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4. THE GELFAND-NAIMARK REPRESENTATION THEOREM FOR COMMUTATIVE B^* -ALGEBRAS

Let us briefly recall the Gelfand theory of commutative Banach algebras (for proofs of this preliminary material see [29, pp. 470-479]).

If A is a commutative Banach algebra denote by \hat{A} the set of all nonzero complex-valued linear functionals ϕ on A satisfying $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$. If $\phi \in \hat{A}$, then $\|\phi\| \leq 1$. For each x in A define a complex-valued function $\hat{x}: \hat{A} \rightarrow C$ by $\hat{x}(\phi) = \phi(x)$ for $\phi \in \hat{A}$; \hat{x} is called the *Gelfand transform* of x .

The *Gelfand topology* on \hat{A} is defined to be the weakest topology on \hat{A} under which all the functions \hat{x} are continuous; it is the relative topology which \hat{A} inherits as a subset of the dual space A' with the weak*-topology. The set \hat{A} endowed with the Gelfand topology is called the *structure space* of A .

If the algebra A has no identity element it is often convenient to adjoin one. This can be done by considering the algebra A_e of ordered pairs (x, λ) with $x \in A, \lambda \in C$. The product in A_e is defined by $(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu)$ and the involution by $(x, \lambda)^* = (x^*, \bar{\lambda})$ if A is a $*$ -algebra. Identifying x in A with $(x, 0)$ in A_e we see that A is a maximal two-sided ideal in A_e with $e = (0, 1)$ as identity. If A is actually a Banach algebra A_e can also be made into a Banach algebra by extending the norm on A to A_e ; for example by defining $\|(x, \lambda)\| = \|x\| + |\lambda|$. Every multiplicative linear functional ϕ on a commutative Banach algebra A can be extended uniquely to a multiplicative linear functional ϕ_e on A_e by setting $\phi_e((x, \lambda)) = \phi(x) + \lambda$ for $(x, \lambda) \in A_e$.

It follows from the Alaoglu theorem [29, p. 458] that the structure space \hat{A} of a commutative Banach algebra A is a locally compact Hausdorff space which is compact if A has an identity. Furthermore the functions \hat{x} on \hat{A} vanish at infinity.

The mapping $x \rightarrow \hat{x}$, called the *Gelfand representation*, is an algebra homomorphism of A into $C_0(\hat{A})$. Moreover, if $\|\cdot\|_\infty$ denotes the sup-norm on $C_0(\hat{A})$, then $\|\hat{x}\|_\infty \leq \|x\|$, and so $\hat{x} \rightarrow x$ is continuous. In general, the Gelfand representation is neither injective, surjective nor norm-preserving.

But in the case of a commutative B^* -algebra it will be seen to be an isometric $*$ -isomorphism of A onto $C_0(\hat{A})$.

For this purpose we introduce the *spectrum of an element* x in an algebra A with identity as the set $\sigma_A(x)$ of all complex λ such that $x - \lambda$ is not invertible in A ; if A has no identity define $\sigma_A(x) = \sigma_{A_e}(x)$. The spectrum of an element x in a Banach algebra A is a compact subset of the complex plane and furthermore the following basic *Beurling-Gelfand* formula holds:

$$|x|_\sigma = \lim_{n \rightarrow \infty} \|x^n\|^{1/n} \leq \|x\|$$

where $|x|_\sigma = \sup \{ |\lambda| : \lambda \in \sigma_A(x) \}$ is called the *spectral radius* of x .

The multiplicative linear functionals on a commutative Banach algebra A are related to the points in the spectrum of elements of A . If $\lambda \neq 0$, then $\lambda \in \sigma_A(x)$ if and only if there exists $\phi \in \hat{A}$ such that $\phi(x) = \lambda$. Hence $\hat{x}(\hat{A}) \cup \{0\} = \sigma_A(x) \cup \{0\}$ and so $\|\hat{x}\|_\infty = |x|_\sigma \leq \|x\|$. Now we are ready to prove the Gelfand-Naimark representation theorem for commutative B^* -algebras.

THEOREM I. *If A is a commutative B^* -algebra, then $x \rightarrow \hat{x}$ is an isometric $*$ -isomorphism of A onto $C_0(\hat{A})$.*

Proof. We have seen that $x \rightarrow \hat{x}$ is a homomorphism of A into $C_0(\hat{A})$. The isometry of the involution in A is proved quite simply by the following argument of Gelfand and Naimark [23]. For every $h \in A$ with $h^* = h$ the B^* -condition gives $\|h^2\| = \|h\|^2$; by iteration $\|h^{2^n}\| = \|h\|^{2^n}$ or $\|h\| = \|h^{2^n}\|^{1/2^n}$ and so $\|h\| = |h|_\sigma$. In particular $\|x^*x\| = |x^*x|_\sigma$. Since $\sigma(x^*) = \overline{\sigma(x)}$ we see that $|x^*|_\sigma = |x|_\sigma$. Hence using the submultiplicativity of the spectral radius on commuting elements $\|x^*\| \cdot \|x\| = \|x^*x\| = |x^*x|_\sigma \leq |x^*|_\sigma |x|_\sigma = |x|_\sigma^2 \leq \|x\|^2$ and so $\|x^*\| \leq \|x\|$. Replacing x by x^* we also have $\|x\| \leq \|x^*\|$; Thus $\|x^*\| = \|x\|$.

If A has an identity element we can now show that $x \rightarrow \hat{x}$ is a $*$ -map. We first show by two different arguments that $\phi(h)$ is real for $h \in A$ with $h^* = h$ and $\phi \in \hat{A}$.

Aren's argument [3]: Set $z = h + ite$ for real t . If $\phi(h) = \alpha + i\beta$ with α and β real then $\phi(z) = \alpha + i(\beta + t)$ and $z^*z = (h - ite)(h + ite) = h^2 + t^2e$ so that

$$\alpha^2 + (\beta + t)^2 = |\phi(z)|^2 \leq \|z\|^2 = \|z^*z\| \leq \|h^2\| + t^2$$

or $\alpha^2 + \beta^2 + 2\beta t \leq \|h^2\|$ for all real t . Thus $\beta = 0$ and $\phi(h)$ is real.

Fukamiya's argument [21]: Recall that in a Banach algebra $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$. Set $u = \exp(ih)$. Then $u^* = \exp(-ih)$ and so $u^*u = e = uu^*$. Since $1 = \|u^*u\| = \|u\|^2$ we see that $\|u\| = 1 = \|u^{-1}\|$. Hence $|\hat{u}(\phi)| \leq 1$ and $|\hat{u}^{-1}(\phi)| \leq 1$ which implies $|\hat{u}(\phi)| = 1$. Since $1 = |\hat{u}(\phi)| = |\phi(u)| = |\exp(i\phi(h))|$, it follows that $\phi(h)$ is real.

Now, if $x \in A$, then $x = h + ik$ with $h = (x + x^*)/2$ and $k = (x - x^*)/2i$. Since $h^* = h$, $k^* = k$, and $x^* = h - ik$ we have for every $\phi \in \hat{A}$,

$$(x^*)^\wedge(\phi) = \phi(x^*) = \phi(h - ik) = \phi(h + ik) = \phi(x) = \overline{\hat{x}(\phi)}.$$

Thus $(x^*)^\wedge = \overline{\hat{x}}$; i.e. the Gelfand representation is a $*$ -map.

Next assume that A has no identity element. Since every $\phi \in \hat{A}$ can be extended to A_e it suffices to show that the norm on A can be extended to a B^* -norm on A_e . Suppose A is a (not necessarily commutative) B^* -algebra with isometric involution. Observe that for every $x \in A$, $\|x\| = \sup \{ \|xy\| : y \in A, \|y\| \leq 1 \}$. Extend the norm on A to A_e by

$$\|x + \lambda e\| = \sup \{ \|(x + \lambda e)y\| : y \in A, \|y\| \leq 1 \}.$$

Then A_e is a Banach $*$ -algebra in which A is isometrically embedded as a closed ideal of codimension one. Since the involution in A is isometric we have

$$\|(x + \lambda e)y\|^2 = \|y^*(x + \lambda e)^*(x + \lambda e)y\| \leq \|(x + \lambda e)^*(x + \lambda e)\| \cdot \|y\|^2.$$

Therefore $\|x + \lambda e\|^2 \leq \|(x + \lambda e)^*(x + \lambda e)\|$; hence A_e is a B^* -algebra with isometric involution.

This shows that $x \rightarrow \hat{x}$ is a $*$ -map even if A has no identity. It is now easily seen that $x \rightarrow \hat{x}$ is an isometry. Indeed:

$$\begin{aligned} \|x\|^2 &= \|x^*x\| = |x^*x|_\sigma = \|(x^*x)^\wedge\|_\infty = \|(x^*)^\wedge \hat{x}\|_\infty = \|\overline{\hat{x}\hat{x}}\|_\infty \\ &= \|\hat{x}\|_\infty^2, \text{ or } \|x\| = \|\hat{x}\|_\infty. \end{aligned}$$

Summarizing, we have shown that the Gelfand representation is an isometric $*$ -isomorphism of A into $C_0(\hat{A})$. Let B denote the range of $x \rightarrow \hat{x}$. Then B is clearly a norm-closed subalgebra of $C_0(\hat{A})$ which separates the points of \hat{A} , vanishes identically at no point of \hat{A} , and is closed under

complex conjugation. By the Stone-Weierstrass theorem [29, p. 151] we conclude that $B = C_0(\hat{A})$ and hence that $x \rightarrow \hat{x}$ is onto. Thus the proof of the representation theorem for commutative B^* -algebras is complete.

The reader who is interested in an unconventional proof of the preceding theorem may consult Edward Nelson [38, p. 78]. Quite simple proofs of the Gelfand-Naimark theorem in the special case of function algebras have been given by Nelson Dunford and Jacob T. Schwartz [14, pp. 274-275] and Karl E. Aubert [5].

5. THE GELFAND-NAIMARK THEOREM FOR ARBITRARY B^* -ALGEBRAS

The proof of the representation theorem for an arbitrary B^* -algebra is much more involved than the commutative case and it will be divided into several steps. After having established that the involution is continuous we will introduce a new equivalent B^* -norm with isometric involution. An investigation of the unitary elements will show that the original norm on the algebra coincides with this new norm. The representation of B^* -algebras will then easily be effected by the well known Gelfand-Naimark-Segal construction. General references for material in this section are [13], [37] and [43].

Step. 1. *The involution in a B^* -algebra A is continuous.*

Proof [39, Lemma 1.3]. First we show that the set $H(A) = \{h \in A : h^* = h\}$ of *hermitian elements* in A is closed. Let $\{h_n\}$ be a convergent sequence in $H(A)$ whose limit is $h + ik$, with $h, k \in H(A)$. Since $h_n - h \rightarrow ik$ we may assume (by putting h_n for $h_n - h$) that h_n converges to ik . The spectral mapping theorem for polynomials [43, p. 32] gives $\sigma_A(h_n^2 - h_n^4) = \{\lambda^2 - \lambda^4 : \lambda \in \sigma_A(h_n)\}$; since $\|h\| = \|h\|_\sigma$ and $\sigma_A(h)$ is real (see the first part of the proof of Theorem I, the Aren's-Fukamiya arguments and recall $\sigma_A(h) = \hat{h}(\hat{A}) \cup \{0\}$) we have

$$\begin{aligned} \|h_n^2 - h_n^4\| &= \sup \{ \lambda^2 - \lambda^4 : \lambda \in \sigma_A(h_n) \} \\ &\leq \sup \{ \lambda^2 : \lambda \in \sigma_A(h_n) \} = \|h_n^2\|. \end{aligned}$$

Letting $n \rightarrow \infty$ we obtain $\| -k^2 - k^4 \| \leq \|k^2\|$. Hence

$$\sup \{ \lambda^2 + \lambda^4 : \lambda \in \sigma_A(k) \} \leq \sup \{ \lambda^2 : \lambda \in \sigma_A(k) \}.$$

Choose $\mu \in \sigma_A(k)$ such that $\mu^2 = \sup \{ \lambda^2 : \lambda \in \sigma_A(k) \}$. Then $\mu^2 + \mu^4 \leq \mu^2$, so $\mu = 0$. It follows that $\|k\| = \|k\|_\sigma = 0$ and hence $k = 0$. This shows that $H(A)$ is closed.