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THE GELFAND-NAIMARK THEOREMS FOR C*-ALGEBRAS

by Robert S. Doran and Josef Wichmann

1. Introduction

Many of the Banach spaces which attract attention are at the same time algebras under some multiplication. In spite of this fact their study from this richer point of view was taken up only after the publication in 1932 of Banach's book [6]. One of the early fundamental results in the general theory of Banach algebras was a generalization of the classical theorem of Frobenius that any finite dimensional division algebra over the complex field is isomorphic to the field of complex numbers. S. Mazur [35] announced in 1938 that every complex normed division algebra is isomorphic to the field of complex numbers. Since the first published proof was given by I. M. Gelfand [22] this result is often called the Mazur-Gelfand theorem [43], [55]. As an immediate consequence one obtains the following beautiful characterization of the complex field among normed algebras: any normed algebra satisfying the norm condition $\| xy \| = \| x \| \cdot \| y \|$ for all elements x and y is isometrically isomorphic to the field of complex numbers.

Many important Banach algebras carry a natural involution. In the case of an algebra of functions the involution is the operation of taking the complex-conjugate and in the case of an algebra of operators on a Hilbert space it is the operation of taking the adjoint operator. Motivated by these observations the Soviet mathematicians Israel M. Gelfand and Mark A. Naimark [23] proved, under some additional assumptions, the following two theorems:

Theorem I. *Let A be a commutative Banach algebra with involution satisfying $\| x^*x \| = \| x^* \| \cdot \| x \|$ for all x in A. Then A is isometrically *-isomorphic to $C_0(X)$, the algebra of all continuous complex-valued functions which vanish at infinity on some locally compact Hausdorff space X.*

Theorem II. *Let A be a Banach algebra with involution satisfying $\| x^*x \| = \| x^* \| \cdot \| x \|$ for all x in A. Then A is isometrically *-iso-*

*morphic to a norm-closed *-subalgebra of bounded linear operators on some Hilbert space.*

The purpose of this paper is to present a thorough discussion of these two representation theorems. We shall trace, as carefully as we have been able, the interesting and rather tangled history which led to their present form. Then proofs of the theorems will be given. Finally, we shall survey some recent developments inspired by the theorems.

2. Definitions and motivation

A **-algebra* is a complex associative linear algebra A with a mapping $x \to x^*$ of A into itself such that for all $x, y \in A$ and complex λ: (a) $x^{**} = x$; (b) $(\lambda x)^* = \bar{\lambda} x^*$; (c) $(x+y)^* = x^* + y^*$; and (d) $(xy)^* = y^* x^*$. The map $x \to x^*$ is called an *involution*; because of (a) it is clearly bijective. A subalgebra B of A is called a **-subalgebra* if $x \in B$ implies $x^* \in B$.

An algebra which is also a Banach space satisfying $\| xy \| \leqslant \| x \| \cdot \| y \|$ for all x and y is called a *Banach algebra.* A Banach algebra which is also a *-algebra is called a *Banach *-algebra.* The involution in a Banach *-algebra is said to be *continuous* if there is a constant M such that $\| x^* \| \leqslant M \| x \|$ for all x; the involution is *isometric* if $\| x^* \| = \| x \|$ for all x.

A norm on a *-algebra is said to satisfy the B*-*condition* if $\| x^* x \| = \| x^* \| \cdot \| x \|$ for all x; a B*-*algebra* is a Banach *-algebra whose norm satisfies the B*-condition. A B*-algebra with isometric involution clearly satisfies the condition $\| x^* x \| = \| x \|^2$. On the other hand, if A is a Banach *-algebra satisfying $\| x \|^2 \leqslant \| x^* x \|$ (in particular if equality holds), then A is easily seen to be a B*-algebra with isometric involution.

The Banach space $C(X)$ of continuous complex-valued functions on a compact Hausdorff space is a commutative B*-algebra under point-wise multiplication $(fg)(t) = f(t)\, g(t)$, involution $f^*(t) = \overline{f(t)}$, and sup-norm. Similarly, the algebra $C_0(X)$ of continuous complex-valued functions which vanish at infinity on a locally compact Hausdorff space is a commutative B*-algebra.

Examples of noncommutative B*-algebras are provided by the algebra $B(H)$ of bounded linear operators on a Hilbert space H. Multiplication in $B(H)$ is operator composition, the involution $T \to T^*$ is the usual adjoint operation, and the norm is the operator norm $\| T \| = \sup \{ \| T\xi \| : \| \xi \| \leqslant 1, \xi \in H \}$. A norm-closed *-subalgebra of $B(H)$ is called a C*-*algebra*; clearly, every C*-algebra is a B*-algebra.

Are there examples of B*-algebras other than the above? Numerous mathematical papers have been devoted to answering this question. In the remainder of this article we shall be occupied not only with its history and solution, but also with recent developments which have been stimulated by it.

3. Historical development

In 1943 the Soviet mathematicians Gelfand and Naimark published (in English!) a ground-breaking paper [23] in which they proved that a Banach *-algebra with an identity element e is isometrically *-isomorphic to a C*-algebra if it satisfies the following three conditions:

1°	$\| x^*x \| = \| x^* \| \cdot \| x \|$	(the B*-condition);
2°	$\| x^* \| = \| x \|$	(isometric involution);
3°	$e + x^*x$ is invertible	(symmetry)

for all x. They immediately asked in a footnote if conditions 2° and 3° could be deleted—apparently recognizing that they were of a different character than condition 1° and were needed primarily because of their method of proof. This indeed turned out to be true after considerable work. To trace the resulting history in detail it is convenient to look at the commutative and noncommutative cases separately.

Commutative algebras: In their paper Gelfand and Naimark first proved that every commutative B*-algebra with identity is a $C(X)$ for some compact Hausdorff space X. In the presence of commutativity they were able to show quite simply that the B*-condition implies the involution is isometric. Utilizing a delicate argument depending on the notion of "Shilov boundary" they proved that every commutative B*-algebra is symmetric. Thus in the commutative case they were able to show that conditions 2° and 3° follow from condition 1°.

A much simpler proof for the symmetry of a commutative B*-algebra was published in 1946 by Richard Arens [3]. It may be of some historical interest to mention that Professor Arens—as he pointed out to the first named author during a conversation—had not seen Gelfand-Naimark's proof when he found his. In 1952, utilizing the exponential function for elements of a Banach algebra, the Japanese mathematician Masanori Fukamiya published [21] yet another beautiful proof of symmetry. These arguments of Arens and Fukamiya will be given in full in the next section.

Noncommutative algebras: The 1952 paper of Fukamiya [21] implicitly contained the key lemma needed to eliminate condition 3° for noncommutative algebras. In essence this lemma states that if x and y are "positive" elements in a B*-algebra with identity and isometric involution, then $x + y$ is also positive. Independently and nearly simultaneously this lemma was discovered by John L. Kelley and Robert L. Vaught [31]. The Kelley-Vaught argument is extremely brief and elegant, and is the one that we shall give in Section 5.

The nontrivial observation that this lemma was the key to eliminating condition 3° was due to Irving Kaplansky. His ingenious argument was recorded in Joseph A. Schatz's review [45] of Fukamiya's paper, making it an amusing instance where a theorem was first "proved" in the Mathematical Reviews.

In marked contrast to the commutative case, the redundancy of condition 2° for noncommutative algebras did not follow easily; in fact, the question remained open until 1960 when a solution for B*-algebras with identity was published by James G. Glimm and Richard V. Kadison [25]. Their proof was based on a deep "n-fold transitivity" theorem for unitary operators in an irreducible C*-algebra. A beautiful theorem of Bernard Russo and Henry A. Dye [44] made it possible to by-pass the Glimm-Kadison transitivity theorem; an elementary proof of their result was given recently by Lawrence A. Harris [28]. We mention that another paper concerning the elimination of 2° (and also 3°) was published by the Japanese mathematician Tamio Ono [39] in 1959. However this paper appeared to have errors in the arguments of both the main theorems (see the review of [39]). Ten years later Ono [40] acknowledged these mistakes and corrected them from the viewpoint of 1959.

The original conjecture of Gelfand and Naimark was, at this time, completely solved for algebras with identity. What about algebras without identity? This question is of considerable importance since most C*-algebras which occur in applications do not possess an identity. An answer was provided in 1967 by B. J. Vowden [54]. He was able to utilize the notion of "approximate identity" and several arguments from Ono [39] to embed a B*-algebra without identity in a B*-algebra with an identity. He then applied the case for algebras with identity to complete the proof. Hence after nearly twenty five years of work the mathematical community had the theorems as we have stated them in the introduction.

4. The Gelfand-Naimark representation theorem for commutative B*-algebras

Let us briefly recall the Gelfand theory of commutative Banach algebras (for proofs of this preliminary material see [29, pp. 470-479]).

If A is a commutative Banach algebra denote by $\hat{A}$ the set of all nonzero complex-valued linear functionals ϕ on A satisfying $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$. If $\phi \in \hat{A}$, then $\|\phi\| \leqslant 1$. For each x in A define a complex-valued function $\hat{x}: \hat{A} \to C$ by $\hat{x}(\phi) = \phi(x)$ for $\phi \in \hat{A}$; $\hat{x}$ is called the *Gelfand transform of* x.

The *Gelfand topology* on $\hat{A}$ is defined to be the weakest topology on $\hat{A}$ under which all the functions $\hat{x}$ are continuous; it is the relative topology which $\hat{A}$ inherits as a subset of the dual space A' with the weak*-topology. The set $\hat{A}$ endowed with the Gelfand topology is called the *structure space of* A.

If the algebra A has no identity element it is often convenient to adjoin one. This can be done by considering the algebra A_e of ordered pairs (x, λ) with $x \in A$, $\lambda \in C$. The product in A_e is defined by $(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu)$ and the involution by $(x, \lambda)^* = (x^*, \bar{\lambda})$ if A is a *-algebra. Identifying x in A with $(x, 0)$ in A_e we see that A is a maximal two-sided ideal in A_e with $e = (0, 1)$ as identity. If A is actually a Banach algebra A_e can also be made into a Banach algebra by extending the norm on A to A_e; for example by defining $\|(x, \lambda)\| = \|x\| + |\lambda|$. Every multiplicative linear functional ϕ on a commutative Banach algebra A can be extended uniquely to a multiplicative linear functional ϕ_e on A_e by setting $\phi_e((x, \lambda)) = \phi(x) + \lambda$ for $(x, \lambda) \in A_e$.

It follows from the Alaoglu theorem [29, p. 458] that the structure space $\hat{A}$ of a commutative Banach algebra A is a locally compact Hausdorff space which is compact if A has an identity. Furthermore the functions $\hat{x}$ on $\hat{A}$ vanish at infinity.

The mapping $x \to \hat{x}$, called the *Gelfand representation*, is an algebra homomorphism of A into $C_0(\hat{A})$. Moreover, if $\|\cdot\|_\infty$ denotes the sup-norm on $C_0(\hat{A})$, then $\|\hat{x}\|_\infty \leqslant \|x\|$, and so $x \to \hat{x}$ is continuous. In general, the Gelfand representation is neither injective, surjective nor norm-preserving.

But in the case of a commutative B*-algebra it will be seen to be an isometric *-isomorphism of A onto $C_0(\hat{A})$.

For this purpose we introduce the *spectrum of an element* x in an algebra A with identity as the set $\sigma_A(x)$ of all complex λ such that $x - \lambda$ is not invertible in A; if A has no identity define $\sigma_A(x) = \sigma_{A_e}(x)$. The spectrum of an element x in a Banach algebra A is a compact subset of the complex plane and furthermore the following basic *Beurling-Gelfand* formula holds:

$$|x|_\sigma = \lim_{n\to\infty} \| x^n \|^{1/n} \leqslant \| x \|$$

where $|x|_\sigma = \sup \{ |\lambda| : \lambda \in \sigma_A(x) \}$ is called the *spectral radius* of x.

The multiplicative linear functionals on a commutative Banach algebra A are related to the points in the spectrum of elements of A. If $\lambda \neq 0$, then $\lambda \in \sigma_A(x)$ if and only if there exists $\phi \in \hat{A}$ such that $\phi(x) = \lambda$. Hence $\hat{x}(\hat{A}) \cup \{0\} = \sigma_A(x) \cup \{0\}$ and so $\| \hat{x} \|_\infty = |x|_\sigma \leqslant \| x \|$. Now we are ready to prove the Gelfand-Naimark representation theorem for commutative B*-algebras.

THEOREM I. *If A is a commutative B*-algebra, then $x \to \hat{x}$ is an isometric *-isomorphism of A onto $C_0(\hat{A})$.*

Proof. We have seen that $x \to \hat{x}$ is a homomorphism of A into $C_0(\hat{A})$. The isometry of the involution in A is proved quite simply by the following argument of Gelfand and Naimark [23]. For every $h \in A$ with $h^* = h$ the B*-condition gives $\| h^2 \| = \| h \|^2$; by iteration $\| h^{2^n} \| = \| h \|^{2^n}$ or $\| h \| = \| h^{2^n} \|^{1/2^n}$ and so $\| h \| = |h|_\sigma$. In particular $\| x^*x \| = |x^*x|_\sigma$. Since $\sigma(x^*) = \overline{\sigma(x)}$ we see that $|x^*|_\sigma = |x|_\sigma$. Hence using the submultiplicativity of the spectral radius on commuting elements $\| x^* \| \cdot \| x \| = \| x^*x \| = |x^*x|_\sigma \leqslant |x^*|_\sigma |x|_\sigma = |x|_\sigma^2 \leqslant \| x \|^2$ and so $\| x^* \| \leqslant \| x \|$. Replacing x by x^* we also have $\| x \| \leqslant \| x^* \|$; Thus $\| x^* \| = \| x \|$.

If A has an identity element we can now show that $x \to \hat{x}$ is a *-map. We first show by two different arguments that $\phi(h)$ is real for $h \in A$ with $h^* = h$ and $\phi \in \hat{A}$.

Aren's argument [3]: Set $z = h + ite$ for real t. If $\phi(h) = \alpha + i\beta$ with α and β real then $\phi(z) = \alpha + i(\beta + t)$ and $z^*z = (h - ite)(h + ite) = h^2 + t^2 e$ so that

$$\alpha^2 + (\beta + t)^2 = |\phi(z)|^2 \leqslant \| z \|^2 = \| z^*z \| \leqslant \| h^2 \| + t^2$$

or $\alpha^2 + \beta^2 + 2\beta t \leqslant \| h^2 \|$ for all real t. Thus $\beta = 0$ and $\phi(h)$ is real.

Fukamiya's argument [21]: Recall that in a Banach algebra $\exp(x) = \Sigma_{n=0}^{\infty} x^n/n!$. Set $u = \exp(ih)$. Then $u^* = \exp(-ih)$ and so $u^*u = e = uu^*$. Since $1 = \| u^*u \| = \| u \|^2$ we see that $\| u \| = 1 = \| u^{-1} \|$. Hence $| \hat{u}(\phi) | \leqslant 1$ and $| \hat{u}^{-1}(\phi) | \leqslant 1$ which implies $| \hat{u}(\phi) | = 1$. Since $1 = | \hat{u}(\phi) | = | \phi(u) | = | \exp(i\phi(h)) |$, it follows that $\phi(h)$ is real.

Now, if $x \in A$, then $x = h + ik$ with $h = (x+x^*)/2$ and $k = (x-x^*)/2i$. Since $h^* = h$, $k^* = k$, and $x^* = h - ik$ we have for every $\phi \in \hat{A}$,

$$(x^*)^\wedge(\phi) = \phi(x^*) = \phi(h-ik) = \overline{\phi(h+ik)} = \overline{\phi(x)} = \overline{\hat{x}(\phi)}\,.$$

Thus $(x^*)^\wedge = \bar{\hat{x}}$; i.e. the Gelfand representation is a *-map.

Next assume that A has no identity element. Since every $\phi \in \hat{A}$ can be extended to A_e it suffices to show that the norm on A can be extended to a B*-norm on A_e. Suppose A is a (not necessarily commutative) B*-algebra with isometric involution. Observe that for every $x \in A$, $\| x \| = \sup \{ \| xy \| : y \in A, \| y \| \leqslant 1 \}$. Extend the norm on A to A_e by

$$\| x + \lambda e \| = \sup \{ \| (x+\lambda e) y \| : y \in A\,, \ \| y \| \leqslant 1 \}\,.$$

Then A_e is a Banach *-algebra in which A is isometrically embedded as a closed ideal of codimension one. Since the involution in A is isometric we have

$$\| (x+\lambda e)\, y \|^2 = \| y^* (x+\lambda e)^* (x+\lambda e)\, y \| \leqslant \| (x+\lambda e)^* (x+\lambda e) \| \cdot \| y \|^2\,.$$

Therefore $\| x + \lambda e \|^2 \leqslant \| (x+\lambda e)^* (x+\lambda e) \|$; hence A_e is a B*-algebra with isometric involution.

This shows that $x \to \hat{x}$ is a *-map even if A has no identity. It is now easily seen that $x \to \hat{x}$ is an isometry. Indeed:

$$\| x \|^2 = \| x^*x \| = | x^*x |_\sigma = \| (x^*x)^\wedge \|_\infty = \| (x^*)^\wedge \hat{x} \|_\infty = \| \bar{\hat{x}} \hat{x} \|_\infty$$
$$= \| \hat{x} \|_\infty^2\,, \text{ or } \| x \| = \| \hat{x} \|_\infty\,.$$

Summarizing, we have shown that the Gelfand representation is an isometric *-isomorphism of A into $C_0(\hat{A})$. Let B denote the range of $x \to \hat{x}$. Then B is clearly a norm-closed subalgebra of $C_0(\hat{A})$ which separates the points of $\hat{A}$, vanishes identically at no point of $\hat{A}$, and is closed under complex conjugation. By the Stone-Weierstrass theorem [29, p. 151] we conclude that $\hat{B} = C_0(\hat{A})$ and hence that $x \to \hat{x}$ is onto. Thus the proof of the representation theorem for commutative B*-algebras is complete.

The reader who is interested in an unconventional proof of the preceding theorem may consult Edward Nelson [38, p. 78]. Quite simple proofs of the Gelfand-Naimark theorem in the special case of function algebras have been given by Nelson Dunford and Jacob T. Schwartz [14, pp. 274-275] and Karl E. Aubert [5].

5. The Gelfand-Naimark theorem for arbitrary B*-algebras

The proof of the representation theorem for an arbitrary B*-algebra is much more involved than the commutative case and it will be divided into several steps. After having established that the involution is continuous we will introduce a new equivalent B*-norm with isometric involution. An investigation of the unitary elements will show that the original norm on the algebra coincides with this new norm. The representation of B*-algebras will then easily be effected by the well known Gelfand-Naimark-Segal construction. General references for material in this section are [13], [37] and [43].

Step. 1. *The involution in a* B*-*algebra* A *is continuous.*

Proof [39, Lemma 1.3]. First we show that the set $H(A) = \{ h \in A : h^* = h \}$ of *hermitian elements* in A is closed. Let $\{ h_n \}$ be a convergent sequence in $H(A)$ whose limit is $h + ik$, with $h, k \in H(A)$. Since $h_n - h \to ik$ we may assume (by putting h_n for $h_n - h$) that h_n converges to ik. The spectral mapping theorem for polynomials [43, p. 32] gives $\sigma_A(h_n^2 - h_n^4) = \{ \lambda^2 - \lambda^4 : \lambda \in \sigma_A(h_n) \}$; since $\| h \| = | h |_\sigma$ and $\sigma_A(h)$ is real (see the first part of the proof of Theorem I, the Aren's-Fukamiya arguments and recall $\sigma_A(h) = \hat{h}(\hat{A}) \cup \{ 0 \})$ we have

$$\begin{aligned} \| h_n^2 - h_n^4 \| &= \sup \{ \lambda^2 - \lambda^4 : \lambda \in \sigma_A(h_n) \} \\ &\leqslant \sup \{ \lambda^2 : \lambda \in \sigma_A(h_n) \} = \| h_n^2 \| . \end{aligned}$$

Letting $n \to \infty$ we obtain $\| - k^2 - k^4 \| \leqslant \| k^2 \|$. Hence

$$\sup \{ \lambda^2 + \lambda^4 : \lambda \in \sigma_A(k) \} \leqslant \sup \{ \lambda^2 : \lambda \in \sigma_A(k) \} .$$

Choose $\mu \in \sigma_A(k)$ such that $\mu^2 = \sup \{ \lambda^2 : \lambda \in \sigma_A(k) \}$. Then $\mu^2 + \mu^4 \leqslant \mu^2$, so $\mu = 0$. It follows that $\| k \| = | k |_\sigma = 0$ and hence $k = 0$. This shows that $H(A)$ is closed.

Now it is easy to prove that the graph of the map $x \to x^*$ of A onto A is closed. For suppose $x_n \to x$ and $x_n^* \to y$. Then $x_n + x_n^* \to x + y$ and $(x_n - x_n^*)/i \to (x-y)/i$. Since $H(A)$ is closed, $x+y$ and $(x-y)/i$ are hermitian and so $x+y = x^* + y^*$ and $x - y = y^* - x^*$, whence $y = x^*$. Thus by the closed graph theorem, valid for conjugate linear maps, the involution in A is continuous.

Step 2. *Let A be a* B*-*algebra. Then $\| x \|_0 = \| x^*x \|^{1/2}$ is an equivalent* B*-*norm on A such that $\| x^* \|_0 = \| x \|_0$ for all $x \in A$, and $\| h \|_0 = \| h \|$ for all hermitian $h \in A$.*

Proof. [2], [53]. By Step 1 there exists $M \geqslant 1$ such that $\| x^* \| \leqslant M \| x \|$ for all $x \in A$. Then

$$M^{-1/2} \| x \| \leqslant \| x^* \|^{1/2} \| x \|^{1/2} = \| x \|_0 \leqslant M^{1/2} \| x \|$$

so that $\| \cdot \|_0$ and $\| \cdot \|$ are equivalent. Clearly $\| \cdot \|_0$ is homogeneous and submultiplicative. To prove the triangle inequality, let $x, y \in A$. Then

$$\| x + y \|_0^2 = \| (x+y)^* (x+y) \| \leqslant \| x^*x \| + \| y^* y \| + \| x^* y + y^* x \|$$

so it is enough to prove that $\| x^*y + y^*x \| \leqslant 2 \| x \|_0 \| y \|_0$. For any positive integer n

$$\begin{aligned} &\| (x^* y)^{2^{n-1}} + (y^*x)^{2^{n-1}} \|^2 \\ &= \| (x^* y)^{2^n} + (y^*x)^{2^n} + (x^* y)^{2^{n-1}} (y^*x)^{2^{n-1}} + (y^*x)^{2^{n-1}} (x^* y)^{2^{n-1}} \| \\ &\leqslant \| (x^* y)^{2^n} + (y^*x)^{2^n} \| + 2 (\|x^*x\| \cdot \| y^* y\|)^{2^{n-1}} . \end{aligned}$$

For every $\varepsilon > 0$ there is an integer n such that

$$\| (x^* y)^{2^n} \| \leqslant (|x^* y|_\sigma^2 + \varepsilon)^{2^{n-1}} \text{ and } \| (y^*x)^{2^n} \| \leqslant (|y^*x|_\sigma^2 + \varepsilon)^{2^{n-1}} .$$

Then

$$\begin{aligned} \| (x^* y)^{2^n} \| &\leqslant (|x^* y|_\sigma |y^*x|_\sigma + \varepsilon)^{2^{n-1}} \leqslant (\|x^* y\| \cdot \| y^*x\| + \varepsilon)^{2^{n-1}} \\ &\leqslant (\|x^*x\| \cdot \| y^* y\| + \varepsilon)^{2^{n-1}} \end{aligned}$$

and similarly

$$\| (y^*x)^{2^n} \| \leqslant (\|x^*x\| \cdot \| y^* y\| + \varepsilon)^{2^{n-1}}$$

so that

$$\| (x^* y)^{2^n} + (y^*x)^{2^n} \|^2 \leqslant 2 (\|x^*x\| \cdot \| y^* y\| + \varepsilon)^{2^{n-1}}$$

Combining these results we recursively obtain

$$\| (x^*y)^{2^{k-1}} + (y^*x)^{2^{k-1}} \|^2 \leqslant 4(\|x^*x\| \cdot \|y^*y\| + \varepsilon)^{2^{k-1}} .$$

for any k, $1 \leqslant k \leqslant n$. Thus

$$\| x^*y + y^*x \|^2 \leqslant 4(\|x^*x\| \cdot \|y^*y\| + \varepsilon)$$

for arbitrary $\varepsilon > 0$. Hence $\| x^*y + y^*x \| \leqslant 2 \| x \|_0 \| y \|_0$. So we have seen that $\| \cdot \|_0$ is an equivalent algebra norm on A. Further, $\| h \|_0 = \| h^*h \|^{1/2} = \| h \|$ for all hermitian $h \in A$ and so $\| x \|_0^2 = \| x^*x \| = \| x^*x \|_0$; i.e., $\| \cdot \|_0$ is a B*-norm on A with $\| x^* \|_0 = \| x \|_0$ for all $x \in A$.

Step 3. *Positive elements and symmetry.* Let A be a B*-algebra with identity e. Then every hermitian $h \in A$ lies in a maximal commutative B*-algebra B with identity e. Observe that $\sigma_B(x) = \sigma_A(x)$ for all $x \in B$ [43, p. 35]. By the characterization of commutative B*-algebras B is isometrically *-isomorphic to $C(\hat{B})$. Hence every hermitian element $h \in A$ has real spectrum.

A hermitian element $x \in A$ is called *positive*, and we write $x \geqslant 0$, if the spectrum of x in A is a subset of the nonnegative reals.

Clearly $x = h^2$ is positive for every hermitian $h \in A$. The set $P = \{ x \in A : x \geqslant 0 \}$ of all positive elements in A is called the *positive cone.* Indeed, P is a cone. For $\lambda \geqslant 0$ and $x \geqslant 0$ then $\lambda x \geqslant 0$ since $\sigma_A(\lambda x) = \lambda \sigma_A(x)$. That $x \geqslant 0$ and $y \geqslant 0$ implies $x + y \geqslant 0$ may be seen by the following *Kelley-Vaught argument* [31]:

Set $\alpha = \| x \|$, $\beta = \| y \|$, $z = x + y$, and $\gamma = \alpha + \beta$. Since $| x |_\sigma = \| x \|$ the assumption $x \geqslant 0$ implies $\sigma_A(x) \subset [0, \alpha]$, so that $\sigma_A(\alpha e - x) \subset [0, \alpha]$ and therefore $\| \alpha e - x \| = | \alpha e - x |_\sigma \leqslant \alpha$. For the same reason $\| \beta e - y \| \leqslant \beta$. Hence

$$\| \gamma e - z \| = \| (\alpha e - x) + (\beta e - y) \| \leqslant \alpha + \beta = \gamma .$$

Since $z^* = z$, $\sigma_A(\gamma e - z)$ is real so that $\sigma_A(\gamma e - z) \subset [-\gamma, \gamma]$ which implies that $\sigma_A(z) \subset [0, 2\gamma]$. Thus $x + y = z \geqslant 0$.

The symmetry of the involution in A now follows readily by *Kaplansky's argument* [45]:

We intend to show $x^*x \geqslant 0$ for all $x \in A$. By observing that a real-valued continuous function is the difference of two nonnegative real-valued continuous functions whose product is zero, we can write the hermitian element x^*x in the form

$$x^*x = u - v , \; u \geqslant 0 , \; v \geqslant 0 , \; uv = 0 = vu .$$

Now $(xv)^* (xv) = v^*x^*xv = vx^*xv = v(u-v)v = -v^3$ so that $(xv)^* (xv) \leqslant 0$. Since $(xv)^* (xv)$ and $(xv)(xv)^*$ have the same nonzero spectrum, also $(xv)(xv)^* \leqslant 0$. Write $xv = h + ik$ with h and k hermitian. Then

$$0 \geqslant (xv)^* (xv) + (xv)(xv)^* = 2(h^2 + k^2) \geqslant 0 .$$

Thus $h = 0 = k$ or $xv = 0$. But then $0 = (xv)^* (xv) = -v^3$ and so $v = 0$. Hence $x^*x = u \geqslant 0$; in particular, $e + x^*x$ is invertible for all $x \in A$.

Step 4. *Let A be a B^*-algebra with isometric involution. Then there exists a net $\{ e_\alpha \}$ of hermitian elements in A, bounded by one, such that* $\lim e_\alpha x = x = \lim x e_\alpha$ *for all $x \in A$. The net $\{ e_\alpha \}$ is called an approximate identity.*

Proof. The following construction is due to Irving E. Segal [50]. If A has no identity, we may embed A in a B*-algebra A_e with identity e (see the proof of Theorem I). Thus in any case we can use the preceding results about positive elements.

For any $\alpha = \{ x_1, ..., x_n \}$ in the class of all finite subsets of A, ordered by inclusion, set $h = x_1^* x_1 + ... + x_n^* x_n$. Then $h \geqslant 0$ and so $e_\alpha = nh(e+nh)^{-1}$ is a well defined element in A. Viewing h as a non-negative function on the structure space of some maximal commutative B*-subalgebra we see that $\| e_\alpha \| = | e_\alpha |_\sigma \leqslant 1$. It remains to show that $\lim e_\alpha x = x = \lim x e_\alpha$. Observe that

$$\begin{aligned}[x_i(e-e_\alpha)]^* [x_i(e-e_\alpha)] &\leqslant \sum_{j=1}^{n} [x_j(e-e_\alpha)]^* [x_j(e-e_\alpha)] \\ &\leqslant (e-e_\alpha) h (e-e_\alpha) \\ &\leqslant h(e+nh)^{-2} \leqslant e/4n\end{aligned}$$

where the last inequality follows from the fact that the real function $t \to t(1+nt)^{-2}$ $(t \geqslant 0)$ has maximum value $1/4n$. Thus

$$\| x_i(e-e_\alpha) \|^2 = \| [x_i(e-e_\alpha)]^* [x_i(e-e_\alpha)] \| \leqslant 1/4n .$$

Now for arbitrary $x \in A$ and $\varepsilon > 0$ choose a finite set α_0 of n elements in A such that $x \in \alpha_0$ and $n > \varepsilon^{-2}$. Then for all $\alpha \geqslant \alpha_0$ we have $\| x - x e_\alpha \| = \| x(e-e_\alpha) \| < \varepsilon$. Hence $\lim x e_\alpha = x$ for every $x \in A$; and by the continuity of the involution also $\lim e_\alpha x = (\lim x^* e_\alpha)^* = (x^*)^* = x$.

Step 5. *Every B^*-algebra without identity can be isometrically embedded in a B^*-algebra with identity.*

Proof. Let A be a B*-algebra without identity. By Step 2, A is a B*-algebra with isometric involution with respect to the equivalent norm $\| x \|_0 = \| x^*x \|^{1/2}$. Hence, by Step 4, A has an approximate identity $\{ e_\alpha \}$ consisting of hermitian elements such that $\| e_\alpha \| = \| e_\alpha \|_0 \leqslant 1$. Now observe that for every $x \in A$,

$$\| x \| = \sup \{ \| xy \| : y \in A , \| y \| \leqslant 1 \} = \sup \{ \| yx \| : y \in A , \| y \| \leqslant 1 \}$$

and extend the norm on A to A_e by

$$\begin{aligned} \| x + \lambda e \| &= \sup \{ \| (x + \lambda e) y \| : y \in A , \| y \| \leqslant 1 \} \\ &= \sup \{ \| y (x + \lambda e) \| : y \in A , \| y \| \leqslant 1 \} . \end{aligned}$$

Then A_e is a Banach *-algebra with identity in which A is isometrically embedded as a closed ideal of codimension one. To see that the B*-condition holds in A_e we first prove that

$$\| x + \lambda e \| = \lim_\alpha \| (x + \lambda e) e_\alpha \| = \lim_\alpha \| e_\alpha (x + \lambda e) \| .$$

Given any $\varepsilon > 0$ there exists $y \in A$ with $\| y \| \leqslant 1$ such that

$$\| (x + \lambda e) y \| > \| x + \lambda e \| - \varepsilon .$$

Since $\lim_\alpha (x + \lambda e) e_\alpha y = (x + \lambda e) y$, there exists α_0 such that for all $\alpha \geqslant \alpha_0$, $\| (x + \lambda e) e_\alpha y \| > \| x + \lambda e \| - \varepsilon$. Since $\| (x + \lambda e) e_\alpha y \| \leqslant \| (x + \lambda e) e_\alpha \| \leqslant \| x + \lambda e \|$, it follows that $\lim_\alpha \| (x + \lambda e) e_\alpha \|$ exists and is equal to $\| x + \lambda e \|$. Similarly $\lim_\alpha \| e_\alpha (x + \lambda e) \| = \| x + \lambda e \|$. Thus

$$\begin{aligned} \| (x + \lambda e)^* \| \cdot \| (x + \lambda e) \| &= \lim_\alpha \| e_\alpha (x + \lambda e)^* \| \cdot \lim_\alpha \| (x + \lambda e) e_\alpha \| \\ &= \lim_\alpha \| e_\alpha (x + \lambda e)^* (x + \lambda e) e_\alpha \| \\ &= \| (x + \lambda e)^* (x + \lambda e) \| . \end{aligned}$$

Therefore $\| (x + \lambda e)^* (x + \lambda e) \| = \| (x + \lambda e)^* \| \cdot \| x + \lambda e \|$, and so A_e is a B*-algebra.

Step 6. *Let A be a B*-algebra with identity e and isometric involution. Denote by $U = \{ u \in A : u^*u = e = uu^* \}$ the group of unitary elements in A. Then every element x in A is a linear combination of unitary elements and $\| x \| = \| x \|_u$, where*

$$\| x \|_u = \inf \{ \sum_{n=1}^{N} | \lambda_n | : x = \sum_{n=1}^{N} \lambda_n u_n , \lambda_n \in C , u_n \in U \} .$$

Proof. To prove that every $x \in A$ is a linear combination of unitary elements it clearly suffices to show that every hermitian $h \in A$ with $\| h \| < 1$

can be written as a linear combination of unitary elements. If $\| h \| < 1$, then $\| h^2 \| \leqslant \| h \|^2 < 1$ and so

$$k = \sum_{n=0}^{\infty} \binom{1/2}{n} (-h^2)^n$$

is a well-defined element in A. Clearly, k is a hermitian element commuting with h such that $k^2 = e - h^2$. Thus $u = h + ik$ is unitary and $h = \frac{1}{2} u + \frac{1}{2} u^*$.

It now follows that $\| x \|_u$ (as given in Step 6) is well-defined for each $x \in A$; further, it is clear from the definition that $\| \cdot \|_u$ is a seminorm on A. We shall call it the *unitary seminorm.* Since the unitary elements form a group under multiplication $\| \cdot \|_u$ is submultiplicative.

Let us compare the unitary seminorm with the B*-norm on A. Observe that $\| h \|_u \leqslant \| h \|$ for every hermitian $h \in A$. Indeed, if $\| h \| < 1$, then $h = \frac{1}{2} u + \frac{1}{2} u^*$ for some unitary $u \in A$ and so $\| h \|_u \leqslant 1$. Thus $\| h \|_u \leqslant \| h \|$ for every hermitian $h \in A$. Further $\| x \|_u \leqslant 2 \| x \|$ for every $x \in A$. For if $x = h + ik$ with hermitian h and k, then $\| x \|_u \leqslant \| h \|_u + \| k \|_u \leqslant 2 \| x \|$. On the other hand $\| x \| \leqslant \| x \|_u$ for all $x \in A$. Indeed, if $x = \Sigma_{n=1}^{N} \lambda_n u_n$, $\lambda_n \in C$, $u_n \in U$, then

$$\| x \| = \| \sum_{n=1}^{N} \lambda_n u_n \| \leqslant \sum_{n=1}^{N} | \lambda_n | \cdot \| u_n \| = \sum_{n=1}^{N} | \lambda_n |$$

since $\| u \|^2 = \| u^* u \| = 1$ for every unitary $u \in A$. Thus $\| x \| \leqslant \| x \|_u$. Hence the unitary seminorm and the B*-norm on A are equivalent norms with $\| x \| \leqslant \| x \|_u \leqslant 2 \| x \|$ for all $x \in A$. To see that these two norms are actually equal we need the following result of Russo and Dye [44] about the closure of the convex hull of the unitary elements in A.

Russo-Dye Theorem. *Let A be a B^*-algebra with identity e and isometric involution. Then then open unit ball of A is contained in the closed convex hull of the unitary elements of A; that is, for each x in A with $\| x \| < 1$ and each $\varepsilon > 0$ there exists a positive integer m and unitary elements u_k such that* $\left\| x - \Sigma_{k=1}^{m} \frac{1}{m} u_k \right\| < \varepsilon$.

The equality of the unitary seminorm and the B*-norm on A is an immediate consequence of this result. Indeed, let $x \in A$ with $\| x \| < 1$.

Then for every $\varepsilon > 0$ there is a positive integer m and unitary elements u_k such that $\left\| x - \Sigma_{k=1}^{m} \frac{1}{m} u_k \right\| < \varepsilon$ and so

$$\| x \|_u \leqslant \left\| \sum_{k=1}^{m} \frac{1}{m} u_k \right\|_u + \left\| x - \sum_{k=1}^{m} \frac{1}{m} u_k \right\|_u$$
$$\leqslant \sum_{k=1}^{m} \frac{1}{m} \| u_k \|_u + 2 \left\| x - \sum_{k=1}^{m} \frac{1}{m} u_k \right\| \leqslant 1 + 2\varepsilon ;$$

since $\varepsilon > 0$ was arbitrary, $\| x \|_u \leqslant 1$. This proves $\| x \|_u \leqslant \| x \|$ and so $\| x \| = \| x \|_u$ for all $x \in A$.

For completeness we will now prove the Russo-Dye Theorem The following elementary proof, valid for arbitrary Banach *-algebras with isometric involution, is based on ideas of Harris [28].

Proof of the Russo-Dye Theorem: Let $x \in A$ with $\| x \| < 1$. Then $\| xx^* \| \leqslant \| x \| \cdot \| x^* \| = \| x \|^2 < 1$. Hence the hermitian element $e - xx^*$ is invertible and has the invertible hermitian square root $(e - xx^*)^{1/2} = \Sigma_{n=0}^{\infty} \binom{1/2}{n} (-xx^*)^n$. Similarly $e - x^*x$ has invertible hermitian square root $(e - x^*x)^{1/2} = \Sigma_{n=0}^{\infty} \binom{1/2}{n} (-x^*x)^n$. For complex λ with $|\lambda| = 1$ define

$$u_\lambda = (e - xx^*)^{-1/2} (x - \lambda e)(e - \lambda x^*)^{-1} (e - x^*x)^{1/2} .$$

We intend to show that u_λ is unitary. Since $\lambda\bar{\lambda} = 1$,

$$\begin{aligned} u_\lambda^* &= (e - x^*x)^{1/2} (e - \bar{\lambda} x)^{-1} (x^* - \bar{\lambda} e)(e - xx^*)^{-1/2} \\ &= (e - x^*x)^{1/2} (\lambda e - x)^{-1} (\lambda x^* - e)(e - xx^*)^{-1/2} . \end{aligned}$$

Observe that

$$\begin{aligned} (\lambda e - x)^{-1} (\lambda x^* - e) &= (\lambda e - x)^{-1} [(\lambda e - x) x^* - (e - xx^*)] \\ &= x^* - (\lambda e - x)^{-1} (e - xx^*) , \\ (e - \lambda x^*)(x - \lambda e)^{-1} &= [x^* (\lambda e - x) - (e - x^*x)] (\lambda e - x)^{-1} \\ &= x^* - (e - x^*x)(\lambda e - x)^{-1} , \end{aligned}$$

and

$$\begin{aligned} x (e - x^*x)^{1/2} &= \sum_{n=0}^{\infty} \binom{1/2}{n} x (-x^*x)^n = \sum_{n=0}^{\infty} \binom{1/2}{n} (-xx^*)^n x \\ &= (e - xx^*)^{1/2} x \end{aligned}$$

which may be conjugated to give the related equality

$$(e - x^*x)^{1/2} x^* = x^* (e - xx^*)^{1/2} .$$

Utilizing these relations it follows easily that $u_\lambda^* = u_\lambda^{-1}$ so u_λ is unitary.

Let $u_{k/m}$ denote the unitary element u_λ with $\lambda = \exp\left(2\pi i \frac{k}{m}\right)$ where k, m are positive integers. We will show that $x = \lim \Sigma_{k=1}^{m} (1/m)\, u_{k/m}$.

With λ as above, let $x_{k/m}$ denote the element

$$x_\lambda = (x - \lambda e)(e - \lambda x^*)^{-1}.$$

Then

$$x - \sum_{k=1}^{m} \frac{1}{m} u_{k/m} = x - \frac{1}{m} \sum_{k=1}^{m} (e - xx^*)^{-1/2} x_{k/m} (e - x^*x)^{1/2}$$
$$= (e - xx^*)^{-1/2} \left[x - \frac{1}{m} \sum_{k=1}^{m} x_{k/m} \right] (e - x^*x)^{1/2}$$

and so

(1) $$\left\| x - \sum_{k=1}^{m} \frac{1}{m} u_{k/m} \right\|$$
$$\leqslant \| (e - xx^*)^{-1/2} \| \cdot \left\| x - \frac{1}{m} \sum_{k=1}^{m} x_{k/m} \right\| \cdot \| (e - x^*x)^{1/2} \|.$$

Observe that

$$x_\lambda = \sum_{n=0}^{\infty} (x - \lambda e)(\lambda x^*)^n = \sum_{n=0}^{\infty} \lambda^n x (x^*)^n - \sum_{n=0}^{\infty} \lambda^{n+1} (x^*)^n$$

and so

$$x - x_\lambda = \sum_{n=0}^{\infty} \lambda^{n+1} (x^*)^n - \sum_{n=1}^{\infty} \lambda^n x (x^*)^n$$
$$= \sum_{n=1}^{\infty} \lambda^n [(x^*)^{n-1} - x (x^*)^n]$$
$$= (e - xx^*) \sum_{n=1}^{\infty} \lambda^n (x^*)^{n-1}.$$

Summing over k, $1 \leqslant k \leqslant m$, and dividing by m we have

$$x - \frac{1}{m} \sum_{k=1}^{m} x_{k/m} = \frac{1}{m} \sum_{k=1}^{m} (x - x_{k/m})$$
$$= (e - xx^*) \sum_{n=1}^{\infty} \frac{1}{m} \sum_{k=1}^{m} \left[\exp\left(2\pi i \frac{k}{m}\right) \right]^n (x^*)^{n-1}$$
$$= (e - xx^*) \sum_{n=1}^{\infty} \frac{1}{m} \sum_{k=1}^{m} \left[\exp\left(2\pi i \frac{n}{m}\right) \right]^k (x^*)^{n-1}.$$

Now, if $1 \leqslant n < m$, then $\exp\left(2\pi i \frac{n}{m}\right) \neq 1$ and so by the sum formula for a finite geometric sum

$$\sum_{k=1}^{m}\left[\exp\left(2\pi i\frac{n}{m}\right)\right]^{k} = \frac{\exp\left(2\pi i\frac{n}{m}\right) - \exp\left(2\pi i\frac{n(m+1)}{m}\right)}{1 - \exp\left(2\pi i\frac{n}{m}\right)} = 0\,;$$

hence we have

$$x - \frac{1}{m}\sum_{k=1}^{m} x_{k/m} = (e - xx^*)\sum_{n=m}^{\infty}\frac{1}{m}\sum_{k=1}^{m}\left[\exp\left(2\pi i\frac{n}{m}\right)\right]^{k}(x^*)^{n-1}.$$

Then

$$\begin{aligned}\left\| x - \frac{1}{m}\sum_{k=1}^{m} x_{k/m}\right\| &\leqslant \| e - xx^*\| \sum_{n=m-1}^{\infty} \|(x^*)^n\| \\ &\leqslant \| e - xx^*\| \sum_{n=m-1}^{\infty} \| x\|^n \\ &\leqslant \| e - xx^*\| \frac{\| x\|^{m-1}}{1 - \| x\|}.\end{aligned}$$

Since the right hand side converges to 0 as $m \to \infty$, the theorem now follows immediately from relation (1) above.

Step 7. *The involution in a B^*-algebra A is isometric.*

Proof. Since every B*-algebra without identity can be isometrically embedded in a B*-algebra with identity we may assume A has an identity. By Step 2 $\| x\|_0 = \| x^*x\|^{1/2}$ is an equivalent B*-norm on A such that $\| x^*\|_0 = \| x\|_0$ for all $x \in A$. Hence, by Step 6, $\| x\|_0 = \| x\|_u$ where $\|\cdot\|_u$ is the unitary seminorm on A.

Observe that $\| u\| = 1$ for every unitary $u \in A$. Indeed, since u and u^* commute, by the argument given in the first step of the proof of Theorem I, we have $\| u^*\| = \| u\|$ and so $\| u\| = 1$.

Now, if $x = \Sigma_{n=1}^{N} \lambda_n u_n$, $\lambda_n \in C$, $u_n \in U$, then

$$\| x\| = \left\| \sum_{n=1}^{N} \lambda_n u_n\right\| \leqslant \sum_{n=1}^{N} |\lambda_n|\cdot\| u_n\| = \sum_{n=1}^{N} |\lambda_n|\,.$$

Thus $\| x\| \leqslant \| x\|_u = \| x\|_0 = \| x^*x\|^{1/2}$ and so $\| x^*\| = \| x\|$.

Step 8. *The Gelfand-Naimark-Segal Construction.* We have seen that the involution in a B*-algebra A is isometric. Further, if A has no identity we can embed A isometrically as a closed ideal of codimension one in the B*-algebra A_e with identity e. Thus we can and will assume without loss of generality that A has an identity e.

The representation of such an algebra A as a norm-closed *-subalgebra of bounded linear operators on a Hilbert space is effected by means of positive functionals on A and a construction due to Gelfand-Naimark [23] and Segal [49].

A *positive functional* on A is a linear functional p such that $p(x^*x) \geqslant 0$ for all $x \in A$. For $x, y \in A$ set $(x, y) = p(y^*x)$. This scalar product on A is linear in x, conjugate linear in y and (x, x) is nonnegative for all x. Thus in particular $p(y^*x) = \overline{p(x^*y)}$ and $|p(y^*x)|^2 \leqslant p(x^*x)\,p(y^*y)$ (Schwarz inequality). Setting $y = e$ we get $p(x^*) = \overline{p(x)}$ and $|p(x)|^2 \leqslant p(e)\,p(x^*x)$.

In general the scalar product on A is degenerate so that a reduction is necessary to obtain nondegeracy. To this end we define the associated *null ideal* $I = \{ x \in A : p(x^*x) = 0 \}$. Since by the above properties of positive functionals

$$I = \{ x \in A : p(y^*x) = 0 \quad \text{for all } y \in A \},$$

the null ideal is clearly a left ideal in A. Then the quotient space $X = A/I$ is a pre-Hilbert space with respect to the induced scalar product

$$(x+I, y+I) = p(y^*x)$$

and, further, for each $a \in A$ we can define a linear operator T_a on X by $T_a(x+I) = ax + I$. The map $a \to T_a$ has the following easily verified properties: $T_{a+b} = T_a + T_b$, $T_{\lambda a} = \lambda T_a$, $T_{ab} = T_a T_b$, and T_e is the identity operator; also

$$\big(T_a(x+I),\ y+I\big) = \big(x+I, T_a^*(y+I)\big)$$

so that $a \to T_a$ is a **-representation* of A on the pre-Hilbert space X.

Let H be the Hilbert space completion of X. We want to show that every operator T_a on X can be extended to a bounded operator on H. We claim that $\| T_a \| \leqslant \| a \|$. Note that $\| T_a(x+I) \|^2 = (ax+I, ax+I) = p(x^*a^*ax)$. For any $\alpha > \| a^*a \| = \| a \|^2$ there exists a hermitian $h \in A$ such that $h^2 = \alpha e - a^*a$. Hence

$$\alpha p(x^*x) - p(x^*a^*ax) = p\big(x^*(\alpha e - a^*a)\,x\big) = p\big((hx)^*(hx)\big) \geqslant 0$$

and so $p(x^*a^*ax) \leqslant \| a \|^2 p(x^*x)$. Thus $\| T_a \| \leqslant \| a \|$. Denote the extended operator on H also by T_a.

The preceding discussion has shown that for every positive functional on A there is associated a *-representation of A as a *-subalgebra of bounded linear operators on a Hilbert space H such that $\| T_a \| \leqslant \| a \|$. In general this representation is neither injective nor norm-preserving. By constructing appropriate positive functionals in the next step we will, however, be able to build a representation with these properties.

Step 9. *Construction of positive functionals.* We will construct for every fixed $z \in A$ a positive functional p on A such that $p(e) = 1$ and $p(z^*z) = \| z \|^2$. Clearly the associated *-representation has the property $\| T_z \| = \| z \|$. Indeed,

$$\| z \|^2 = p(z^*z) = (T_z(e+I),\ T_z(e+I)) = \| T_z(e+I) \|^2$$
$$\leqslant \| T_z \|^2 \| e + I \|^2 = \| T_z \|^2 p(e) = \| T_z \|^2$$

which together with $\| T_z \| \leqslant \| z \|$ gives $\| T_z \| = \| z \|$.

The following construction of the desired positive functional is a special case of an extension theorem for positive functionals due to M. Krein [32].

Construction: Let $H(A)$ be the real vector space of hermitian elements in A and P the positive cone of all positive elements in A. On the subspace $Re + Rz^*z$ of $H(A)$ generated by e and z^*z define p by

$$p(\alpha e + \beta z^*z) = \alpha + \beta \| z^*z \| .$$

Note that p is well-defined on $Re + Rz^*z$ even if e and z^*z are linearly dependent. Since $\| z^*z \| = | z^*z |_\sigma \in \sigma_A(z^*z)$ we have that $\alpha + \beta \| z^*z \|$ lies in $\sigma_A(\alpha e + \beta z^*z)$. In other words, $p(x) \in \sigma_A(x)$ if $x \in Re + Rz^*z$ so that $p(x) \geqslant 0$ for all $x \in P \cap (Re + Rz^*z)$.

Assume p has been extended to a real-linear functional on a subspace W of $H(A)$ such that $p(x) \geqslant 0$ for all $x \in P \cap W$ and assume that there is a $y \in H(A)$ with $y \notin W$. Set

$$a = \inf \{ p(v) \colon y \leqslant v \in W \} \text{ and } b = \sup \{ p(u) \colon y \geqslant u \in W \} .$$

Since $y \leqslant \| y \| e$ and $y \geqslant - \| y \| e$ the infimum and supremum are taken over nonempty sets, and are therefore finite numbers, clearly satisfying $a \geqslant b$. Define p on the subspace of $H(A)$ generated by W and y by

$$p(x + \alpha y) = p(x) + \alpha c \quad (x \in W,\ \alpha \in R),$$

where c is any fixed number such that $a \geqslant c \geqslant b$.

Suppose that $x + \alpha y \geqslant 0$ $(x \in W, \alpha \in R)$. We shall show that $p(x+y) \geqslant 0$. If $\alpha = 0$, then $p(x+\alpha y) = p(x) \geqslant 0$ by assumption.

If $\alpha > 0$, then $x + \alpha y \geqslant 0$ implies $y \geqslant -\dfrac{x}{\alpha} \in W$, so that $p\left(-\dfrac{x}{\alpha}\right) \leqslant c$, or $p(x+\alpha y) \geqslant 0$.

If $\alpha < 0$, then $x + \alpha y \geqslant 0$ implies $y \leqslant -\dfrac{x}{\alpha} \in W$, so that $p\left(-\dfrac{x}{\alpha}\right) \geqslant c$, or $p(x+\alpha y) \geqslant 0$.

By Zorn's Lemma we conclude that p can be extended to a real linear functional p on $H(A)$ such that $p(x) \geqslant 0$ for all $x \in P$.

Finally set $p(x) = p(h) + ip(k)$ if $x = h + ik$ with $h, k \in H(A)$. Then p is a positive functional on A such that $p(e) = 1$ and $p(z^*z) = \| z^*z \| = \| z \|^2$. This completes the construction.

Step 10. *The isometric *-representation.* In the preceding step we constructed for every $z \in A$ a positive functional on A such that the associated *-representation $T^{(z)}$ of A on the Hilbert space $H^{(z)}$ is norm-decreasing and $\| T_z^{(z)} \| = \| z \|$.

Let H be the direct sum of the Hilbert spaces $H^{(z)}$. The *direct sum* of the family $H^{(z)}$, $z \in A$, is defined as the set of all mappings f on A with $f(z) \in H^{(z)}$ such that $\sum\limits_{z \in A} (f(z), f(z)) < \infty$. The algebraic operations in H are pointwise and the scalar product is given by $(f, g) = \sum\limits_{z \in A} (f(z), g(z))$. The reader may easily verify that all Hilbert space axioms are satisfied by H (see [14]).

Define the *-representation T of A on H by

$$(T_a f)(z) = T_a^{(z)}(f(z)) .$$

Note that the inequality

$$\sum_{z \in A} ((T_a f)(z), (T_a f)(z)) \leqslant \| a \|^2 \sum_{z \in A} (f(z), f(z))$$

shows that with f also $T_a f$ belongs to H. Then T_a is a bounded operator on H such that

$$\| T_a \| = \sup_{z \in A} \| T_a^{(z)} \| = \| T_a^{(a)} \| = \| a \| .$$

Hence the map $a \to T_a$ is a norm-preserving *-representation of A on H. This completes the proof of Theorem II as stated in the introduction.

6. Geometrical characterizations of B*-algebras

The first step to a geometrical characterization of B*-algebras among complex Banach algebras was taken in 1956 by Ivan Vidav [52]. To state his result in an appropriate form let us collect some basic ideas and results. For details the reader is referred to the well-written monographs of Frank F. Bonsall and John Duncan [10], [11] on numerical ranges.

Let A be a *unital Banach algebra*, i.e., a Banach algebra with an identity 1 of norm one. A continuous linear functional f on A is called a *state* if $\|f\| = f(1)$. This definition exploits an earlier involution-independent geometrical characterization of the positive functionals on a C*-algebra due to H. Frederic Bohnenblust and Samuel Karlin [9]: a continuous linear functional f on a unital C*-algebra is positive if and only if $\|f\| = f(1)$. Gunter Lumer [34] made strikingly successful use of the generalization of this to define hermitian elements in an arbitrary unital Banach algebra. An element x of a unital Banach algebra A is called *hermitian* if $f(x)$ is real for every state f on A. Clearly, in the special case where A is a C*-algebra, an element $x \in A$ is hermitian if and only if $x^* = x$. Further it turned out that the following conditions for an element x of a unital Banach algebra are equivalent:

1. $f(x)$ is real for every state f on A;
2. $\| 1 + i\alpha x \| = 1 + o(\alpha)$ (α real);
3. $\| \exp(i\alpha x) \| = 1$ (α real).

In fact, Vidav [52] used the second condition to define hermitian elements in unital Banach algebras. Obviously in the algebra of complex numbers we have

$$| 1 + i\alpha x | = 1 + o(\alpha) \quad (\alpha \text{ real})$$

if and only if x is a real number. In the C*-algebra of all bounded operators on a Hilbert space the *self-adjoint operators* (the operators x with $x^* = x$) play the same role as the real numbers in the algebra of complex numbers. Motivated by this observation, Vidav—as he pointed out in a letter to the second named author—asked if the self-adjoint operators could be characterized in a similar way. And, indeed, he was able to show quite easily that an element x in a C*-algebra is self-adjoint if and only if

$$\| 1 + i\alpha x \| = 1 + o(\alpha) \quad (\alpha \text{ real}) .$$

Here is his short argument. Let x be any bounded operator on a Hilbert space and write $x = h + ik$, where h and k are self-adjoint. For all real α we have: $\| 1 + i\alpha x \|^2 = \sup \| \xi + i\alpha x\xi \|^2$, where the supremum is taken over all vectors ξ of norm one. We can write:

$$\| \xi + i\alpha x\xi \|^2 = (\xi, \xi) - 2\alpha(k\xi, \xi) + \alpha^2 [\|h\xi\|^2 + \|k\xi\|^2 + i(k\xi, h\xi) - i(h\xi, k\xi)]$$

Hence if $\| \xi \| = 1$, then

$$\| \xi + i\alpha x\xi \|^2 = 1 - 2\alpha(k\xi, \xi) + O(\alpha^2).$$

Thus $\| 1 + i\alpha x \| = 1 + o(\alpha)$ only if $(k\xi, \xi) = 0$ for every vector ξ. This implies $k = 0$; i.e., x is self-adjoint.

Conversely, if $x = h$ is self-adjoint, then

$$\| \xi + i\alpha h\xi \|^2 = (\xi, \xi) + \alpha^2 \| h\xi \|^2$$

and so $\| 1 + i\alpha h \|^2 = 1 + \alpha^2 \| h \|^2$, which implies $\| 1 + i\alpha h \| = 1 + o(\alpha)$. Thus an element x in a unital C*-algebra is self-adjoint if and only if

$$\| 1 + i\alpha x \| = 1 + o(\alpha) \quad (\alpha \text{ real}).$$

Further investigations of the set $H(A)$ of hermitian elements in a unital Banach algebra A led Vidav [52] to a rather deep geometrical characterization of B*-algebras.

THEOREM. *Let A be a unital Banach algebra such that:*

i) $A = H(A) + iH(A)$;

ii) *if $h \in H(A)$ then $h^2 = a + ib$ for some $a, b \in H(A)$ with $ab = ba$.*

Then the algebra A has the following properties.

1. *The decomposition $x = h + ik$ with $h, k \in H(A)$ is unique.*

2. *Setting $x^* = h - ik$ if $x = h + ik$ the map $x \to x^*$ is an involution on A. Furthermore for $h \in H(A)$ we have $\| h^2 \| = \| h \|^2$.*

3. *$\| x \|_0 = \| x^*x \|^{1/2}$ defines a B*-norm on A which is equivalent to the original norm.*

Nearly ten years later Barnett W. Glickfeld [24] and Earl Berkson [8] showed independently that A is actually a B*-algebra under its original norm. Their proofs in the commutative case are quite different. Berkson utilized the notion of a semi-inner-product space introduced by Lumer [34] and the theory of scalar type operators as developed by Nelson Dunford [14],

[15], [16]. Glickfeld recognized the importance of the exponential function and obtained the commutative theorem via the hermiticity condition $\| \exp(i\alpha x) \| = 1$ (α real) for $x \in A$. A simplification of his proof was pointed out by Robert B. Burckel [12]. The extension to arbitrary (possibly non-commutative) unital Banach algebras is an immediate consequence of a result of Russo and Dye [44] on unitary operators in C*-algebras (see also Step 6 of the preceding section).

Based on a refinement of the Russo-Dye Theorem, Theodore W. Palmer [41] finally showed that condition ii) in Vidav's theorem is unnecessary and also gave the simplest proof that A is already a B*-algebra under its original norm. Thus the following elegant characterization of B*-algebras was established.

THEOREM. *A unital Banach algebra A admits an involution with respect to which it is a B*-algebra if and only if $A = H(A) + iH(A)$.*

Recently Robert T. Moore [36] gave deep duality characterizations of B*-algebras. He defines *hermitian functionals* on an arbitrary unital Banach algebra A to be those in the real span $H(A')$ of the states on A. It is shown that every functional f in the dual A' of A can be decomposed as $f = h + ik$, where h and k are hermitian functionals. Moore's proof of this uses the usual decomposition of measures. Independently Allan M. Sinclair [51] has given an interesting direct proof in which the measure theory is replaced by convexity and Hahn-Banach separation arguments. Their result is a useful strengthening of the Bohnenblust-Karlin vertex theorem [9] which asserts that the states on a unital Banach algebra separate points in A. Substantial simplifications of the proofs of Moore and Sinclair have been given by L. A. Asimow and A. J. Ellis [4].

Clearly, in the special case where A is a C*-algebra, a continuous linear functional f on A is hermitian if and only if $f(x^*) = \overline{f(x)}$ for all $x \in A$. Moreover, every hermitian functional on a C*-algebra is the difference of two positive functionals (see Corollary 2.6.4 of [13]). We have seen that B*-algebras are characterized among unital Banach algebras as those for which there are *enough* hermitian elements. Moore's duality characterization shows that they may also be characterized as those for which there are *too many* hermitian functionals.

THEOREM. *A unital Banach algebra A admits an involution with respect to which it is a B*-algebra if and only if the dual A' decomposes as a real* *direct sum* $A' = H(A') + iH(A')$; *or, equivalently, iff the hermitian elements in* A *separate points in* A'.

This result reduces an important property of a Banach algebra to properties of its dual space and may play a crucial role in further investigations.

7. Further weakening of the B*-axioms

The result of Russo and Dye on the closed convex hull of the unitaries had an immediate consequence for the further weakening of the B*-axioms. Based on Vidav's theorem [52] or on Glimm-Kadison's proof in [25], as Jacob Feldman [19] observed, the following conclusion results (see [8], [24]).

THEOREM. *A Banach *-algebra* A *with identity is a B*-algebra if and only if* $\| x^*x \| = \| x^* \| \cdot \| x \|$ *whenever* x *and* x^* *commute.*

The assumption of an identity was removed in 1970 by George A. Elliott [17]. A result of Johan F. Aarnes and R. V. Kadison [1] on the existence of an approximate identity in a C*-algebra commuting with a given strictly positive element enabled him to extend the norm on A to A_e so that the algebra A_e still satisfied the B*-condition for normal elements.

In 1972 Vlastimil Pták [42] presented in an excellent forty-five page survey article a simplified treatment of the theory of hermitian Banach *-algebras (that is, Banach *-algebras in which all self-adjoint elements have real spectrum) based on the fundamental spectral inequality

$$| x |_\sigma^2 \leqslant | x^*x |_\sigma .$$

Investigating their connections with C*-algebras, he derived several characterizations of B*-algebras in an elegant way. His article circumvented many difficulties by assuming throughout that the algebras possess an identity element.

In an informal conversation during an ergodic theory conference at Texas Christian University in the summer of 1972 the first named author asked Husihiro Araki if the submultiplicativity condition $\| xy \| \leqslant \| x \| \cdot \| y \|$ was actually necessary in the axioms of a B*-algebra. Some months later Araki and Elliott [2] proved the following two results.

THEOREM 1. *Let A be a *-algebra with a complete linear space norm such that $\| x^*x \| = \| x \|^2$ for all $x \in A$. Then A is a B^*-algebra.*

THEOREM 2. *Let A be a *-algebra with a complete linear space norm such that $\| x^*x \| = \| x^* \| \cdot \| x \|$ for all $x \in A$. Suppose that the involution is continuous. Then A is a B^*-algebra.*

Actually the assertion of Theorem 1 was already implicit under some additional assumptions in a 1961 paper of Vidav [53]. Without knowledge of this interesting paper Araki and Elliott were able to give a rather simple proof of it. On the other hand, Theorem 2 was only established after very long and tedious arguments involving the second dual of a C*-algebra and it would be desirable to have a more elegant proof of it.

Araki and Elliott asked at once if in Theorem 1 or 2 it is enough to assume $\| x^*x \| = \| x \|^2$, respectively $\| x^*x \| = \| x^* \| \cdot \| x \|$, only for all *normal* x (all x with $x^*x = xx^*$). Apparently they were not aware of the following well known counterexample [47]. Let $B(H)$ be the *-algebra of all bounded operators on a Hilbert space H of dimension $\geqslant 2$. The numerical radius of an operator x on H is defined by

$$\| x \|_1 = \sup \{ |(x\xi, \xi)| : \xi \in H, \; \| \xi \| = 1 \}.$$

It is easily seen that $\| \cdot \|_1$ is a complete linear space norm on $B(H)$ with $\frac{1}{2} \| x \| \leqslant \| x \|_1 \leqslant \| x \|$ for all $x \in B(H)$ and $\| x \|_1 = \| x \|$ for all normal $x \in B(H)$, where $\| \cdot \|$ is the usual operator norm (see Chapter 17 of [27]). The norm $\| \cdot \|_1$ has the following properties:

$$\| x^* \|_1 = \| x \|_1 \quad \text{for all} \quad x \in B(H),$$
$$\| x^*x \| \geqslant \| x \|_1^2 = \| x^* \|_1 \| x \|_1 \quad \text{for all} \quad x \in B(H);$$

and

$$\| x^*x \|_1 = \| x \|_1^2 = \| x^* \|_1 \| x \|_1$$

for all normal $x \in B(H)$ but not for all $x \in B(H)$. For another counterexample see the addenda to [2].

In [46] Zoltán Sebestyén proved the following general characterization of B*-algebras.

THEOREM. *Let A be a *-algebra with complete linear space norm such that*

$$\| x^*x \| \leqslant \| x \|^2 \quad \textit{for all} \quad x \in A$$

and

$$\| x^*x \| = \| x \|^2 \quad \textit{for all normal} \quad x \in A\,.$$

Then A *is a* B^**-algebra.*

In a later paper [48] Sebestyén claimed to prove that continuity of the involution can be dropped from Theorem 2 above. However, G. A. Elliott has pointed out an error in [48]; indeed, on line four of page 212 the series displayed, although convergent, is not shown to converge to the quasi-inverse of $\lambda^{-1}x$. The paper does reduce the problem to the commutative case; but in this case it remains an interesting open question.

8. Applications

Numerous applications of the Gelfand-Naimark theorems appear in the literature. Indeed, utilizing the representation theorem for commutative algebras important theorems in abstract harmonic analysis can be established. For example both the Plancherel theorem and the Pontryagin duality theorem are proved in [33] via the commutative theorem. Further applications to harmonic analysis can be found in [15], [30], [33] and [37]. The representation theorem for commutative algebras can be used to establish important results on compactifications of topological spaces and locally compact abelian groups (see [15], [30] and [33]); it also provides the most elegant method of proof of the spectral theorem for normal operators on a Hilbert space ([15], [30], [33]).

Applications to group representations and von Neumann algebras can be found in [13] and [37]. For applications to numerical ranges of operators see [7], [9], [10], [11] and [34].

In recent years the theory of C*-algebras has entered into the study of statistical mechanics and quantum theory. The basic principle of the algebraic approach is to avoid starting with a specific Hilbert space scheme and rather to emphasize that the primary objects of the theory are the fields (or observables) considered as purely algebraic quantities, together with their linear combinations, products, and limits in an appropriate topology. The representations of these algebraic objects as operators acting on a suitable Hilbert space can then be obtained in a way that depends essentially only on the states of the physical system under investigation. The principal tool needed to build the required Hilbert space and associated representation is the Gelfand-Naimark-Segal construction discussed earlier in this article.

A substantial literature has now emerged from this new algebraic point of view and a recent book by G. Emch [18] has been written with the express purpose of offering a systematic introduction to the ideas and techniques of the C*-algebra approach to physical problems. The authors recommend this book to the reader who would like to pursue this subject further. The book contains a bibliography of more than four hundred items which should aid the interested reader in his study of this new and interesting application of operator algebras.

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