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*morphic to a norm-closed *-subalgebra of bounded linear operators on some Hilbert space.*

The purpose of this paper is to present a thorough discussion of these two representation theorems. We shall trace, as carefully as we have been able, the interesting and rather tangled history which led to their present form. Then proofs of the theorems will be given. Finally, we shall survey some recent developments inspired by the theorems.

2. Definitions and motivation

A **-algebra* is a complex associative linear algebra A with a mapping $x \to x^*$ of A into itself such that for all $x, y \in A$ and complex λ: (a) $x^{**} = x$; (b) $(\lambda x)^* = \bar{\lambda} x^*$; (c) $(x+y)^* = x^* + y^*$; and (d) $(xy)^* = y^* x^*$. The map $x \to x^*$ is called an *involution*; because of (a) it is clearly bijective. A subalgebra B of A is called a **-subalgebra* if $x \in B$ implies $x^* \in B$.

An algebra which is also a Banach space satisfying $\| xy \| \leqslant \| x \| \cdot \| y \|$ for all x and y is called a *Banach algebra.* A Banach algebra which is also a *-algebra is called a *Banach *-algebra.* The involution in a Banach *-algebra is said to be *continuous* if there is a constant M such that $\| x^* \| \leqslant M \| x \|$ for all x; the involution is *isometric* if $\| x^* \| = \| x \|$ for all x.

A norm on a *-algebra is said to satisfy the B*-*condition* if $\| x^* x \| = \| x^* \| \cdot \| x \|$ for all x; a B*-*algebra* is a Banach *-algebra whose norm satisfies the B*-condition. A B*-algebra with isometric involution clearly satisfies the condition $\| x^* x \| = \| x \|^2$. On the other hand, if A is a Banach *-algebra satisfying $\| x \|^2 \leqslant \| x^* x \|$ (in particular if equality holds), then A is easily seen to be a B*-algebra with isometric involution.

The Banach space $C(X)$ of continuous complex-valued functions on a compact Hausdorff space is a commutative B*-algebra under point-wise multiplication $(fg)(t) = f(t)\, g(t)$, involution $f^*(t) = \overline{f(t)}$, and sup-norm. Similarly, the algebra $C_0(X)$ of continuous complex-valued functions which vanish at infinity on a locally compact Hausdorff space is a commutative B*-algebra.

Examples of noncommutative B*-algebras are provided by the algebra $B(H)$ of bounded linear operators on a Hilbert space H. Multiplication in $B(H)$ is operator composition, the involution $T \to T^*$ is the usual adjoint operation, and the norm is the operator norm $\| T \| = \sup \{ \| T\xi \| : \| \xi \| \leqslant 1, \xi \in H \}$. A norm-closed *-subalgebra of $B(H)$ is called a C*-*algebra*; clearly, every C*-algebra is a B*-algebra.

Are there examples of B*-algebras other than the above? Numerous mathematical papers have been devoted to answering this question. In the remainder of this article we shall be occupied not only with its history and solution, but also with recent developments which have been stimulated by it.

3. Historical development

In 1943 the Soviet mathematicians Gelfand and Naimark published (in English!) a ground-breaking paper [23] in which they proved that a Banach *-algebra with an identity element e is isometrically *-isomorphic to a C*-algebra if it satisfies the following three conditions:

1^o	$\| x^*x \| = \| x^* \| \cdot \| x \|$	(the B*-condition);
2^o	$\| x^* \| = \| x \|$	(isometric involution);
3^o	$e + x^*x$ is invertible	(symmetry)

for all x. They immediately asked in a footnote if conditions 2^o and 3^o could be deleted—apparently recognizing that they were of a different character than condition 1^o and were needed primarily because of their method of proof. This indeed turned out to be true after considerable work. To trace the resulting history in detail it is convenient to look at the commutative and noncommutative cases separately.

Commutative algebras: In their paper Gelfand and Naimark first proved that every commutative B*-algebra with identity is a $C(X)$ for some compact Hausdorff space X. In the presence of commutativity they were able to show quite simply that the B*-condition implies the involution is isometric. Utilizing a delicate argument depending on the notion of "Shilov boundary" they proved that every commutative B*-algebra is symmetric. Thus in the commutative case they were able to show that conditions 2^o and 3^o follow from condition 1^o.

A much simpler proof for the symmetry of a commutative B*-algebra was published in 1946 by Richard Arens [3]. It may be of some historical interest to mention that Professor Arens—as he pointed out to the first named author during a conversation—had not seen Gelfand-Naimark's proof when he found his. In 1952, utilizing the exponential function for elements of a Banach algebra, the Japanese mathematician Masanori Fukamiya published [21] yet another beautiful proof of symmetry. These arguments of Arens and Fukamiya will be given in full in the next section.