and $p(z^*z) = \|z\|^2$. Clearly the associated $*$ -representation has the property $\|T_z\| = \|z\|$. Indeed,

$$\begin{aligned} \|z\|^2 &= p(z^*z) = (T_z(e+I), T_z(e+I)) = \|T_z(e+I)\|^2 \\ &\leq \|T_z\|^2 \|e+I\|^2 = \|T_z\|^2 p(e) = \|T_z\|^2 \end{aligned}$$

which together with $\|T_z\| \leq \|z\|$ gives $\|T_z\| = \|z\|$.

The following construction of the desired positive functional is a special case of an extension theorem for positive functionals due to M. Krein [32].

Construction: Let $H(A)$ be the real vector space of hermitian elements in A and P the positive cone of all positive elements in A . On the subspace $Re + Rz^*z$ of $H(A)$ generated by e and z^*z define p by

$$p(\alpha e + \beta z^*z) = \alpha + \beta \|z^*z\|.$$

Note that p is well-defined on $Re + Rz^*z$ even if e and z^*z are linearly dependent. Since $\|z^*z\| = |z^*z|_\sigma \in \sigma_A(z^*z)$ we have that $\alpha + \beta \|z^*z\|$ lies in $\sigma_A(\alpha e + \beta z^*z)$. In other words, $p(x) \in \sigma_A(x)$ if $x \in Re + Rz^*z$ so that $p(x) \geq 0$ for all $x \in P \cap (Re + Rz^*z)$.

Assume p has been extended to a real-linear functional on a subspace W of $H(A)$ such that $p(x) \geq 0$ for all $x \in P \cap W$ and assume that there is a $y \in H(A)$ with $y \notin W$. Set

$$a = \inf \{ p(v) : y \leq v \in W \} \text{ and } b = \sup \{ p(u) : y \geq u \in W \}.$$

Since $y \leq \|y\|e$ and $y \geq -\|y\|e$ the infimum and supremum are taken over nonempty sets, and are therefore finite numbers, clearly satisfying $a \geq b$. Define p on the subspace of $H(A)$ generated by W and y by

$$p(x + \alpha y) = p(x) + \alpha c \quad (x \in W, \alpha \in \mathbb{R}),$$

where c is any fixed number such that $a \geq c \geq b$.

Suppose that $x + \alpha y \geq 0$ ($x \in W, \alpha \in R$). We shall show that $p(x + y) \geq 0$. If $\alpha = 0$, then $p(x + \alpha y) = p(x) \geq 0$ by assumption.

If $\alpha > 0$, then $x + \alpha y \geq 0$ implies ' $y \geq -\frac{x}{\alpha} \in W$, so that $p\left(-\frac{x}{\alpha}\right) \leq c$, or $p(x + \alpha y) \geq 0$.

If $\alpha < 0$, then $x + \alpha y \geq 0$ implies $y \leq -\frac{x}{\alpha} \in W$, so that $p\left(-\frac{x}{\alpha}\right) \geq c$, or $p(x + \alpha y) \geq 0$.

By Zorn's Lemma we conclude that p can be extended to a real linear functional p on $H(A)$ such that $p(x) \geq 0$ for all $x \in P$.

Finally set $p(x) = p(h) + ip(k)$ if $x = h + ik$ with $h, k \in H(A)$. Then p is a positive functional on A such that $p(e) = 1$ and $p(z^*z) = \|z^*z\| = \|z\|^2$. This completes the construction.

Step 10. *The isometric *-representation.* In the preceding step we constructed for every $z \in A$ a positive functional on A such that the associated *-representation $T^{(z)}$ of A on the Hilbert space $H^{(z)}$ is norm-decreasing and $\|T_z^{(z)}\| = \|z\|$.

Let H be the direct sum of the Hilbert spaces $H^{(z)}$. The *direct sum* of the family $H^{(z)}$, $z \in A$, is defined as the set of all mappings f on A with $f(z) \in H^{(z)}$ such that $\sum_{z \in A} (f(z), f(z)) < \infty$. The algebraic operations in H are pointwise and the scalar product is given by $(f, g) = \sum_{z \in A} (f(z), g(z))$.

The reader may easily verify that all Hilbert space axioms are satisfied by H (see [14]).

Define the *-representation T of A on H by

$$(T_a f)(z) = T_a^{(z)}(f(z)).$$

Note that the inequality

$$\sum_{z \in A} ((T_a f)(z), (T_a f)(z)) \leq \|a\|^2 \sum_{z \in A} (f(z), f(z))$$

shows that with f also $T_a f$ belongs to H . Then T_a is a bounded operator on H such that

$$\|T_a\| = \sup_{z \in A} \|T_a^{(z)}\| = \|T_a^{(a)}\| = \|a\|.$$

Hence the map $a \rightarrow T_a$ is a norm-preserving *-representation of A on H . This completes the proof of Theorem II as stated in the introduction.