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complex conjugation. By the Stone-Weierstrass theorem [29, p. 151] we conclude that $\hat{B} = C_0(\hat{A})$ and hence that $x \to \hat{x}$ is onto. Thus the proof of the representation theorem for commutative B*-algebras is complete.

The reader who is interested in an unconventional proof of the preceding theorem may consult Edward Nelson [38, p. 78]. Quite simple proofs of the Gelfand-Naimark theorem in the special case of function algebras have been given by Nelson Dunford and Jacob T. Schwartz [14, pp. 274-275] and Karl E. Aubert [5].

## 5. The Gelfand-Naimark theorem for arbitrary B*-algebras

The proof of the representation theorem for an arbitrary B*-algebra is much more involved than the commutative case and it will be divided into several steps. After having established that the involution is continuous we will introduce a new equivalent B*-norm with isometric involution. An investigation of the unitary elements will show that the original norm on the algebra coincides with this new norm. The representation of B*-algebras will then easily be effected by the well known Gelfand-Naimark-Segal construction. General references for material in this section are [13], [37] and [43].

Step. 1. *The involution in a* B**-algebra* $A$ *is continuous.*

*Proof* [39, Lemma 1.3]. First we show that the set $H(A) = \{ h \in A : h^* = h \}$ of *hermitian elements* in $A$ is closed. Let $\{ h_n \}$ be a convergent sequence in $H(A)$ whose limit is $h + ik$, with $h, k \in H(A)$. Since $h_n - h \to ik$ we may assume (by putting $h_n$ for $h_n - h$) that $h_n$ converges to $ik$. The spectral mapping theorem for polynomials [43, p. 32] gives $\sigma_A(h_n^2 - h_n^4) = \{ \lambda^2 - \lambda^4 : \lambda \in \sigma_A(h_n) \}$; since $\| h \| = | h |_\sigma$ and $\sigma_A(h)$ is real (see the first part of the proof of Theorem I, the Aren's-Fukamiya arguments and recall $\sigma_A(h) = \hat{h}(\hat{A}) \cup \{ 0 \}$) we have

$$\| h_n^2 - h_n^4 \| = \sup \{ \lambda^2 - \lambda^4 : \lambda \in \sigma_A(h_n) \} \leqslant \sup \{ \lambda^2 : \lambda \in \sigma_A(h_n) \} = \| h_n^2 \| .$$

Letting $n \to \infty$ we obtain $\| - k^2 - k^4 \| \leqslant \| k^2 \|$. Hence

$$\sup \{ \lambda^2 + \lambda^4 : \lambda \in \sigma_A(k) \} \leqslant \sup \{ \lambda^2 : \lambda \in \sigma_A(k) \} .$$

Choose $\mu \in \sigma_A(k)$ such that $\mu^2 = \sup \{ \lambda^2 : \lambda \in \sigma_A(k) \}$. Then $\mu^2 + \mu^4 \leqslant \mu^2$, so $\mu = 0$. It follows that $\| k \| = | k |_\sigma = 0$ and hence $k = 0$. This shows that $H(A)$ is closed.

Now it is easy to prove that the graph of the map $x \to x^*$ of $A$ onto $A$ is closed. For suppose $x_n \to x$ and $x_n^* \to y$. Then $x_n + x_n^* \to x + y$ and $(x_n - x_n^*)/i \to (x-y)/i$. Since $H(A)$ is closed, $x+y$ and $(x-y)/i$ are hermitian and so $x + y = x^* + y^*$ and $x - y = y^* - x^*$, whence $y = x^*$. Thus by the closed graph theorem, valid for conjugate linear maps, the involution in $A$ is continuous.

Step 2. *Let $A$ be a* B*-*algebra. Then $\| x \|_0 = \| x^*x \|^{1/2}$ is an equivalent* B*-*norm on $A$ such that $\| x^* \|_0 = \| x \|_0$ for all $x \in A$, and $\| h \|_0 = \| h \|$ for all hermitian $h \in A$.*

*Proof.* [2], [53]. By Step 1 there exists $M \geqslant 1$ such that $\| x^* \| \leqslant M \| x \|$ for all $x \in A$. Then

$$M^{-1/2} \| x \| \leqslant \| x^* \|^{1/2} \| x \|^{1/2} = \| x \|_0 \leqslant M^{1/2} \| x \|$$

so that $\| \cdot \|_0$ and $\| \cdot \|$ are equivalent. Clearly $\| \cdot \|_0$ is homogeneous and submultiplicative. To prove the triangle inequality, let $x, y \in A$. Then

$$\| x + y \|_0^2 = \| (x+y)^* (x+y) \| \leqslant \| x^*x \| + \| y^*y \| + \| x^*y + y^*x \|$$

so it is enough to prove that $\| x^*y + y^*x \| \leqslant 2 \| x \|_0 \| y \|_0$. For any positive integer $n$

$$\begin{aligned} &\| (x^*y)^{2^{n-1}} + (y^*x)^{2^{n-1}} \|^2 \\ &= \| (x^*y)^{2^n} + (y^*x)^{2^n} + (x^*y)^{2^{n-1}} (y^*x)^{2^{n-1}} + (y^*x)^{2^{n-1}} (x^*y)^{2^{n-1}} \| \\ &\leqslant \| (x^*y)^{2^n} + (y^*x)^{2^n} \| + 2 (\|x^*x\| \cdot \| y^*y\|)^{2^{n-1}} . \end{aligned}$$

For every $\varepsilon > 0$ there is an integer $n$ such that

$$\| (x^*y)^{2^n} \| \leqslant (|x^*y|_\sigma^2 + \varepsilon)^{2^{n-1}} \text{ and } \| (y^*x)^{2^n} \| \leqslant (|y^*x|_\sigma^2 + \varepsilon)^{2^{n-1}} .$$

Then

$$\begin{aligned} \| (x^*y)^{2^n} \| &\leqslant (|x^*y|_\sigma |y^*x|_\sigma + \varepsilon)^{2^{n-1}} \leqslant (\|x^*y\| \cdot \| y^*x\| + \varepsilon)^{2^{n-1}} \\ &\leqslant (\|x^*x\| \cdot \| y^*y\| + \varepsilon)^{2^{n-1}} \end{aligned}$$

and similarly

$$\| (y^*x)^{2^n} \| \leqslant (\|x^*x\| \cdot \| y^*y\| + \varepsilon)^{2^{n-1}}$$

so that

$$\| (x^*y)^{2^n} + (y^*x)^{2^n} \|^2 \leqslant 2 (\|x^*x\| \cdot \| y^*y\| + \varepsilon)^{2^{n-1}}$$

Combining these results we recursively obtain

$$\| (x^* y)^{2^{k-1}} + (y^* x)^{2^{k-1}} \|^2 \leqslant 4 (\|x^* x\| \cdot \|y^* y\| + \varepsilon)^{2^{k-1}} .$$

for any $k$, $1 \leqslant k \leqslant n$. Thus

$$\| x^* y + y^* x \|^2 \leqslant 4 (\|x^* x\| \cdot \|y^* y\| + \varepsilon)$$

for arbitrary $\varepsilon > 0$. Hence $\| x^* y + y^* x \| \leqslant 2 \| x \|_0 \| y \|_0$. So we have seen that $\| \cdot \|_0$ is an equivalent algebra norm on $A$. Further, $\| h \|_0 = \| h^* h \|^{1/2} = \| h \|$ for all hermitian $h \in A$ and so $\| x \|_0^2 = \| x^* x \| = \| x^* x \|_0$; i.e., $\| \cdot \|_0$ is a B*-norm on $A$ with $\| x^* \|_0 = \| x \|_0$ for all $x \in A$.

Step 3. *Positive elements and symmetry.* Let $A$ be a B*-algebra with identity $e$. Then every hermitian $h \in A$ lies in a maximal commutative B*-algebra $B$ with identity $e$. Observe that $\sigma_B(x) = \sigma_A(x)$ for all $x \in B$ [43, p. 35]. By the characterization of commutative B*-algebras $B$ is isometrically *-isomorphic to $C(\hat{B})$. Hence every hermitian element $h \in A$ has real spectrum.

A hermitian element $x \in A$ is called *positive*, and we write $x \geqslant 0$, if the spectrum of $x$ in $A$ is a subset of the nonnegative reals.

Clearly $x = h^2$ is positive for every hermitian $h \in A$. The set $P = \{ x \in A : x \geqslant 0 \}$ of all positive elements in $A$ is called the *positive cone*. Indeed, $P$ is a cone. For $\lambda \geqslant 0$ and $x \geqslant 0$ then $\lambda x \geqslant 0$ since $\sigma_A(\lambda x) = \lambda \sigma_A(x)$. That $x \geqslant 0$ and $y \geqslant 0$ implies $x + y \geqslant 0$ may be seen by the following *Kelley-Vaught argument* [31]:

Set $\alpha = \| x \|$, $\beta = \| y \|$, $z = x + y$, and $\gamma = \alpha + \beta$. Since $| x |_\sigma = \| x \|$ the assumption $x \geqslant 0$ implies $\sigma_A(x) \subset [0, \alpha]$, so that $\sigma_A(\alpha e - x) \subset [0, \alpha]$ and therefore $\| \alpha e - x \| = | \alpha e - x |_\sigma \leqslant \alpha$. For the same reason $\| \beta e - y \| \leqslant \beta$. Hence

$$\| \gamma e - z \| = \| (\alpha e - x) + (\beta e - y) \| \leqslant \alpha + \beta = \gamma .$$

Since $z^* = z$, $\sigma_A(\gamma e - z)$ is real so that $\sigma_A(\gamma e - z) \subset [-\gamma, \gamma]$ which implies that $\sigma_A(z) \subset [0, 2\gamma]$. Thus $x + y = z \geqslant 0$.

The symmetry of the involution in $A$ now follows readily by *Kaplansky's argument* [45]:

We intend to show $x^* x \geqslant 0$ for all $x \in A$. By observing that a real-valued continuous function is the difference of two nonnegative real-valued continuous functions whose product is zero, we can write the hermitian element $x^* x$ in the form

$$x^* x = u - v , \; u \geqslant 0 , \; v \geqslant 0 , \; uv = 0 = vu .$$

Now $(xv)^*(xv) = v^*x^*xv = vx^*xv = v(u-v)v = -v^3$ so that $(xv)^*(xv) \leqslant 0$. Since $(xv)^*(xv)$ and $(xv)(xv)^*$ have the same nonzero spectrum, also $(xv)(xv)^* \leqslant 0$. Write $xv = h + ik$ with $h$ and $k$ hermitian. Then

$$0 \geqslant (xv)^*(xv) + (xv)(xv)^* = 2(h^2 + k^2) \geqslant 0 .$$

Thus $h = 0 = k$ or $xv = 0$. But then $0 = (xv)^*(xv) = -v^3$ and so $v = 0$. Hence $x^*x = u \geqslant 0$; in particular, $e + x^*x$ is invertible for all $x \in A$.

Step 4. *Let $A$ be a B*-algebra with isometric involution. Then there exists a net $\{e_\alpha\}$ of hermitian elements in $A$, bounded by one, such that* $\lim e_\alpha x = x = \lim xe_\alpha$ *for all $x \in A$. The net $\{e_\alpha\}$ is called an approximate identity.*

*Proof.* The following construction is due to Irving E. Segal [50]. If $A$ has no identity, we may embed $A$ in a B*-algebra $A_e$ with identity $e$ (see the proof of Theorem I). Thus in any case we can use the preceding results about positive elements.

For any $\alpha = \{x_1, \ldots, x_n\}$ in the class of all finite subsets of $A$, ordered by inclusion, set $h = x_1^*x_1 + \ldots + x_n^*x_n$. Then $h \geqslant 0$ and so $e_\alpha = nh(e+nh)^{-1}$ is a well defined element in $A$. Viewing $h$ as a non-negative function on the structure space of some maximal commutative B*-subalgebra we see that $\| e_\alpha \| = | e_\alpha |_\sigma \leqslant 1$. It remains to show that $\lim e_\alpha x = x = \lim xe_\alpha$. Observe that

$$\begin{aligned}[x_i(e-e_\alpha)]^*[x_i(e-e_\alpha)] &\leqslant \sum_{j=1}^{n} [x_j(e-e_\alpha)]^*[x_j(e-e_\alpha)] \\ &\leqslant (e-e_\alpha)h(e-e_\alpha) \\ &\leqslant h(e+nh)^{-2} \leqslant e/4n\end{aligned}$$

where the last inequality follows from the fact that the real function $t \to t(1+nt)^{-2}$ $(t \geqslant 0)$ has maximum value $1/4n$. Thus

$$\| x_i(e-e_\alpha) \|^2 = \| [x_i(e-e_\alpha)]^*[x_i(e-e_\alpha)] \| \leqslant 1/4n .$$

Now for arbitrary $x \in A$ and $\varepsilon > 0$ choose a finite set $\alpha_0$ of $n$ elements in $A$ such that $x \in \alpha_0$ and $n > \varepsilon^{-2}$. Then for all $\alpha \geqslant \alpha_0$ we have $\| x - xe_\alpha \| = \| x(e-e_\alpha) \| < \varepsilon$. Hence $\lim xe_\alpha = x$ for every $x \in A$; and by the continuity of the involution also $\lim e_\alpha x = (\lim x^*e_\alpha)^* = (x^*)^* = x$.

Step 5. *Every B*-algebra without identity can be isometrically embedded in a B*-algebra with identity.*

*Proof.* Let $A$ be a B*-algebra without identity. By Step 2, $A$ is a B*-algebra with isometric involution with respect to the equivalent norm $\| x \|_0 = \| x^*x \|^{1/2}$. Hence, by Step 4, $A$ has an approximate identity $\{ e_\alpha \}$ consisting of hermitian elements such that $\| e_\alpha \| = \| e_\alpha \|_0 \leqslant 1$. Now observe that for every $x \in A$,

$$\| x \| = \sup \{ \| xy \| : y \in A , \| y \| \leqslant 1 \} = \sup \{ \| yx \| : y \in A , \| y \| \leqslant 1 \}$$

and extend the norm on $A$ to $A_e$ by

$$\begin{aligned} \| x + \lambda e \| &= \sup \{ \| (x + \lambda e) y \| : y \in A , \| y \| \leqslant 1 \} \\ &= \sup \{ \| y (x + \lambda e) \| : y \in A , \| y \| \leqslant 1 \} . \end{aligned}$$

Then $A_e$ is a Banach *-algebra with identity in which $A$ is isometrically embedded as a closed ideal of codimension one. To see that the B*-condition holds in $A_e$ we first prove that

$$\| x + \lambda e \| = \lim_\alpha \| (x + \lambda e) e_\alpha \| = \lim_\alpha \| e_\alpha (x + \lambda e) \| .$$

Given any $\varepsilon > 0$ there exists $y \in A$ with $\| y \| \leqslant 1$ such that

$$\| (x + \lambda e) y \| > \| x + \lambda e \| - \varepsilon .$$

Since $\lim_\alpha (x + \lambda e) e_\alpha y = (x + \lambda e) y$, there exists $\alpha_0$ such that for all $\alpha \geqslant \alpha_0$, $\| (x + \lambda e) e_\alpha y \| > \| x + \lambda e \| - \varepsilon$. Since $\| (x + \lambda e) e_\alpha y \| \leqslant \| (x + \lambda e) e_\alpha \| \leqslant \| x + \lambda e \|$, it follows that $\lim_\alpha \| (x + \lambda e) e_\alpha \|$ exists and is equal to $\| x + \lambda e \|$. Similarly $\lim_\alpha \| e_\alpha (x + \lambda e) \| = \| x + \lambda e \|$. Thus

$$\begin{aligned} \| (x + \lambda e)^* \| \cdot \| (x + \lambda e) \| &= \lim_\alpha \| e_\alpha (x + \lambda e)^* \| \cdot \lim_\alpha \| (x + \lambda e) e_\alpha \| \\ &= \lim_\alpha \| e_\alpha (x + \lambda e)^* (x + \lambda e) e_\alpha \| \\ &= \| (x + \lambda e)^* (x + \lambda e) \| . \end{aligned}$$

Therefore $\| (x + \lambda e)^* (x + \lambda e) \| = \| (x + \lambda e)^* \| \cdot \| x + \lambda e \|$, and so $A_e$ is a B*-algebra.

Step 6. *Let $A$ be a B*-algebra with identity $e$ and isometric involution. Denote by $U = \{ u \in A : u^*u = e = uu^* \}$ the group of unitary elements in $A$. Then every element $x$ in $A$ is a linear combination of unitary elements and $\| x \| = \| x \|_u$, where*

$$\| x \|_u = \inf \{ \sum_{n=1}^{N} | \lambda_n | : x = \sum_{n=1}^{N} \lambda_n u_n , \lambda_n \in C , u_n \in U \} .$$

*Proof.* To prove that every $x \in A$ is a linear combination of unitary elements it clearly suffices to show that every hermitian $h \in A$ with $\| h \| < 1$

can be written as a linear combination of unitary elements. If $\| h \| < 1$, then $\| h^2 \| \leqslant \| h \|^2 < 1$ and so

$$k = \sum_{n=0}^{\infty} \binom{1/2}{n} (-h^2)^n$$

is a well-defined element in $A$. Clearly, $k$ is a hermitian element commuting with $h$ such that $k^2 = e - h^2$. Thus $u = h + ik$ is unitary and $h = \frac{1}{2} u + \frac{1}{2} u^*$.

It now follows that $\| x \|_u$ (as given in Step 6) is well-defined for each $x \in A$; further, it is clear from the definition that $\| \cdot \|_u$ is a seminorm on $A$. We shall call it the *unitary seminorm*. Since the unitary elements form a group under multiplication $\| \cdot \|_u$ is submultiplicative.

Let us compare the unitary seminorm with the B*-norm on $A$. Observe that $\| h \|_u \leqslant \| h \|$ for every hermitian $h \in A$. Indeed, if $\| h \| < 1$, then $h = \frac{1}{2} u + \frac{1}{2} u^*$ for some unitary $u \in A$ and so $\| h \|_u \leqslant 1$. Thus $\| h \|_u \leqslant \| h \|$ for every hermitian $h \in A$. Further $\| x \|_u \leqslant 2 \| x \|$ for every $x \in A$. For if $x = h + ik$ with hermitian $h$ and $k$, then $\| x \|_u \leqslant \| h \|_u + \| k \|_u \leqslant 2 \| x \|$. On the other hand $\| x \| \leqslant \| x \|_u$ for all $x \in A$. Indeed, if $x = \Sigma_{n=1}^{N} \lambda_n u_n$, $\lambda_n \in C$, $u_n \in U$, then

$$\| x \| = \| \sum_{n=1}^{N} \lambda_n u_n \| \leqslant \sum_{n=1}^{N} | \lambda_n | \cdot \| u_n \| = \sum_{n=1}^{N} | \lambda_n |$$

since $\| u \|^2 = \| u^* u \| = 1$ for every unitary $u \in A$. Thus $\| x \| \leqslant \| x \|_u$. Hence the unitary seminorm and the B*-norm on $A$ are equivalent norms with $\| x \| \leqslant \| x \|_u \leqslant 2 \| x \|$ for all $x \in A$. To see that these two norms are actually equal we need the following result of Russo and Dye [44] about the closure of the convex hull of the unitary elements in $A$.

Russo-Dye Theorem. *Let $A$ be a B*-algebra with identity $e$ and isometric involution. Then then open unit ball of $A$ is contained in the closed convex hull of the unitary elements of $A$; that is, for each $x$ in $A$ with $\| x \| < 1$ and each $\varepsilon > 0$ there exists a positive integer $m$ and unitary elements $u_k$ such that* $\left\| x - \Sigma_{k=1}^{m} \frac{1}{m} u_k \right\| < \varepsilon$.

The equality of the unitary seminorm and the B*-norm on $A$ is an immediate consequence of this result. Indeed, let $x \in A$ with $\| x \| < 1$.

Then for every $\varepsilon > 0$ there is a positive integer $m$ and unitary elements $u_k$ such that $\left\| x - \Sigma_{k=1}^{m} \frac{1}{m} u_k \right\| < \varepsilon$ and so

$$\| x \|_u \leqslant \left\| \sum_{k=1}^{m} \frac{1}{m} u_k \right\|_u + \left\| x - \sum_{k=1}^{m} \frac{1}{m} u_k \right\|_u$$

$$\leqslant \sum_{k=1}^{m} \frac{1}{m} \| u_k \|_u + 2 \left\| x - \sum_{k=1}^{m} \frac{1}{m} u_k \right\| \leqslant 1 + 2\varepsilon ;$$

since $\varepsilon > 0$ was arbitrary, $\| x \|_u \leqslant 1$. This proves $\| x \|_u \leqslant \| x \|$ and so $\| x \| = \| x \|_u$ for all $x \in A$.

For completeness we will now prove the Russo-Dye Theorem The following elementary proof, valid for arbitrary Banach *-algebras with isometric involution, is based on ideas of Harris [28].

*Proof of the Russo-Dye Theorem:* Let $x \in A$ with $\| x \| < 1$. Then $\| xx^* \| \leqslant \| x \| \cdot \| x^* \| = \| x \|^2 < 1$. Hence the hermitian element $e - xx^*$ is invertible and has the invertible hermitian square root $(e - xx^*)^{1/2} = \Sigma_{n=0}^{\infty} \binom{1/2}{n} (-xx^*)^n$. Similarly $e - x^*x$ has invertible hermitian square root $(e - x^*x)^{1/2} = \Sigma_{n=0}^{\infty} \binom{1/2}{n} (-x^*x)^n$. For complex $\lambda$ with $|\lambda| = 1$ define

$$u_\lambda = (e - xx^*)^{-1/2} (x - \lambda e)(e - \lambda x^*)^{-1} (e - x^*x)^{1/2} .$$

We intend to show that $u_\lambda$ is unitary. Since $\lambda\bar{\lambda} = 1$,

$$\begin{aligned} u_\lambda^* &= (e - x^*x)^{1/2} (e - \bar{\lambda} x)^{-1} (x^* - \bar{\lambda} e)(e - xx^*)^{-1/2} \\ &= (e - x^*x)^{1/2} (\lambda e - x)^{-1} (\lambda x^* - e)(e - xx^*)^{-1/2} . \end{aligned}$$

Observe that

$$\begin{aligned} (\lambda e - x)^{-1} (\lambda x^* - e) &= (\lambda e - x)^{-1} [(\lambda e - x) x^* - (e - xx^*)] \\ &= x^* - (\lambda e - x)^{-1} (e - xx^*) , \\ (e - \lambda x^*)(x - \lambda e)^{-1} &= [x^* (\lambda e - x) - (e - x^*x)] (\lambda e - x)^{-1} \\ &= x^* - (e - x^*x)(\lambda e - x)^{-1} , \end{aligned}$$

and

$$\begin{aligned} x (e - x^*x)^{1/2} &= \sum_{n=0}^{\infty} \binom{1/2}{n} x (-x^*x)^n = \sum_{n=0}^{\infty} \binom{1/2}{n} (-xx^*)^n x \\ &= (e - xx^*)^{1/2} x \end{aligned}$$

which may be conjugated to give the related equality

$$(e - x^*x)^{1/2} x^* = x^* (e - xx^*)^{1/2} .$$

Utilizing these relations it follows easily that $u_\lambda^* = u_\lambda^{-1}$ so $u_\lambda$ is unitary.

Let $u_{k/m}$ denote the unitary element $u_\lambda$ with $\lambda = \exp\left(2\pi i \frac{k}{m}\right)$ where $k$, $m$ are positive integers. We will show that $x = \lim \Sigma_{k=1}^{m} (1/m)\, u_{k/m}$.

With $\lambda$ as above, let $x_{k/m}$ denote the element

$$x_\lambda = (x - \lambda e)(e - \lambda x^*)^{-1}.$$

Then

$$x - \sum_{k=1}^{m} \frac{1}{m} u_{k/m} = x - \frac{1}{m} \sum_{k=1}^{m} (e - xx^*)^{-1/2} x_{k/m} (e - x^*x)^{1/2}$$
$$= (e - xx^*)^{-1/2} \left[ x - \frac{1}{m} \sum_{k=1}^{m} x_{k/m} \right] (e - x^*x)^{1/2}$$

and so

(1) $$\left\| x - \sum_{k=1}^{m} \frac{1}{m} u_{k/m} \right\|$$
$$\leqslant \| (e - xx^*)^{-1/2} \| \cdot \left\| x - \frac{1}{m} \sum_{k=1}^{m} x_{k/m} \right\| \cdot \| (e - x^*x)^{1/2} \|.$$

Observe that

$$x_\lambda = \sum_{n=0}^{\infty} (x - \lambda e)(\lambda x^*)^n = \sum_{n=0}^{\infty} \lambda^n x (x^*)^n - \sum_{n=0}^{\infty} \lambda^{n+1} (x^*)^n$$

and so

$$x - x_\lambda = \sum_{n=0}^{\infty} \lambda^{n+1} (x^*)^n - \sum_{n=1}^{\infty} \lambda^n x (x^*)^n$$
$$= \sum_{n=1}^{\infty} \lambda^n [(x^*)^{n-1} - x(x^*)^n]$$
$$= (e - xx^*) \sum_{n=1}^{\infty} \lambda^n (x^*)^{n-1}.$$

Summing over $k$, $1 \leqslant k \leqslant m$, and dividing by $m$ we have

$$x - \frac{1}{m} \sum_{k=1}^{m} x_{k/m} = \frac{1}{m} \sum_{k=1}^{m} (x - x_{k/m})$$
$$= (e - xx^*) \sum_{n=1}^{\infty} \frac{1}{m} \sum_{k=1}^{m} \left[ \exp\left(2\pi i \frac{k}{m}\right) \right]^n (x^*)^{n-1}$$
$$= (e - xx^*) \sum_{n=1}^{\infty} \frac{1}{m} \sum_{k=1}^{m} \left[ \exp\left(2\pi i \frac{n}{m}\right) \right]^k (x^*)^{n-1}.$$

Now, if $1 \leqslant n < m$, then $\exp\left(2\pi i \frac{n}{m}\right) \neq 1$ and so by the sum formula for a finite geometric sum

$$\sum_{k=1}^{m}\left[\exp\left(2\pi i\,\frac{n}{m}\right)\right]^{k} = \frac{\exp\left(2\pi i\,\frac{n}{m}\right) - \exp\left(2\pi i\,\frac{n\,(m+1)}{m}\right)}{1 - \exp\left(2\pi i\,\frac{n}{m}\right)} = 0\,;$$

hence we have

$$x - \frac{1}{m}\sum_{k=1}^{m} x_{k/m} = (e - xx^*)\sum_{n=m}^{\infty}\frac{1}{m}\sum_{k=1}^{m}\left[\exp\left(2\pi i\,\frac{n}{m}\right)\right]^{k}(x^*)^{n-1}\,.$$

Then

$$\left\| x - \frac{1}{m}\sum_{k=1}^{m} x_{k/m}\right\| \leqslant \| e - xx^* \| \sum_{n=m-1}^{\infty} \| (x^*)^n \|$$

$$\leqslant \| e - xx^* \| \sum_{n=m-1}^{\infty} \| x \|^n$$

$$\leqslant \| e - xx^* \| \frac{\| x \|^{m-1}}{1 - \| x \|}\,.$$

Since the right hand side converges to 0 as $m \to \infty$, the theorem now follows immediately from relation (1) above.

Step 7. *The involution in a B*-algebra $A$ is isometric.*

*Proof.* Since every B*-algebra without identity can be isometrically embedded in a B*-algebra with identity we may assume $A$ has an identity. By Step 2 $\| x \|_0 = \| x^*x \|^{1/2}$ is an equivalent B*-norm on $A$ such that $\| x^* \|_0 = \| x \|_0$ for all $x \in A$. Hence, by Step 6, $\| x \|_0 = \| x \|_u$ where $\| \cdot \|_u$ is the unitary seminorm on $A$.

Observe that $\| u \| = 1$ for every unitary $u \in A$. Indeed, since $u$ and $u^*$ commute, by the argument given in the first step of the proof of Theorem I, we have $\| u^* \| = \| u \|$ and so $\| u \| = 1$.

Now, if $x = \Sigma_{n=1}^{N} \lambda_n u_n$, $\lambda_n \in C$, $u_n \in U$, then

$$\| x \| = \left\| \sum_{n=1}^{N} \lambda_n u_n \right\| \leqslant \sum_{n=1}^{N} | \lambda_n | \cdot \| u_n \| = \sum_{n=1}^{N} | \lambda_n |\,.$$

Thus $\| x \| \leqslant \| x \|_u = \| x \|_0 = \| x^*x \|^{1/2}$ and so $\| x^* \| = \| x \|$.

Step 8. *The Gelfand-Naimark-Segal Construction.* We have seen that the involution in a B*-algebra $A$ is isometric. Further, if $A$ has no identity we can embed $A$ isometrically as a closed ideal of codimension one in the B*-algebra $A_e$ with identity $e$. Thus we can and will assume without loss of generality that $A$ has an identity $e$.

The representation of such an algebra $A$ as a norm-closed *-subalgebra of bounded linear operators on a Hilbert space is effected by means of positive functionals on $A$ and a construction due to Gelfand-Naimark [23] and Segal [49].

A *positive functional* on $A$ is a linear functional $p$ such that $p(x^*x) \geqslant 0$ for all $x \in A$. For $x, y \in A$ set $(x, y) = p(y^*x)$. This scalar product on $A$ is linear in $x$, conjugate linear in $y$ and $(x, x)$ is nonnegative for all $x$. Thus in particular $p(y^*x) = \overline{p(x^*y)}$ and $|p(y^*x)|^2 \leqslant p(x^*x)\,p(y^*y)$ (Schwarz inequality). Setting $y = e$ we get $p(x^*) = \overline{p(x)}$ and $|p(x)|^2 \leqslant p(e)\,p(x^*x)$.

In general the scalar product on $A$ is degenerate so that a reduction is necessary to obtain nondegeracy. To this end we define the associated *null ideal* $I = \{x \in A : p(x^*x) = 0\}$. Since by the above properties of positive functionals

$$I = \{x \in A : p(y^*x) = 0 \quad \text{for all } y \in A\},$$

the null ideal is clearly a left ideal in $A$. Then the quotient space $X = A/I$ is a pre-Hilbert space with respect to the induced scalar product

$$(x+I, y+I) = p(y^*x)$$

and, further, for each $a \in A$ we can define a linear operator $T_a$ on $X$ by $T_a(x+I) = ax + I$. The map $a \to T_a$ has the following easily verified properties: $T_{a+b} = T_a + T_b$, $T_{\lambda a} = \lambda T_a$, $T_{ab} = T_a T_b$, and $T_e$ is the identity operator; also

$$\left(T_a(x+I),\ y+I\right) = \left(x+I,\ T_a{}^*(y+I)\right)$$

so that $a \to T_a$ is a **-representation* of $A$ on the pre-Hilbert space $X$.

Let $H$ be the Hilbert space completion of $X$. We want to show that every operator $T_a$ on $X$ can be extended to a bounded operator on $H$. We claim that $\| T_a \| \leqslant \| a \|$. Note that $\| T_a(x+I) \|^2 = (ax+I, ax+I) = p(x^*a^*ax)$. For any $\alpha > \| a^*a \| = \| a \|^2$ there exists a hermitian $h \in A$ such that $h^2 = \alpha e - a^*a$. Hence

$$\alpha p(x^*x) - p(x^*a^*ax) = p\left(x^*(\alpha e - a^*a)\,x\right) = p\left((hx)^*(hx)\right) \geqslant 0$$

and so $p(x^*a^*ax) \leqslant \| a \|^2 p(x^*x)$. Thus $\| T_a \| \leqslant \| a \|$. Denote the extended operator on $H$ also by $T_a$.

The preceding discussion has shown that for every positive functional on $A$ there is associated a $*$-representation of $A$ as a $*$-subalgebra of bounded linear operators on a Hilbert space $H$ such that $\| T_a \| \leqslant \| a \|$. In general this representation is neither injective nor norm-preserving. By constructing appropriate positive functionals in the next step we will, however, be able to build a representation with these properties.

Step 9. *Construction of positive functionals.* We will construct for every fixed $z \in A$ a positive functional $p$ on $A$ such that $p(e) = 1$ and $p(z^*z) = \| z \|^2$. Clearly the associated $*$-representation has the property $\| T_z \| = \| z \|$. Indeed,

$$\| z \|^2 = p(z^*z) = (T_z(e+I), \; T_z(e+I)) = \| T_z(e+I) \|^2$$
$$\leqslant \| T_z \|^2 \| e + I \|^2 = \| T_z \|^2 p(e) = \| T_z \|^2$$

which together with $\| T_z \| \leqslant \| z \|$ gives $\| T_z \| = \| z \|$.

The following construction of the desired positive functional is a special case of an extension theorem for positive functionals due to M. Krein [32].

Construction: Let $H(A)$ be the real vector space of hermitian elements in $A$ and $P$ the positive cone of all positive elements in $A$. On the subspace $Re + Rz^*z$ of $H(A)$ generated by $e$ and $z^*z$ define $p$ by

$$p(\alpha e + \beta z^*z) = \alpha + \beta \| z^*z \| .$$

Note that $p$ is well-defined on $Re + Rz^*z$ even if $e$ and $z^*z$ are linearly dependent. Since $\| z^*z \| = | z^*z |_\sigma \in \sigma_A(z^*z)$ we have that $\alpha + \beta \| z^*z \|$ lies in $\sigma_A(\alpha e + \beta z^*z)$. In other words, $p(x) \in \sigma_A(x)$ if $x \in Re + Rz^*z$ so that $p(x) \geqslant 0$ for all $x \in P \cap (Re + Rz^*z)$.

Assume $p$ has been extended to a real-linear functional on a subspace $W$ of $H(A)$ such that $p(x) \geqslant 0$ for all $x \in P \cap W$ and assume that there is a $y \in H(A)$ with $y \notin W$. Set

$$a = \inf \{ p(v) : y \leqslant v \in W \} \text{ and } b = \sup \{ p(u) : y \geqslant u \in W \} .$$

Since $y \leqslant \| y \| e$ and $y \geqslant - \| y \| e$ the infimum and supremum are taken over nonempty sets, and are therefore finite numbers, clearly satisfying $a \geqslant b$. Define $p$ on the subspace of $H(A)$ generated by $W$ and $y$ by

$$p(x + \alpha y) = p(x) + \alpha c \quad (x \in W, \; \alpha \in R) ,$$

where $c$ is any fixed number such that $a \geqslant c \geqslant b$.

Suppose that $x + \alpha y \geqslant 0$ ($x \in W, \alpha \in R$). We shall show that $p(x+y) \geqslant 0$. If $\alpha = 0$, then $p(x+\alpha y) = p(x) \geqslant 0$ by assumption.

If $\alpha > 0$, then $x + \alpha y \geqslant 0$ implies $y \geqslant -\frac{x}{\alpha} \in W$, so that $p\left(-\frac{x}{\alpha}\right) \leqslant c$, or $p(x+\alpha y) \geqslant 0$.

If $\alpha < 0$, then $x + \alpha y \geqslant 0$ implies $y \leqslant -\frac{x}{\alpha} \in W$, so that $p\left(-\frac{x}{\alpha}\right) \geqslant c$, or $p(x+\alpha y) \geqslant 0$.

By Zorn's Lemma we conclude that $p$ can be extended to a real linear functional $p$ on $H(A)$ such that $p(x) \geqslant 0$ for all $x \in P$.

Finally set $p(x) = p(h) + ip(k)$ if $x = h + ik$ with $h, k \in H(A)$. Then $p$ is a positive functional on $A$ such that $p(e) = 1$ and $p(z^*z) = \| z^*z \| = \| z \|^2$. This completes the construction.

Step 10. *The isometric *-representation.* In the preceding step we constructed for every $z \in A$ a positive functional on $A$ such that the associated *-representation $T^{(z)}$ of $A$ on the Hilbert space $H^{(z)}$ is norm-decreasing and $\| T_z^{(z)} \| = \| z \|$.

Let $H$ be the direct sum of the Hilbert spaces $H^{(z)}$. The *direct sum* of the family $H^{(z)}$, $z \in A$, is defined as the set of all mappings $f$ on $A$ with $f(z) \in H^{(z)}$ such that $\sum_{z \in A} (f(z), f(z)) < \infty$. The algebraic operations in $H$ are pointwise and the scalar product is given by $(f, g) = \sum_{z \in A} (f(z), g(z))$. The reader may easily verify that all Hilbert space axioms are satisfied by $H$ (see [14]).

Define the *-representation $T$ of $A$ on $H$ by

$$(T_a f)(z) = T_a^{(z)}(f(z)) .$$

Note that the inequality

$$\sum_{z \in A} ((T_a f)(z), (T_a f)(z)) \leqslant \| a \|^2 \sum_{z \in A} (f(z), f(z))$$

shows that with $f$ also $T_a f$ belongs to $H$. Then $T_a$ is a bounded operator on $H$ such that

$$\| T_a \| = \sup_{z \in A} \| T_a^{(z)} \| = \| T_a^{(a)} \| = \| a \| .$$

Hence the map $a \to T_a$ is a norm-preserving *-representation of $A$ on $H$. This completes the proof of Theorem II as stated in the introduction.