Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 23 (1977)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE GELFAND-NAIMARK THEOREMS FOR C*-ALGEBRAS

Autor: Doran, Robert S. / Wichmann, Josef

Kapitel: 4. The Gelfand-Naimark representation theorem for commutative b*-algebras

DOI: https://doi.org/10.5169/seals-48924

4. The Gelfand-Naimark representation theorem for commutative B*-algebras

Let us briefly recall the Gelfand theory of commutative Banach algebras (for proofs of this preliminary material see [29, pp. 470-479]).

If A is a commutative Banach algebra denote by $\hat{A}$ the set of all nonzero complex-valued linear functionals ϕ on A satisfying $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$. If $\phi \in \hat{A}$, then $\|\phi\| \leqslant 1$. For each x in A define a complex-valued function $\hat{x}: \hat{A} \to C$ by $\hat{x}(\phi) = \phi(x)$ for $\phi \in \hat{A}$; $\hat{x}$ is called the *Gelfand transform of* x.

The *Gelfand topology* on $\hat{A}$ is defined to be the weakest topology on $\hat{A}$ under which all the functions $\hat{x}$ are continuous; it is the relative topology which $\hat{A}$ inherits as a subset of the dual space A' with the weak*-topology. The set $\hat{A}$ endowed with the Gelfand topology is called the *structure space of* A.

If the algebra A has no identity element it is often convenient to adjoin one. This can be done by considering the algebra A_e of ordered pairs (x, λ) with $x \in A$, $\lambda \in C$. The product in A_e is defined by $(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu)$ and the involution by $(x, \lambda)^* = (x^*, \bar{\lambda})$ if A is a *-algebra. Identifying x in A with $(x, 0)$ in A_e we see that A is a maximal two-sided ideal in A_e with $e = (0, 1)$ as identity. If A is actually a Banach algebra A_e can also be made into a Banach algebra by extending the norm on A to A_e; for example by defining $\|(x, \lambda)\| = \|x\| + |\lambda|$. Every multiplicative linear functional ϕ on a commutative Banach algebra A can be extended uniquely to a multiplicative linear functional ϕ_e on A_e by setting $\phi_e((x, \lambda)) = \phi(x) + \lambda$ for $(x, \lambda) \in A_e$.

It follows from the Alaoglu theorem [29, p. 458] that the structure space $\hat{A}$ of a commutative Banach algebra A is a locally compact Hausdorff space which is compact if A has an identity. Furthermore the functions $\hat{x}$ on $\hat{A}$ vanish at infinity.

The mapping $x \to \hat{x}$, called the *Gelfand representation*, is an algebra homomorphism of A into $C_0(\hat{A})$. Moreover, if $\|\cdot\|_\infty$ denotes the sup-norm on $C_0(\hat{A})$, then $\|\hat{x}\|_\infty \leqslant \|x\|$, and so $x \to \hat{x}$ is continuous. In general, the Gelfand representation is neither injective, surjective nor norm-preserving.

But in the case of a commutative B*-algebra it will be seen to be an isometric *-isomorphism of A onto $C_0(\hat{A})$.

For this purpose we introduce the *spectrum of an element* x in an algebra A with identity as the set $\sigma_A(x)$ of all complex λ such that $x - \lambda$ is not invertible in A; if A has no identity define $\sigma_A(x) = \sigma_{A_e}(x)$. The spectrum of an element x in a Banach algebra A is a compact subset of the complex plane and furthermore the following basic *Beurling-Gelfand* formula holds:

$$|x|_\sigma = \lim_{n\to\infty} \|x^n\|^{1/n} \leqslant \|x\|$$

where $|x|_\sigma = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$ is called the *spectral radius* of x.

The multiplicative linear functionals on a commutative Banach algebra A are related to the points in the spectrum of elements of A. If $\lambda \neq 0$, then $\lambda \in \sigma_A(x)$ if and only if there exists $\phi \in \hat{A}$ such that $\phi(x) = \lambda$. Hence $\hat{x}(\hat{A}) \cup \{0\} = \sigma_A(x) \cup \{0\}$ and so $\|\hat{x}\|_\infty = |x|_\sigma \leqslant \|x\|$. Now we are ready to prove the Gelfand-Naimark representation theorem for commutative B*-algebras.

THEOREM I. *If* A *is a commutative* B**-algebra, then* $x \to \hat{x}$ *is an isometric *-isomorphism of* A *onto* $C_0(\hat{A})$.

Proof. We have seen that $x \to \hat{x}$ is a homomorphism of A into $C_0(\hat{A})$. The isometry of the involution in A is proved quite simply by the following argument of Gelfand and Naimark [23]. For every $h \in A$ with $h^* = h$ the B*-condition gives $\|h^2\| = \|h\|^2$; by iteration $\|h^{2^n}\| = \|h\|^{2^n}$ or $\|h\| = \|h^{2^n}\|^{1/2^n}$ and so $\|h\| = |h|_\sigma$. In particular $\|x^*x\| = |x^*x|_\sigma$. Since $\sigma(x^*) = \overline{\sigma(x)}$ we see that $|x^*|_\sigma = |x|_\sigma$. Hence using the submultiplicativity of the spectral radius on commuting elements $\|x^*\| \cdot \|x\| = \|x^*x\| = |x^*x|_\sigma \leqslant |x^*|_\sigma |x|_\sigma = |x|_\sigma^2 \leqslant \|x\|^2$ and so $\|x^*\| \leqslant \|x\|$. Replacing x by x^* we also have $\|x\| \leqslant \|x^*\|$; Thus $\|x^*\| = \|x\|$.

If A has an identity element we can now show that $x \to \hat{x}$ is a *-map. We first show by two different arguments that $\phi(h)$ is real for $h \in A$ with $h^* = h$ and $\phi \in \hat{A}$.

Aren's argument [3]: Set $z = h + ite$ for real t. If $\phi(h) = \alpha + i\beta$ with α and β real then $\phi(z) = \alpha + i(\beta + t)$ and $z^*z = (h - ite)(h + ite) = h^2 + t^2 e$ so that

$$\alpha^2 + (\beta + t)^2 = |\phi(z)|^2 \leqslant \|z\|^2 = \|z^*z\| \leqslant \|h^2\| + t^2$$

or $\alpha^2 + \beta^2 + 2\beta t \leqslant \| h^2 \|$ for all real t. Thus $\beta = 0$ and $\phi(h)$ is real.

Fukamiya's argument [21]: Recall that in a Banach algebra $\exp(x) = \Sigma_{n=0}^{\infty} x^n/n!$. Set $u = \exp(ih)$. Then $u^* = \exp(-ih)$ and so $u^* u = e = uu^*$. Since $1 = \| u^* u \| = \| u \|^2$ we see that $\| u \| = 1 = \| u^{-1} \|$. Hence $| \hat{u}(\phi) | \leqslant 1$ and $| \hat{u}^{-1}(\phi) | \leqslant 1$ which implies $| \hat{u}(\phi) | = 1$. Since $1 = | \hat{u}(\phi) | = | \phi(u) | = | \exp(i\phi(h)) |$, it follows that $\phi(h)$ is real.

Now, if $x \in A$, then $x = h + ik$ with $h = (x + x^*)/2$ and $k = (x - x^*)/2i$. Since $h^* = h$, $k^* = k$, and $x^* = h - ik$ we have for every $\phi \in \hat{A}$,

$$(x^*)^\wedge(\phi) = \phi(x^*) = \phi(h - ik) = \overline{\phi(h + ik)} = \overline{\phi(x)} = \overline{\hat{x}(\phi)}\,.$$

Thus $(x^*)^\wedge = \bar{\hat{x}}$; i.e. the Gelfand representation is a *-map.

Next assume that A has no identity element. Since every $\phi \in \hat{A}$ can be extended to A_e it suffices to show that the norm on A can be extended to a B*-norm on A_e. Suppose A is a (not necessarily commutative) B*-algebra with isometric involution. Observe that for every $x \in A$, $\| x \| = \sup \{ \| xy \| : y \in A, \| y \| \leqslant 1 \}$. Extend the norm on A to A_e by

$$\| x + \lambda e \| = \sup \{ \| (x + \lambda e) y \| : y \in A\,,\ \| y \| \leqslant 1 \}\,.$$

Then A_e is a Banach *-algebra in which A is isometrically embedded as a closed ideal of codimension one. Since the involution in A is isometric we have

$$\| (x + \lambda e)\, y \|^2 = \| y^* (x + \lambda e)^* (x + \lambda e)\, y \| \leqslant \| (x + \lambda e)^* (x + \lambda e) \| \cdot \| y \|^2\,.$$

Therefore $\| x + \lambda e \|^2 \leqslant \| (x + \lambda e)^* (x + \lambda e) \|$; hence A_e is a B*-algebra with isometric involution.

This shows that $x \to \hat{x}$ is a *-map even if A has no identity. It is now easily seen that $x \to \hat{x}$ is an isometry. Indeed:

$$\| x \|^2 = \| x^* x \| = | x^* x |_\sigma = \| (x^* x)^\wedge \|_\infty = \| (x^*)^\wedge \hat{x} \|_\infty = \| \bar{\hat{x}} \hat{x} \|_\infty$$
$$= \| \hat{x} \|_\infty^2\,, \text{ or } \| x \| = \| \hat{x} \|_\infty\,.$$

Summarizing, we have shown that the Gelfand representation is an isometric *-isomorphism of A into $C_0(\hat{A})$. Let B denote the range of $x \to \hat{x}$. Then B is clearly a norm-closed subalgebra of $C_0(\hat{A})$ which separates the points of $\hat{A}$, vanishes identically at no point of $\hat{A}$, and is closed under complex conjugation. By the Stone-Weierstrass theorem [29, p. 151] we conclude that $B = C_0(\hat{A})$ and hence that $x \to \hat{x}$ is onto. Thus the proof of the representation theorem for commutative B*-algebras is complete.

The reader who is interested in an unconventional proof of the preceding theorem may consult Edward Nelson [38, p. 78]. Quite simple proofs of the Gelfand-Naimark theorem in the special case of function algebras have been given by Nelson Dunford and Jacob T. Schwartz [14, pp. 274-275] and Karl E. Aubert [5].

5. The Gelfand-Naimark theorem for arbitrary B*-algebras

The proof of the representation theorem for an arbitrary B*-algebra is much more involved than the commutative case and it will be divided into several steps. After having established that the involution is continuous we will introduce a new equivalent B*-norm with isometric involution. An investigation of the unitary elements will show that the original norm on the algebra coincides with this new norm. The representation of B*-algebras will then easily be effected by the well known Gelfand-Naimark-Segal construction. General references for material in this section are [13], [37] and [43].

Step. 1. *The involution in a* B*-*algebra* A *is continuous.*

Proof [39, Lemma 1.3]. First we show that the set $H(A) = \{ h \in A : h^* = h \}$ of *hermitian elements* in A is closed. Let $\{ h_n \}$ be a convergent sequence in $H(A)$ whose limit is $h + ik$, with $h, k \in H(A)$. Since $h_n - h \to ik$ we may assume (by putting h_n for $h_n - h$) that h_n converges to ik. The spectral mapping theorem for polynomials [43, p. 32] gives $\sigma_A(h_n^2 - h_n^4) = \{ \lambda^2 - \lambda^4 : \lambda \in \sigma_A(h_n) \}$; since $\| h \| = | h |_\sigma$ and $\sigma_A(h)$ is real (see the first part of the proof of Theorem I, the Aren's-Fukamiya arguments and recall $\sigma_A(h) = \hat{h}(\hat{A}) \cup \{ 0 \}$) we have

$$\begin{aligned} \| h_n^2 - h_n^4 \| &= \sup \{ \lambda^2 - \lambda^4 : \lambda \in \sigma_A(h_n) \} \\ &\leqslant \sup \{ \lambda^2 : \lambda \in \sigma_A(h_n) \} = \| h_n^2 \| . \end{aligned}$$

Letting $n \to \infty$ we obtain $\| -k^2 - k^4 \| \leqslant \| k^2 \|$. Hence

$$\sup \{ \lambda^2 + \lambda^4 : \lambda \in \sigma_A(k) \} \leqslant \sup \{ \lambda^2 : \lambda \in \sigma_A(k) \} .$$

Choose $\mu \in \sigma_A(k)$ such that $\mu^2 = \sup \{ \lambda^2 : \lambda \in \sigma_A(k) \}$. Then $\mu^2 + \mu^4 \leqslant \mu^2$, so $\mu = 0$. It follows that $\| k \| = | k |_\sigma = 0$ and hence $k = 0$. This shows that $H(A)$ is closed.