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#### 4. THE GELFAND-NAIMARK REPRESENTATION THEOREM FOR COMMUTATIVE B\*-ALGEBRAS

Let us briefly recall the Gelfand theory of commutative Banach algebras (for proofs of this preliminary material see [29, pp. 470-479]).

If  $A$  is a commutative Banach algebra denote by  $\hat{A}$  the set of all nonzero complex-valued linear functionals  $\phi$  on  $A$  satisfying  $\phi(xy) = \phi(x)\phi(y)$  for all  $x, y \in A$ . If  $\phi \in \hat{A}$ , then  $\|\phi\| \leq 1$ . For each  $x$  in  $A$  define a complex-valued function  $\hat{x}: \hat{A} \rightarrow C$  by  $\hat{x}(\phi) = \phi(x)$  for  $\phi \in \hat{A}$ ;  $\hat{x}$  is called the *Gelfand transform* of  $x$ .

The *Gelfand topology* on  $\hat{A}$  is defined to be the weakest topology on  $\hat{A}$  under which all the functions  $\hat{x}$  are continuous; it is the relative topology which  $\hat{A}$  inherits as a subset of the dual space  $A'$  with the weak\*-topology. The set  $\hat{A}$  endowed with the Gelfand topology is called the *structure space* of  $A$ .

If the algebra  $A$  has no identity element it is often convenient to adjoin one. This can be done by considering the algebra  $A_e$  of ordered pairs  $(x, \lambda)$  with  $x \in A, \lambda \in C$ . The product in  $A_e$  is defined by  $(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu)$  and the involution by  $(x, \lambda)^* = (x^*, \bar{\lambda})$  if  $A$  is a \*-algebra. Identifying  $x$  in  $A$  with  $(x, 0)$  in  $A_e$  we see that  $A$  is a maximal two-sided ideal in  $A_e$  with  $e = (0, 1)$  as identity. If  $A$  is actually a Banach algebra  $A_e$  can also be made into a Banach algebra by extending the norm on  $A$  to  $A_e$ ; for example by defining  $\|(x, \lambda)\| = \|x\| + |\lambda|$ . Every multiplicative linear functional  $\phi$  on a commutative Banach algebra  $A$  can be extended uniquely to a multiplicative linear functional  $\phi_e$  on  $A_e$  by setting  $\phi_e((x, \lambda)) = \phi(x) + \lambda$  for  $(x, \lambda) \in A_e$ .

It follows from the Alaoglu theorem [29, p. 458] that the structure space  $\hat{A}$  of a commutative Banach algebra  $A$  is a locally compact Hausdorff space which is compact if  $A$  has an identity. Furthermore the functions  $\hat{x}$  on  $\hat{A}$  vanish at infinity.

The mapping  $x \rightarrow \hat{x}$ , called the *Gelfand representation*, is an algebra homomorphism of  $A$  into  $C_0(\hat{A})$ . Moreover, if  $\|\cdot\|_\infty$  denotes the sup-norm on  $C_0(\hat{A})$ , then  $\|\hat{x}\|_\infty \leq \|x\|$ , and so  $\hat{x} \rightarrow x$  is continuous. In general, the Gelfand representation is neither injective, surjective nor norm-preserving.

But in the case of a commutative  $B^*$ -algebra it will be seen to be an isometric  $*$ -isomorphism of  $A$  onto  $C_0(\hat{A})$ .

For this purpose we introduce the *spectrum of an element*  $x$  in an algebra  $A$  with identity as the set  $\sigma_A(x)$  of all complex  $\lambda$  such that  $x - \lambda$  is not invertible in  $A$ ; if  $A$  has no identity define  $\sigma_A(x) = \sigma_{A_e}(x)$ . The spectrum of an element  $x$  in a Banach algebra  $A$  is a compact subset of the complex plane and furthermore the following basic *Beurling-Gelfand* formula holds:

$$|x|_\sigma = \lim_{n \rightarrow \infty} \|x^n\|^{1/n} \leq \|x\|$$

where  $|x|_\sigma = \sup \{ |\lambda| : \lambda \in \sigma_A(x) \}$  is called the *spectral radius* of  $x$ .

The multiplicative linear functionals on a commutative Banach algebra  $A$  are related to the points in the spectrum of elements of  $A$ . If  $\lambda \neq 0$ , then  $\lambda \in \sigma_A(x)$  if and only if there exists  $\phi \in \hat{A}$  such that  $\phi(x) = \lambda$ . Hence  $\hat{x}(\hat{A}) \cup \{0\} = \sigma_A(x) \cup \{0\}$  and so  $\|\hat{x}\|_\infty = |x|_\sigma \leq \|x\|$ . Now we are ready to prove the Gelfand-Naimark representation theorem for commutative  $B^*$ -algebras.

**THEOREM I.** *If  $A$  is a commutative  $B^*$ -algebra, then  $x \rightarrow \hat{x}$  is an isometric  $*$ -isomorphism of  $A$  onto  $C_0(\hat{A})$ .*

*Proof.* We have seen that  $x \rightarrow \hat{x}$  is a homomorphism of  $A$  into  $C_0(\hat{A})$ . The isometry of the involution in  $A$  is proved quite simply by the following argument of Gelfand and Naimark [23]. For every  $h \in A$  with  $h^* = h$  the  $B^*$ -condition gives  $\|h^2\| = \|h\|^2$ ; by iteration  $\|h^{2^n}\| = \|h\|^{2^n}$  or  $\|h\| = \|h^{2^n}\|^{1/2^n}$  and so  $\|h\| = |h|_\sigma$ . In particular  $\|x^*x\| = |x^*x|_\sigma$ . Since  $\sigma(x^*) = \overline{\sigma(x)}$  we see that  $|x^*|_\sigma = |x|_\sigma$ . Hence using the submultiplicativity of the spectral radius on commuting elements  $\|x^*\| \cdot \|x\| = \|x^*x\| = |x^*x|_\sigma \leq |x^*|_\sigma |x|_\sigma = |x|_\sigma^2 \leq \|x\|^2$  and so  $\|x^*\| \leq \|x\|$ . Replacing  $x$  by  $x^*$  we also have  $\|x\| \leq \|x^*\|$ ; Thus  $\|x^*\| = \|x\|$ .

If  $A$  has an identity element we can now show that  $x \rightarrow \hat{x}$  is a  $*$ -map. We first show by two different arguments that  $\phi(h)$  is real for  $h \in A$  with  $h^* = h$  and  $\phi \in \hat{A}$ .

*Aren's argument* [3]: Set  $z = h + ite$  for real  $t$ . If  $\phi(h) = \alpha + i\beta$  with  $\alpha$  and  $\beta$  real then  $\phi(z) = \alpha + i(\beta + t)$  and  $z^*z = (h - ite)(h + ite) = h^2 + t^2e$  so that

$$\alpha^2 + (\beta + t)^2 = |\phi(z)|^2 \leq \|z\|^2 = \|z^*z\| \leq \|h^2\| + t^2$$

or  $\alpha^2 + \beta^2 + 2\beta t \leq \|h^2\|$  for all real  $t$ . Thus  $\beta = 0$  and  $\phi(h)$  is real.

*Fukamiya's argument* [21]: Recall that in a Banach algebra  $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$ . Set  $u = \exp(ih)$ . Then  $u^* = \exp(-ih)$  and so  $u^*u = e = uu^*$ . Since  $1 = \|u^*u\| = \|u\|^2$  we see that  $\|u\| = 1 = \|u^{-1}\|$ . Hence  $|\hat{u}(\phi)| \leq 1$  and  $|\hat{u}^{-1}(\phi)| \leq 1$  which implies  $|\hat{u}(\phi)| = 1$ . Since  $1 = |\hat{u}(\phi)| = |\phi(u)| = |\exp(i\phi(h))|$ , it follows that  $\phi(h)$  is real.

Now, if  $x \in A$ , then  $x = h + ik$  with  $h = (x + x^*)/2$  and  $k = (x - x^*)/2i$ . Since  $h^* = h$ ,  $k^* = k$ , and  $x^* = h - ik$  we have for every  $\phi \in \hat{A}$ ,

$$(x^*)^\wedge(\phi) = \phi(x^*) = \phi(h - ik) = \overline{\phi(h + ik)} = \overline{\phi(x)} = \overline{\hat{x}(\phi)}.$$

Thus  $(x^*)^\wedge = \overline{\hat{x}}$ ; i.e. the Gelfand representation is a \*-map.

Next assume that  $A$  has no identity element. Since every  $\phi \in \hat{A}$  can be extended to  $A_e$  it suffices to show that the norm on  $A$  can be extended to a B\*-norm on  $A_e$ . Suppose  $A$  is a (not necessarily commutative) B\*-algebra with isometric involution. Observe that for every  $x \in A$ ,  $\|x\| = \sup \{ \|xy\| : y \in A, \|y\| \leq 1 \}$ . Extend the norm on  $A$  to  $A_e$  by

$$\|x + \lambda e\| = \sup \{ \|(x + \lambda e)y\| : y \in A, \|y\| \leq 1 \}.$$

Then  $A_e$  is a Banach \*-algebra in which  $A$  is isometrically embedded as a closed ideal of codimension one. Since the involution in  $A$  is isometric we have

$$\|(x + \lambda e)y\|^2 = \|y^*(x + \lambda e)^*(x + \lambda e)y\| \leq \|(x + \lambda e)^*(x + \lambda e)\| \cdot \|y\|^2.$$

Therefore  $\|x + \lambda e\|^2 \leq \|(x + \lambda e)^*(x + \lambda e)\|$ ; hence  $A_e$  is a B\*-algebra with isometric involution.

This shows that  $x \rightarrow \hat{x}$  is a \*-map even if  $A$  has no identity. It is now easily seen that  $x \rightarrow \hat{x}$  is an isometry. Indeed:

$$\begin{aligned} \|x\|^2 &= \|x^*x\| = |x^*x|_\sigma = \|(x^*x)^\wedge\|_\infty = \|(x^*)^\wedge \hat{x}\|_\infty = \|\overline{\hat{x}\hat{x}}\|_\infty \\ &= \|\hat{x}\|_\infty^2, \text{ or } \|x\| = \|\hat{x}\|_\infty. \end{aligned}$$

Summarizing, we have shown that the Gelfand representation is an isometric \*-isomorphism of  $A$  into  $C_0(\hat{A})$ . Let  $B$  denote the range of  $x \rightarrow \hat{x}$ . Then  $B$  is clearly a norm-closed subalgebra of  $C_0(\hat{A})$  which separates the points of  $\hat{A}$ , vanishes identically at no point of  $\hat{A}$ , and is closed under

complex conjugation. By the Stone-Weierstrass theorem [29, p. 151] we conclude that  $B = C_0(\hat{A})$  and hence that  $x \rightarrow \hat{x}$  is onto. Thus the proof of the representation theorem for commutative  $B^*$ -algebras is complete.

The reader who is interested in an unconventional proof of the preceding theorem may consult Edward Nelson [38, p. 78]. Quite simple proofs of the Gelfand-Naimark theorem in the special case of function algebras have been given by Nelson Dunford and Jacob T. Schwartz [14, pp. 274-275] and Karl E. Aubert [5].

### 5. THE GELFAND-NAIMARK THEOREM FOR ARBITRARY $B^*$ -ALGEBRAS

The proof of the representation theorem for an arbitrary  $B^*$ -algebra is much more involved than the commutative case and it will be divided into several steps. After having established that the involution is continuous we will introduce a new equivalent  $B^*$ -norm with isometric involution. An investigation of the unitary elements will show that the original norm on the algebra coincides with this new norm. The representation of  $B^*$ -algebras will then easily be effected by the well known Gelfand-Naimark-Segal construction. General references for material in this section are [13], [37] and [43].

Step. 1. *The involution in a  $B^*$ -algebra  $A$  is continuous.*

*Proof* [39, Lemma 1.3]. First we show that the set  $H(A) = \{ h \in A : h^* = h \}$  of *hermitian elements* in  $A$  is closed. Let  $\{ h_n \}$  be a convergent sequence in  $H(A)$  whose limit is  $h + ik$ , with  $h, k \in H(A)$ . Since  $h_n - h \rightarrow ik$  we may assume (by putting  $h_n$  for  $h_n - h$ ) that  $h_n$  converges to  $ik$ . The spectral mapping theorem for polynomials [43, p. 32] gives  $\sigma_A(h_n^2 - h_n^4) = \{ \lambda^2 - \lambda^4 : \lambda \in \sigma_A(h_n) \}$ ; since  $\| h \| = |h|_\sigma$  and  $\sigma_A(h)$  is real (see the first part of the proof of Theorem I, the Aren's-Fukamiya arguments and recall  $\sigma_A(h) = \hat{h}(\hat{A}) \cup \{ 0 \}$ ) we have

$$\begin{aligned} \| h_n^2 - h_n^4 \| &= \sup \{ \lambda^2 - \lambda^4 : \lambda \in \sigma_A(h_n) \} \\ &\leq \sup \{ \lambda^2 : \lambda \in \sigma_A(h_n) \} = \| h_n^2 \|. \end{aligned}$$

Letting  $n \rightarrow \infty$  we obtain  $\| -k^2 - k^4 \| \leq \| k^2 \|. Hence$

$$\sup \{ \lambda^2 + \lambda^4 : \lambda \in \sigma_A(k) \} \leq \sup \{ \lambda^2 : \lambda \in \sigma_A(k) \}.$$

Choose  $\mu \in \sigma_A(k)$  such that  $\mu^2 = \sup \{ \lambda^2 : \lambda \in \sigma_A(k) \}$ . Then  $\mu^2 + \mu^4 \leq \mu^2$ , so  $\mu = 0$ . It follows that  $\| k \| = |k|_\sigma = 0$  and hence  $k = 0$ . This shows that  $H(A)$  is closed.