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# EXTENSION AND LIFTING OF $\mathcal{C}^\infty$ WHITNEY FIELDS

by Edward BIERSTONE and Pierre MILMAN

Whitney's Extension Theorem [10] provides a continuous linear extension operator from the space of  $\mathcal{C}^m$  Whitney fields ( $m < \infty$ ) on a closed subset  $X$  of  $\mathbf{R}^n$ , to the space of  $\mathcal{C}^m$  functions on  $\mathbf{R}^n$ . For  $\mathcal{C}^\infty$  Whitney fields, however, there does not in general exist a continuous linear extension operator [3]. Hence an *extension* problem arises: Under what conditions on  $X$  does there exist a continuous linear extension operator from the space  $\mathcal{E}(X)$  of  $\mathcal{C}^\infty$  Whitney fields on  $X$  to the space  $\mathcal{E}(\mathbf{R}^n)$  of  $\mathcal{C}^\infty$  functions? In fact we can formulate a more general *lifting* problem (cf. [4, Section 7]): Let  $T_X: \mathcal{E}(\mathbf{R}^n) \rightarrow \mathcal{E}(X)$  be the canonical projection, associating to each  $\mathcal{C}^\infty$  function its jet of infinite order on  $X$ . If  $E$  is a topological vector space, and  $G: E \rightarrow \mathcal{E}(X)$  a continuous linear map, then under what conditions is there a continuous linear map  $\tilde{G}: E \rightarrow \mathcal{E}(\mathbf{R}^n)$  such that the following diagram commutes?

$$(1) \quad \begin{array}{ccc} & & \mathcal{E}(\mathbf{R}^n) \\ & \nearrow \tilde{G} & \downarrow T_X \\ E & \xrightarrow{G} & \mathcal{E}(X) \end{array}$$

By a lifting of  $G$  at the point  $a \in X$ , we will mean a continuous linear map  $G_a: E \rightarrow \mathcal{E}(\mathbf{R}^n)$  such that  $G(\xi) - T_X \circ G_a(\xi)$  is flat at  $a$ , for all  $\xi \in E$ . In this paper we prove that if  $E$  is a locally convex topological vector space, then a lifting  $\tilde{G}$  of  $G$  exists provided that there exist pointwise lifts  $G_a: E \rightarrow \mathcal{E}(\mathbf{R}^n)$ , uniformly in  $a \in X$ . The uniformity of the pointwise lifts is the key ingredient in the proof, which is a simple argument using a Whitney partition of unity, analogous to the proof of Whitney's theorem in the  $\mathcal{C}^m$  case ( $m < \infty$ ). Nevertheless the result is a useful technical lemma.

Corollary 1 extends Mather's variant of Borel's Lemma [4, Section 7] to  $\mathcal{C}^\infty$  Whitney fields on an arbitrary closed subset  $X$  of  $\mathbf{R}^n$ . Corollary 2,

together with the well-known extension of  $\mathcal{C}^\infty$  functions defined on a half-space [7], [6], provides a new proof of Stein's extension theorem for  $\mathcal{C}^\infty$  functions on a domain with boundary which is Lipschitz of order 1 [8, Chapter VI, Theorem 5]. Corollary 2 is also used by one of the authors in [1], where Stein's theorem, for  $\mathcal{C}^\infty$  Whitney fields, is extended to the case of a domain with boundary which is Lipschitz of any order, and this result is applied to the extension of  $\mathcal{C}^\infty$  Whitney fields from a semianalytic subset  $X \subset \mathbf{R}^n$  which is the closure of an open set.

*Notation.* Our notation is that of [9, Chapter IV]. If  $k = (k_1, \dots, k_n) \in \mathbf{N}^n$ ,  $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ , write  $|k| = k_1 + \dots + k_n$ ,  $k! = k_1! \dots k_n!$ ,  $x^k = x_1^{k_1}, \dots, x_n^{k_n}$ .  $\mathbf{N}^n$  is partially ordered by the relation:  $k \leq l$  if and only if  $k_j \leq l_j$ ,  $j = 1, \dots, n$ . Write  $\binom{l}{k} = \frac{l!}{k!(l-k)!}$  if  $k \leq l$ ,  $\binom{l}{k} = 0$  otherwise.

If  $\Omega$  is an open subset of  $\mathbf{R}^n$ , then  $\mathcal{E}(\Omega)$  denotes the space of  $\mathcal{C}^\infty$  functions on  $\Omega$ .  $\mathcal{E}(\Omega)$  is a Fréchet space; its topology is defined by the seminorms

$$|f|_m^K = \sup_{\substack{x \in K \\ |k| \leq m}} \left| \frac{\partial^{|k|} f}{\partial x^k}(x) \right|,$$

where  $m \in \mathbf{N}$  and  $K \subset \Omega$  is compact.

Let  $X$  be a closed subset of  $\Omega$ . A *jet of infinite order* on  $X$  is a sequence of continuous functions  $F = (F^k)_{k \in \mathbf{N}^n}$  on  $X$ .  $J(X)$  denotes the space of such jets. Write  $|F|_m^K = \sup_{\substack{x \in K \\ |k| \leq m}} |F^k(x)|$ , and  $F(x) = F^0(x)$ ,  $x \in X$ .

There is a linear map  $J: \mathcal{E}(\Omega) \rightarrow J(X)$ , associating to each  $f \in \mathcal{E}(\Omega)$  the jet  $J(f) = \left( \frac{\partial^{|k|} f}{\partial x^k} \Big|_X \right)_{k \in \mathbf{N}^n}$ . For each  $k \in \mathbf{N}^n$ , there is a linear map  $D^k: J(X) \rightarrow J(X)$ , defined by  $D^k F = (F^{k+l})_{l \in \mathbf{N}^n}$ . We also denote by  $D^k$  the map of  $\mathcal{E}(\Omega)$  to itself, given by  $D^k f = \frac{\partial^{|k|} f}{\partial x^k}$ . This should cause no confusion since  $D^k \circ J = J \circ D^k$ .

If  $a \in X$ ,  $m \in \mathbf{N}$ ,  $F \in J(X)$ , then the *Taylor polynomial of order  $m$  of  $F$  at  $a$*  is the polynomial

$$T_a^m F(x) = \sum_{|k| \leq m} \frac{F^k(a)}{k!} (x-a)^k$$

of degree  $\leq m$ . Define  $R_a^m F = F - J(T_a^m F)$ , so that

$$(R_a^m F)^k(x) = F^k(x) - \sum_{|l| \leq m-|k|} \frac{F^{k+l}(a)}{l!} (x-a)^l$$

if  $|k| \leq m$ . Note that  $D^k \circ R_a^m F(a) = (R_a^m F)^k(a) = 0$ ,  $|k| \leq m$ .

We say that  $F \in J(X)$  is a *Whitney field of class  $\mathcal{C}^\infty$*  on  $X$  if for each  $m \in \mathbb{N}$ ,  $|k| \leq m$ :

$$(R_x^m F)^k(y) = o(|x-y|^{m-|k|})$$

as  $|x-y| \rightarrow 0$ ,  $x, y \in X$ .  $\mathcal{E}(X) \subset J(X)$  denotes the subspace of Whitney fields of class  $\mathcal{C}^\infty$ .  $\mathcal{E}(X)$  is a Fréchet space, with the seminorms

$$\|F\|_m^K = \|F\|_m^K + \sup_{\substack{x, y \in K \\ x \neq y \\ |k| \leq m}} \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}},$$

where  $m \in \mathbb{N}$  and  $K \subset X$  is compact.

*Remarks 1.* If  $F \in J(\Omega)$ , and for all  $x \in \mathbb{R}^n$ ,  $m \in \mathbb{N}$ ,  $|k| \leq m$  we have

$$\lim_{y \rightarrow x} \frac{|(R_x^m F)^k(y)|}{|y-x|^{m-|k|}} = 0,$$

then there exists  $f \in \mathcal{E}(\Omega)$  such that  $F = J(f)$ . This simple converse of Taylor's Theorem shows, in particular, that the two spaces we have denoted  $\mathcal{E}(\Omega)$  are equivalent. On  $\mathcal{E}(\Omega)$ , the topologies defined by the seminorms  $\|\cdot\|_m^K$ ,  $\|\cdot\|_m^K$  are equivalent (by the Open Mapping Theorem).

2. The norms  $\|\cdot\|_m^K$ ,  $\|\cdot\|_m^K$  are not in general equivalent. They are, however, if the compact set  $K$  is connected by rectifiable arcs, and the geodesic distance on  $K$  is equivalent to the Euclidean distance (e.g. if  $K$  is convex) [9, Chapter IV, Proposition 2.6].

**THEOREM.** Let  $X$  be a closed subset of  $\mathbb{R}^n$ , and  $E$  a topological vector space, topologized by a family of seminorms  $\|\cdot\|_{\lambda \in \Lambda}$ . Let  $G: E \rightarrow \mathcal{E}(X)$  be a continuous linear map. Suppose that for each  $a \in X$ , there is a continuous linear map  $G_a: E \rightarrow \mathcal{E}(\mathbb{R}^n)$  such that

a)  $G_a(\xi)^k(a) = G(\xi)^k(a)$  for all  $\xi \in E$ ,  $k \in \mathbb{N}^n$ ;

b) for each  $m \in \mathbb{N}$  and  $L \subset \mathbb{R}^n$  compact, there exists  $\lambda = \lambda(m, L) \in \Lambda$  and a constant  $c = c(m, L)$  such that for all  $\xi \in E$ ,

$$(2) \quad \|G_a(\xi)\|_m^L \leq c(m, L) \|\xi\|_{\lambda(m, L)}.$$

Then there exists a continuous linear map  $\tilde{G}: E \rightarrow \mathcal{E}(\mathbb{R}^n)$  such that  $\tilde{G}(\xi)|_X = G(\xi)$ ,  $\xi \in E$ ; i.e. the diagram (1) commutes.



To state Corollary 1, let  $X$  be a closed subset of  $\mathbf{R}^n$ , and  $F: \mathcal{E}(\mathbf{R}^k) \rightarrow \mathcal{E}(X)$  a continuous linear map. As in [4, Section 7], we say  $F$  is *null* at  $x \in \mathbf{R}^k$  if there exists a neighbourhood  $U$  of  $x$  such that if  $f \in \mathcal{E}(\mathbf{R}^k)$  and  $\text{supp } f \subset U$ , then  $F(f) = 0$ . The *support* of  $F$  is the complement of the set of points where  $F$  is null. Clearly  $\text{supp } F$  is closed.

COROLLARY 1. *If  $F$  has compact support, then there is a continuous linear map  $\tilde{F}: \mathcal{E}(\mathbf{R}^k) \rightarrow \mathcal{E}(\mathbf{R}^n)$  such that  $\tilde{F}(f)|_X = F(f)$  for all  $f \in \mathcal{E}(\mathbf{R}^k)$ ; i.e. the following diagram commutes:*

$$\begin{array}{ccc}
 & & \mathcal{E}(\mathbf{R}^n) \\
 & \nearrow \tilde{F} & \downarrow T_X \\
 \mathcal{E}(\mathbf{R}^k) & \xrightarrow{F} & \mathcal{E}(X)
 \end{array}$$

*Proof.* It suffices to assume  $X = K$ , a compact subset of  $\mathbf{R}^n$ . Let  $a \in K$ . Mather's variant of Borel's Lemma [4, Section 7] provides a continuous linear map  $F_a: \mathcal{E}(\mathbf{R}^k) \rightarrow \mathcal{E}(\mathbf{R}^n)$  such that  $F(f) - T_X \circ F_a(f)$  is flat at  $a$ , for all  $f \in \mathcal{E}(\mathbf{R}^k)$ . Let  $L$  be a cube in  $\mathbf{R}^k$  such that  $\text{supp } F \subset \text{Int } L$ . For each  $r \in \mathbf{N}$ , there exists  $s(r) \in \mathbf{N}$  and a constant  $c(r)$ , such that for all  $a \in K$ ,

$$\sup_{|k|=r} |F(f)^k(a)| \leq |F(f)|_r^K \leq c(r) \|f\|_{s(r)}^L.$$

The uniformity condition (2) for the pointwise lifts  $F_a$  then follows from Mather's estimates in [4]. Hence Corollary 1 follows from the Theorem, with the pointwise lifts given by the maps  $F_a$ .

*Remark 3.* If  $Y$  is a closed subspace of  $\mathbf{R}^k$  for which there exists a continuous linear extension operator  $\mathcal{E}(Y) \rightarrow \mathcal{E}(\mathbf{R}^k)$ , then Corollary 1 holds more generally with  $\mathcal{E}(\mathbf{R}^k)$  replaced by  $\mathcal{E}(Y)$ .

COROLLARY 2. *Let  $X$  be a closed subset of  $\mathbf{R}^n$ . Suppose that for each  $a \in X$ , there is a continuous linear map  $W_a: \mathcal{E}(X) \rightarrow \mathcal{E}(\mathbf{R}^n)$  such that*

a)  $W_a(F)^k(a) = F^k(a)$  for all  $F \in \mathcal{E}(X)$  and  $k \in \mathbf{N}^n$ ;

b) for each  $m \in \mathbf{N}^n$  and  $L \subset \mathbf{R}^n$  compact, there exists  $\lambda = \lambda(m, L) \in \mathbf{N}$ ,  $K = K(m, L) \subset X$  compact, and a constant  $c = c(m, L)$ , such that for all  $F \in \mathcal{E}(X)$ ,

$$|W_a(F)|_m^L \leq c \|F\|_\lambda^K.$$

Then there exists a continuous linear map  $W: \mathcal{E}(X) \rightarrow \mathcal{E}(\mathbf{R}^n)$  such that  $W(F)|_X = F$  for all  $F \in \mathcal{E}(X)$ .

This extension result follows immediately from the Theorem, with  $G$  given by the identity map of  $\mathcal{E}(X)$ .

*Remarks 4.* Corollary 2 may be used to prove Stein's extension theorem [8, Chapter VI, Theorem 5] for  $\mathcal{C}^\infty$  functions. Let  $y = \phi(x_1, \dots, x_n)$  be a continuous function which satisfies the Lipschitz condition

$$(3) \quad |\phi(x) - \phi(x')| \leq M |x - x'|$$

for all  $x, x' \in \mathbf{R}^n$ . We consider extension of  $\mathcal{C}^\infty$  Whitney fields from the closed set

$$X = \{(x, y) \in \mathbf{R}^{n+1} \mid y \geq \phi(x)\}.$$

Let  $\Gamma$  be the closed half-cone defined by  $y \geq M(|x_1| + \dots + |x_n|)$ , and let  $\Gamma(a) = a + \Gamma$  for any  $a \in \mathbf{R}^{n+1}$ . The Lipschitz condition (3) implies that  $\Gamma(a) \subset X$  for any  $a \in X$ . Since  $\Gamma$  is defined by linear inequalities, Seeley's extension theorem [7] provides a continuous linear extension operator  $S': \mathcal{E}(\Gamma) \rightarrow \mathcal{E}(\mathbf{R}^{n+1})$ . Let  $\rho: \mathbf{R}^{n+1} \rightarrow \mathbf{R}$  be a compactly supported  $\mathcal{C}^\infty$  function which equals 1 in a neighborhood of 0. Define a continuous linear operator  $S: \mathcal{E}(\Gamma) \rightarrow \mathcal{E}(\mathbf{R}^{n+1})$  by  $S(F) = S'(\rho \cdot F)$ ,  $F \in \mathcal{E}(\Gamma)$ . The operators  $W_a: \mathcal{E}(\Gamma(a)) \rightarrow \mathcal{E}(\mathbf{R}^{n+1})$ , obtained by translating  $S$  to  $\Gamma(a)$  for each  $a \in X$ , provide the pointwise extensions needed to apply Corollary 2.

5. Let  $\mathcal{E}_p$  be the ring of germs at  $0 \in \mathbf{R}^p$  of  $\mathcal{C}^\infty$  functions, and  $\mathfrak{m}$  its maximal ideal. Let  $\phi: \mathbf{R}^n \rightarrow \mathbf{R}^p$  be a  $\mathcal{C}^\infty$  map such that  $\phi(0) = 0$ . Then  $\phi$  induces a ring homomorphism  $\phi^*: \mathcal{E}(\mathbf{R}^p) \rightarrow \mathcal{E}(\mathbf{R}^n)$ , defined by  $\phi^*(f) = f \circ \phi$ ,  $f \in \mathcal{E}(\mathbf{R}^p)$ . We also denote by  $\phi^*$  the induced homomorphism  $\phi^*: \mathcal{E}_p \rightarrow \mathcal{E}_n$ . We say  $\phi$  is *finite* at 0 if  $\mathcal{E}_n / \phi^*(\mathfrak{m}) \cdot \mathcal{E}_n$  is a finite dimensional real vector space. Let  $b_1, \dots, b_k \in \mathcal{E}(\mathbf{R}^n)$  represent a basis of this vector space; we take  $b_1 \equiv 1$ . By the Malgrange Preparation Theorem [9, Chapter IX, Theorem 3.2], the germs of  $b_1, \dots, b_k$  at 0 generate  $\mathcal{E}_n$  over  $\mathcal{E}_p$ ; i.e. for all  $f \in \mathcal{E}(\mathbf{R}^n)$ , there exist  $g_1, \dots, g_k \in \mathcal{E}(\mathbf{R}^p)$  such that  $f = \sum_{j=1}^k \phi^*(g_j) \cdot b_j$  in some neighborhood of 0. A careful study of Mather's proof of this result ([5, Section 6] or [9, Chapter IX, Section 3]) shows, in fact, that there exist a neighborhood  $U$  of 0 in  $\mathbf{R}^n$ , and continuous linear operators  $G_j: \mathcal{E}(\mathbf{R}^n) \rightarrow \mathcal{E}(\mathbf{R}^p)$ ,  $j = 1, \dots, k$ , such that  $f = \sum_{j=1}^k (\phi^* \circ G_j(f)) \cdot b_j$  in  $U$ , for all  $f \in \mathcal{E}(\mathbf{R}^n)$ .

Consider a  $\mathcal{C}^\infty$  map  $\phi: \mathbf{R}^n \rightarrow \mathbf{R}^n$  such that  $\phi(0) = 0$ . Let  $X, X'$  be closed subsets of  $\mathbf{R}^n$  containing 0, such that  $\phi(X') = X$ . Suppose there is a

continuous linear operator  $W': \mathcal{E}(X') \rightarrow \mathcal{E}(\mathbf{R}^n)$  such that  $g - T_{X'} \circ W'(g)$  is flat at 0, for all  $g \in \mathcal{E}(\mathbf{R}^n)$ . If  $\phi$  is finite at 0, then there exists a continuous linear operator  $W: \mathcal{E}(X) \rightarrow \mathcal{E}(\mathbf{R}^n)$  such that  $f - T_X \circ W(f)$  is flat at 0, for all  $f \in \mathcal{E}(\mathbf{R}^n)$ .

To see this, choose  $b_j \in \mathcal{E}(\mathbf{R}^n)$  and  $G_j: \mathcal{E}(\mathbf{R}^n) \rightarrow \mathcal{E}(\mathbf{R}^n)$ ,  $j = 1, \dots, k$ , as above. Let  $W = G_1 \circ W' \circ \phi^*$ . That  $f - T_X \circ W(f)$  is flat at 0,  $f \in \mathcal{E}(\mathbf{R}^n)$ , follows from the fact that for all  $g \in \mathcal{E}(\mathbf{R}^n)$ , the jets of  $G_j(g)$  at 0,  $j = 1, \dots, k$ , are uniquely determined by that of  $g$  (by [2, Proposition 5.2]). This remark might be useful in constructing the pointwise extensions needed to apply Corollary 2.

*Proof of the Theorem.* By an easy partition of unity argument, it suffices to assume  $X = K$ , a compact subset of  $\mathbf{R}^n$ . Let  $\{\Phi_i \mid i \in I\}$  be a Whitney partition of unity on  $\mathbf{R}^n - K$  (as in [9, Chapter IV, Lemma 2.1]); i.e. a family of functions  $\Phi_i \in \mathcal{E}(\mathbf{R}^n - K)$  satisfying the following conditions:

- i)  $\{\text{supp } \Phi_i \mid i \in I\}$  is a locally finite family. If  $N(x)$  is the number of  $\text{supp } \Phi_i$  to which  $x$  belongs, then  $N(x) \leq 4^n$ .
- ii)  $\Phi_i \geq 0$  for all  $i \in I$ .  $\sum_{i \in I} \Phi_i(x) = 1$  for all  $x \in \mathbf{R}^n - K$ .
- iii)  $2d(\text{supp } \Phi_i, K) \geq \text{diam}(\text{supp } \Phi_i)$  for all  $i \in I$ .
- iv) There exists a constant  $C_k$ , depending only on  $k$  and  $n$ , such that for all  $x \in \mathbf{R}^n - K$ ,

$$|D^k \Phi_i(x)| \leq C_k \left( 1 + \frac{1}{d(x, K)^{|k|}} \right).$$

Let  $F = G(\xi) \in \mathcal{E}(K)$ . For each  $i \in I$ , choose a point  $a_i \in K$  such that  $d(\text{supp } \Phi_i, K) = d(\text{supp } \Phi_i, a_i)$ . Define  $f = \tilde{G}(\xi) \in \mathcal{E}(\mathbf{R}^n)$  by

$$\begin{aligned} f(x) &= F^0(x), & x \in K, \\ f(x) &= \sum_{i \in I} \Phi_i(x) G_{a_i}(\xi)(x), & x \notin K. \end{aligned}$$

Then  $f = \tilde{G}(\xi)$  clearly depends linearly on  $\xi$ , and is  $\mathcal{C}^\infty$  on  $\mathbf{R}^n - K$ . We must show that  $f$  is  $\mathcal{C}^\infty$ ,  $D^k f|_K = F^k$ , and that  $\tilde{G}$  is continuous. We write

$$\begin{aligned} f^k(x) &= F^k(x), & x \in K, \\ f^k(x) &= D^k f(x), & x \notin K. \end{aligned}$$

Let  $m \in \mathbf{N}$ , and  $L$  be a cube in  $\mathbf{R}^n$  such that  $K \subset \text{Int } L$ . There is a constant  $c_1 = c_1(m, L)$  such that if  $g \in \mathcal{E}(L)$ ,  $|k| \leq m$ , then

$$(4) \quad |(R_a^m g)^k(x)| \leq c_1 |g|_m^L \cdot |x - a|^{m-|k|}$$

for all  $a, x \in L$  (for example by [9, Chapter IV, (1.5.2)] and Remark 2 above).

Recall that a *modulus of continuity* is a continuous increasing function  $\alpha: [0, \infty[ \rightarrow [0, \infty[$  such that  $\alpha$  is concave downwards and  $\alpha(0) = 0$ . By [9, Chapter IV, Remark 1.8] there exists a modulus of continuity  $\alpha$  such that

$$(5) \quad |(R_a^m F)^k(x)| \leq \alpha(|x - a|) \cdot |x - a|^{m-|k|}$$

if  $a, x \in K$ ,  $|k| \leq m$ ; and

$$(6) \quad \begin{aligned} \alpha(t) &= \alpha(\text{diam } K) \quad \text{if } t \geq \text{diam } K, \\ \|F\|_m^K &= |F|_m^K + \alpha(\text{diam } K). \end{aligned}$$

It follows from (5) that if  $a, b \in K$ ,  $|k| \leq m$ , then

$$(7) \quad \begin{aligned} &|D^k(T_a^m F)(x) - D^k(T_b^m F)(x)| \\ &\leq 2^{m-|k|} e^{n/2} \alpha(|a - b|) \cdot (|x - a|^{m-|k|} + |x - b|^{m-|k|}) \end{aligned}$$

for all  $x \in \mathbf{R}^n$  [9, Chapter IV, Remark 1.7].

*Claim.* There exists a constant  $c_2 = c_2(m, L)$  such that if  $|k| \leq m$ ,  $a \in K$ ,  $x \in L$ , then

$$(8) \quad \begin{aligned} &|f^k(x) - D^k \circ G_a(\xi)(x)| \\ &\leq c_2 \cdot (\|\xi\|_{\lambda(m, L)} + \alpha(|x - a|)) \cdot |x - a|^{m-|k|}. \end{aligned}$$

Once the claim is established, the proof of the theorem may be completed as follows. Let  $(j)$  be the multiindex whose  $j$ 'th component is 1 and whose other components are 0. Let  $k \in \mathbf{N}^n$ ,  $a \in K$ ,  $x \notin K$ . Then

$$\begin{aligned} &|f^k(x) - f^k(a) - \sum_{j=1}^n (x_j - a_j) \cdot f^{k+(j)}(a)| \\ &\leq |f^k(x) - D^k \circ G_a(\xi)(x)| \\ &+ |D^k \circ G_a(\xi)(x) - D^k \circ G_a(\xi)(a) - \sum_{j=1}^n (x_j - a_j) \cdot D^{k+(j)} \circ G_a(\xi)(a)|. \end{aligned}$$

The second term in the right hand side is  $o(|x - a|)$  since  $G_a(\xi) \in \mathcal{C}(\mathbf{R}^n)$ , while the first is  $o(|x - a|)$  by the claim. Hence  $f^k$  is continuously differentiable, and  $\frac{\partial f^k}{\partial x_j} = f^{k+(j)}$ .

Let  $\mu = \sup_{x \in L} d(x, K)$ ,  $m \in \mathbf{N}$ ,  $|k| \leq m$ . Applying the claim to a point  $x \in L$  and a point  $a \in K$  such that  $d(x, K) = d(x, a)$ , we have

$$\begin{aligned} |D^k f(x)| &\leq |D^k \circ G_a(\xi)(x)| + c_2 \cdot (\|\xi\|_{\lambda(m,L)} + \alpha(\mu)) \cdot \mu^{m-|k|} \\ &\leq c \|\xi\|_{\lambda(m,L)} + c_2 \mu^{m-|k|} \cdot (\|\xi\|_{\lambda(m,L)} + \|G(\xi)\|_m^K) \end{aligned}$$

by (8), (6). Hence there is a constant  $c_3 = c_3(m, L)$  such that

$$|\tilde{G}(\xi)|_m^L \leq c_3 \cdot (\|\xi\|_{\lambda(m,L)} + \|G(\xi)\|_m^K).$$

It follows that  $\tilde{G}$  is continuous.

*Proof of claim.* We may assume  $x \notin K$ . Then

$$f(x) - G_a(\xi)(x) = \sum_{i \in I} \Phi_i(x) \cdot (G_{a_i}(\xi)(x) - G_a(\xi)(x)).$$

Hence

$$f^k(x) - D^k \circ G_a(\xi)(x) = \sum_{l \leq k} \binom{k}{l} S_l(x),$$

where

$$S_l(x) = \sum_{i \in I} D^l \Phi_i(x) \cdot D^{k-l}(G_{a_i}(\xi)(x) - G_a(\xi)(x)).$$

If  $a, b \in K$ ,  $|j| \leq m$ , write

$$\begin{aligned} G_b(\xi)^j(x) - G_a(\xi)^j(x) &= G_b(\xi)^j(x) - (T_b^m \circ G_b(\xi))^j(x) \\ &+ (T_a^m \circ G_a(\xi))^j(x) - G_a(\xi)^j(x) + (T_b^m \circ G_b(\xi))^j(x) - (T_a^m \circ G_a(\xi))^j(x). \end{aligned}$$

Since  $G_a(\xi)^j(a) = F^j(a)$ , then

$$\begin{aligned} (9) \quad &|G_b(\xi)^j(x) - G_a(\xi)^j(x)| \\ &\leq c_1 |G_b(\xi)|_m^L \cdot |x - b|^{m-j} + c_1 |G_a(\xi)|_m^L \cdot |x - a|^{m-|j|} \\ &\quad + 2^{m-|j|} e^{n/2} \alpha(|a - b|) \cdot (|x - a|^{m-|j|} + |x - b|^{m-|j|}) \\ &\quad \text{by (4), (7)} \\ &\leq (cc_1 \|\xi\|_{\lambda(m,L)} + 2^{m-|j|} e^{n/2} \alpha(|a - b|)) \cdot (|x - a|^{m-|j|} + |x - b|^{m-|j|}) \\ &\quad \text{by (2).} \end{aligned}$$

To estimate  $|S_0(x)|$ , note that if  $x \in \text{supp } \Phi_i$ , then  $|x - a_i| \leq 3|x - a|$  by iii), so that  $|a - a_i| \leq 4|x - a|$  and  $\alpha(|a - a_i|) \leq 4\alpha(|x - a|)$ . Hence

$$\begin{aligned} |S_0(x)| &\leq 4^n (3^{m-|k|} + 1) \cdot (cc_1 \|\xi\|_{\lambda(m,L)} + 2^{m-|k|+2} e^{n/2} \alpha(|x - a|)) \\ &\quad \cdot |x - a|^{m-|k|} \end{aligned}$$

by i), ii).

Now consider  $|S_l(x)|$ ,  $l \neq 0$ . For all  $b \in K$ ,

$$S_l(x) = \sum_{i \in I} D^l \Phi_i(x) \cdot D^{k-l}(G_{a_i}(\xi)(x) - G_b(\xi)(x)),$$

since  $\sum_{i \in I} D^l \Phi_i(x) = 0$ . Choose  $b$  so that  $|x - b| = d(x, K)$ . As before, then  $|x - a_i| \leq 3|x - b| \leq 3d(x, K)$ ,  $|b - a_i| \leq 4d(x, K)$ ,  $\alpha(|b - a_i|) \leq 4\alpha(d(x, K))$ . By (9) and iv), there exist constants  $c'$ ,  $c''$  depending only on  $m, L$ , such that

$$\begin{aligned} |S_l(x)| &\leq [c' \|\xi\|_{\lambda(m, L)} + c'' \alpha(d(x, K))] \cdot d(x, K)^{m-|k|} \\ &\leq (c' \|\xi\|_{\lambda(m, L)} + c'' \alpha(|x - a|)) \cdot |x - a|^{m-|k|}. \end{aligned}$$

This completes the proof of the claim, and the theorem.

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