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EXTENSION AND LIFTING OF \mathcal{C}^∞ WHITNEY FIELDS

by Edward BIERSTONE and Pierre MILMAN

Whitney's Extension Theorem [10] provides a continuous linear extension operator from the space of \mathcal{C}^m Whitney fields ($m < \infty$) on a closed subset X of \mathbf{R}^n , to the space of \mathcal{C}^m functions on \mathbf{R}^n . For \mathcal{C}^∞ Whitney fields, however, there does not in general exist a continuous linear extension operator [3]. Hence an *extension* problem arises: Under what conditions on X does there exist a continuous linear extension operator from the space $\mathcal{E}(X)$ of \mathcal{C}^∞ Whitney fields on X to the space $\mathcal{E}(\mathbf{R}^n)$ of \mathcal{C}^∞ functions? In fact we can formulate a more general *lifting* problem (cf. [4, Section 7]): Let $T_X: \mathcal{E}(\mathbf{R}^n) \rightarrow \mathcal{E}(X)$ be the canonical projection, associating to each \mathcal{C}^∞ function its jet of infinite order on X . If E is a topological vector space, and $G: E \rightarrow \mathcal{E}(X)$ a continuous linear map, then under what conditions is there a continuous linear map $\tilde{G}: E \rightarrow \mathcal{E}(\mathbf{R}^n)$ such that the following diagram commutes?

$$(1) \quad \begin{array}{ccc} & & \mathcal{E}(\mathbf{R}^n) \\ & \nearrow \tilde{G} & \downarrow T_X \\ E & \xrightarrow{G} & \mathcal{E}(X) \end{array}$$

By a lifting of G at the point $a \in X$, we will mean a continuous linear map $G_a: E \rightarrow \mathcal{E}(\mathbf{R}^n)$ such that $G(\xi) - T_X \circ G_a(\xi)$ is flat at a , for all $\xi \in E$. In this paper we prove that if E is a locally convex topological vector space, then a lifting \tilde{G} of G exists provided that there exist pointwise lifts $G_a: E \rightarrow \mathcal{E}(\mathbf{R}^n)$, uniformly in $a \in X$. The uniformity of the pointwise lifts is the key ingredient in the proof, which is a simple argument using a Whitney partition of unity, analogous to the proof of Whitney's theorem in the \mathcal{C}^m case ($m < \infty$). Nevertheless the result is a useful technical lemma.

Corollary 1 extends Mather's variant of Borel's Lemma [4, Section 7] to \mathcal{C}^∞ Whitney fields on an arbitrary closed subset X of \mathbf{R}^n . Corollary 2,

together with the well-known extension of \mathcal{C}^∞ functions defined on a half-space [7], [6], provides a new proof of Stein's extension theorem for \mathcal{C}^∞ functions on a domain with boundary which is Lipschitz of order 1 [8, Chapter VI, Theorem 5]. Corollary 2 is also used by one of the authors in [1], where Stein's theorem, for \mathcal{C}^∞ Whitney fields, is extended to the case of a domain with boundary which is Lipschitz of any order, and this result is applied to the extension of \mathcal{C}^∞ Whitney fields from a semianalytic subset $X \subset \mathbf{R}^n$ which is the closure of an open set.

Notation. Our notation is that of [9, Chapter IV]. If $k = (k_1, \dots, k_n) \in \mathbf{N}^n$, $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, write $|k| = k_1 + \dots + k_n$, $k! = k_1! \dots k_n!$, $x^k = x_1^{k_1}, \dots, x_n^{k_n}$. \mathbf{N}^n is partially ordered by the relation: $k \leq l$ if and only if $k_j \leq l_j, j = 1, \dots, n$. Write $\binom{l}{k} = \frac{l!}{k!(l-k)!}$ if $k \leq l$, $\binom{l}{k} = 0$ otherwise.

If Ω is an open subset of \mathbf{R}^n , then $\mathcal{E}(\Omega)$ denotes the space of \mathcal{C}^∞ functions on Ω . $\mathcal{E}(\Omega)$ is a Fréchet space; its topology is defined by the seminorms

$$|f|_m^K = \sup_{\substack{x \in K \\ |k| \leq m}} \left| \frac{\partial^{|k|} f}{\partial x^k}(x) \right|,$$

where $m \in \mathbf{N}$ and $K \subset \Omega$ is compact.

Let X be a closed subset of Ω . A *jet of infinite order* on X is a sequence of continuous functions $F = (F^k)_{k \in \mathbf{N}^n}$ on X . $J(X)$ denotes the space of such jets. Write $|F|_m^K = \sup_{\substack{x \in K \\ |k| \leq m}} |F^k(x)|$, and $F(x) = F^0(x), x \in X$.

There is a linear map $J: \mathcal{E}(\Omega) \rightarrow J(X)$, associating to each $f \in \mathcal{E}(\Omega)$ the jet $J(f) = \left(\frac{\partial^{|k|} f}{\partial x^k} \Big|_X \right)_{k \in \mathbf{N}^n}$. For each $k \in \mathbf{N}^n$, there is a linear map $D^k: J(X) \rightarrow J(X)$, defined by $D^k F = (F^{k+l})_{l \in \mathbf{N}^n}$. We also denote by D^k the map of $\mathcal{E}(\Omega)$ to itself, given by $D^k f = \frac{\partial^{|k|} f}{\partial x^k}$. This should cause no confusion since $D^k \circ J = J \circ D^k$.

If $a \in X, m \in \mathbf{N}, F \in J(X)$, then the *Taylor polynomial of order m of F at a* is the polynomial

$$T_a^m F(x) = \sum_{|k| \leq m} \frac{F^k(a)}{k!} (x - a)^k$$

of degree $\leq m$. Define $R_a^m F = F - J(T_a^m F)$, so that

$$(R_a^m F)^k(x) = F^k(x) - \sum_{|l| \leq m - |k|} \frac{F^{k+l}(a)}{l!} (x - a)^l$$

if $|k| \leq m$. Note that $D^k \circ R_a^m F(a) = (R_a^m F)^k(a) = 0$, $|k| \leq m$.

We say that $F \in J(X)$ is a *Whitney field of class \mathcal{C}^∞* on X if for each $m \in \mathbf{N}$, $|k| \leq m$:

$$(R_x^m F)^k(y) = o(|x - y|^{m - |k|})$$

as $|x - y| \rightarrow 0$, $x, y \in X$. $\mathcal{E}(X) \subset J(X)$ denotes the subspace of Whitney fields of class \mathcal{C}^∞ . $\mathcal{E}(X)$ is a Fréchet space, with the seminorms

$$\|F\|_m^K = |F|_m^K + \sup_{\substack{x, y \in K \\ x \neq y \\ |k| \leq m}} \frac{|(R_x^m F)^k(y)|}{|x - y|^{m - |k|}},$$

where $m \in \mathbf{N}$ and $K \subset X$ is compact.

Remarks 1. If $F \in J(\Omega)$, and for all $x \in \mathbf{R}^n$, $m \in \mathbf{N}$, $|k| \leq m$ we have

$$\lim_{y \rightarrow x} \frac{|(R_x^m F)^k(y)|}{|y - x|^{m - |k|}} = 0,$$

then there exists $f \in \mathcal{E}(\Omega)$ such that $F = J(f)$. This simple converse of Taylor's Theorem shows, in particular, that the two spaces we have denoted $\mathcal{E}(\Omega)$ are equivalent. On $\mathcal{E}(\Omega)$, the topologies defined by the seminorms $|\cdot|_m^K, \|\cdot\|_m^K$ are equivalent (by the Open Mapping Theorem).

2. The norms $|\cdot|_m^K, \|\cdot\|_m^K$ are not in general equivalent. They are, however, if the compact set K is connected by rectifiable arcs, and the geodesic distance on K is equivalent to the Euclidean distance (e.g. if K is convex) [9, Chapter IV, Proposition 2.6].

THEOREM. *Let X be a closed subset of \mathbf{R}^n , and E a topological vector space, topologized by a family of seminorms $\|\cdot\|_{\lambda \in \Lambda}$. Let $G: E \rightarrow \mathcal{E}(X)$ be a continuous linear map. Suppose that for each $a \in X$, there is a continuous linear map $G_a: E \rightarrow \mathcal{E}(\mathbf{R}^n)$ such that*

a) $G_a(\xi)^k(a) = G(\xi)^k(a)$ for all $\xi \in E, k \in \mathbf{N}^n$;

b) for each $m \in \mathbf{N}$ and $L \subset \mathbf{R}^n$ compact, there exists $\lambda = \lambda(m, L) \in \Lambda$ and a constant $c = c(m, L)$ such that for all $\xi \in E$,

$$(2) \quad |G_a(\xi)|_m^L \leq c(m, L) \|\xi\|_{\lambda(m, L)}.$$

Then there exists a continuous linear map $\tilde{G}: E \rightarrow \mathcal{E}(\mathbf{R}^n)$ such that $\tilde{G}(\xi)|_X = G(\xi)$, $\xi \in E$; i.e. the diagram (1) commutes.

To state Corollary 1, let X be a closed subset of \mathbf{R}^n , and $F: \mathcal{E}(\mathbf{R}^k) \rightarrow \mathcal{E}(X)$ a continuous linear map. As in [4, Section 7], we say F is *null* at $x \in \mathbf{R}^k$ if there exists a neighbourhood U of x such that if $f \in \mathcal{E}(\mathbf{R}^k)$ and $\text{supp } f \subset U$, then $F(f) = 0$. The *support* of F is the complement of the set of points where F is null. Clearly $\text{supp } F$ is closed.

COROLLARY 1. *If F has compact support, then there is a continuous linear map $\tilde{F}: \mathcal{E}(\mathbf{R}^k) \rightarrow \mathcal{E}(\mathbf{R}^n)$ such that $\tilde{F}(f)|_X = F(f)$ for all $f \in \mathcal{E}(\mathbf{R}^k)$; i.e. the following diagram commutes:*

$$\begin{array}{ccc}
 & & \mathcal{E}(\mathbf{R}^n) \\
 & \nearrow \tilde{F} & \downarrow T_X \\
 \mathcal{E}(\mathbf{R}^k) & \xrightarrow{F} & \mathcal{E}(X)
 \end{array}$$

Proof. It suffices to assume $X = K$, a compact subset of \mathbf{R}^n . Let $a \in K$. Mather's variant of Borel's Lemma [4, Section 7] provides a continuous linear map $F_a: \mathcal{E}(\mathbf{R}^k) \rightarrow \mathcal{E}(\mathbf{R}^n)$ such that $F(f) - T_X \circ F_a(f)$ is flat at a , for all $f \in \mathcal{E}(\mathbf{R}^k)$. Let L be a cube in \mathbf{R}^k such that $\text{supp } F \subset \text{Int } L$. For each $r \in \mathbf{N}$, there exists $s(r) \in \mathbf{N}$ and a constant $c(r)$, such that for all $a \in K$,

$$\sup_{|k|=r} |F(f)^k(a)| \leq |F(f)|_r^K \leq c(r) \|f\|_{s(r)}^L.$$

The uniformity condition (2) for the pointwise lifts F_a then follows from Mather's estimates in [4]. Hence Corollary 1 follows from the Theorem, with the pointwise lifts given by the maps F_a .

Remark 3. If Y is a closed subspace of \mathbf{R}^k for which there exists a continuous linear extension operator $\mathcal{E}(Y) \rightarrow \mathcal{E}(\mathbf{R}^k)$, then Corollary 1 holds more generally with $\mathcal{E}(\mathbf{R}^k)$ replaced by $\mathcal{E}(Y)$.

COROLLARY 2. *Let X be a closed subset of \mathbf{R}^n . Suppose that for each $a \in X$, there is a continuous linear map $W_a: \mathcal{E}(X) \rightarrow \mathcal{E}(\mathbf{R}^n)$ such that*

a) $W_a(F)^k(a) = F^k(a)$ for all $F \in \mathcal{E}(X)$ and $k \in \mathbf{N}^n$;

b) for each $m \in \mathbf{N}^n$ and $L \subset \mathbf{R}^n$ compact, there exists $\lambda = \lambda(m, L) \in \mathbf{N}$, $K = K(m, L) \subset X$ compact, and a constant $c = c(m, L)$, such that for all $F \in \mathcal{E}(X)$,

$$|W_a(F)|_m^L \leq c \|F\|_\lambda^K.$$

Then there exists a continuous linear map $W: \mathcal{E}(X) \rightarrow \mathcal{E}(\mathbf{R}^n)$ such that $W(F)|_X = F$ for all $F \in \mathcal{E}(X)$.

This extension result follows immediately from the Theorem, with G given by the identity map of $\mathcal{E}(X)$.

Remarks 4. Corollary 2 may be used to prove Stein's extension theorem [8, Chapter VI, Theorem 5] for \mathcal{C}^∞ functions. Let $y = \phi(x_1, \dots, x_n)$ be a continuous function which satisfies the Lipschitz condition

$$(3) \quad |\phi(x) - \phi(x')| \leq M |x - x'|$$

for all $x, x' \in \mathbf{R}^n$. We consider extension of \mathcal{C}^∞ Whitney fields from the closed set

$$X = \{(x, y) \in \mathbf{R}^{n+1} \mid y \geq \phi(x)\}.$$

Let Γ be the closed half-cone defined by $y \geq M(|x_1| + \dots + |x_n|)$, and let $\Gamma(a) = a + \Gamma$ for any $a \in \mathbf{R}^{n+1}$. The Lipschitz condition (3) implies that $\Gamma(a) \subset X$ for any $a \in X$. Since Γ is defined by linear inequalities, Seeley's extension theorem [7] provides a continuous linear extension operator $S': \mathcal{E}(\Gamma) \rightarrow \mathcal{E}(\mathbf{R}^{n+1})$. Let $\rho: \mathbf{R}^{n+1} \rightarrow \mathbf{R}$ be a compactly supported \mathcal{C}^∞ function which equals 1 in a neighborhood of 0. Define a continuous linear operator $S: \mathcal{E}(\Gamma) \rightarrow \mathcal{E}(\mathbf{R}^{n+1})$ by $S(F) = S'(\rho \cdot F)$, $F \in \mathcal{E}(\Gamma)$. The operators $W_a: \mathcal{E}(\Gamma(a)) \rightarrow \mathcal{E}(\mathbf{R}^{n+1})$, obtained by translating S to $\Gamma(a)$ for each $a \in X$, provide the pointwise extensions needed to apply Corollary 2.

5. Let \mathcal{E}_p be the ring of germs at $0 \in \mathbf{R}^p$ of \mathcal{C}^∞ functions, and \mathfrak{m} its maximal ideal. Let $\phi: \mathbf{R}^n \rightarrow \mathbf{R}^p$ be a \mathcal{C}^∞ map such that $\phi(0) = 0$. Then ϕ induces a ring homomorphism $\phi^*: \mathcal{E}(\mathbf{R}^p) \rightarrow \mathcal{E}(\mathbf{R}^n)$, defined by $\phi^*(f) = f \circ \phi$, $f \in \mathcal{E}(\mathbf{R}^p)$. We also denote by ϕ^* the induced homomorphism $\phi^*: \mathcal{E}_p \rightarrow \mathcal{E}_n$. We say ϕ is *finite* at 0 if $\mathcal{E}_n / \phi^*(\mathfrak{m}) \cdot \mathcal{E}_n$ is a finite dimensional real vector space. Let $b_1, \dots, b_k \in \mathcal{E}(\mathbf{R}^n)$ represent a basis of this vector space; we take $b_1 \equiv 1$. By the Malgrange Preparation Theorem [9, Chapter IX, Theorem 3.2], the germs of b_1, \dots, b_k at 0 generate \mathcal{E}_n over \mathcal{E}_p ; i.e. for all $f \in \mathcal{E}(\mathbf{R}^n)$, there exist $g_1, \dots, g_k \in \mathcal{E}(\mathbf{R}^p)$ such that $f = \sum_{j=1}^k \phi^*(g_j) \cdot b_j$ in some neighborhood of 0. A careful study of Mather's proof of this result ([5, Section 6] or [9, Chapter IX, Section 3]) shows, in fact, that there exist a neighborhood U of 0 in \mathbf{R}^n , and continuous linear operators $G_j: \mathcal{E}(\mathbf{R}^n) \rightarrow \mathcal{E}(\mathbf{R}^p)$, $j = 1, \dots, k$, such that $f = \sum_{j=1}^k (\phi^* \circ G_j(f)) \cdot b_j$ in U , for all $f \in \mathcal{E}(\mathbf{R}^n)$.

Consider a \mathcal{C}^∞ map $\phi: \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $\phi(0) = 0$. Let X, X' be closed subsets of \mathbf{R}^n containing 0, such that $\phi(X') = X$. Suppose there is a

continuous linear operator $W': \mathcal{E}(X') \rightarrow \mathcal{E}(\mathbf{R}^n)$ such that $g - T_{X'} \circ W'(g)$ is flat at 0, for all $g \in \mathcal{E}(\mathbf{R}^n)$. If ϕ is finite at 0, then there exists a continuous linear operator $W: \mathcal{E}(X) \rightarrow \mathcal{E}(\mathbf{R}^n)$ such that $f - T_X \circ W(f)$ is flat at 0, for all $f \in \mathcal{E}(\mathbf{R}^n)$.

To see this, choose $b_j \in \mathcal{E}(\mathbf{R}^n)$ and $G_j: \mathcal{E}(\mathbf{R}^n) \rightarrow \mathcal{E}(\mathbf{R}^n)$, $j = 1, \dots, k$, as above. Let $W = G_1 \circ W' \circ \phi^*$. That $f - T_X \circ W(f)$ is flat at 0, $f \in \mathcal{E}(\mathbf{R}^n)$, follows from the fact that for all $g \in \mathcal{E}(\mathbf{R}^n)$, the jets of $G_j(g)$ at 0, $j = 1, \dots, k$, are uniquely determined by that of g (by [2, Proposition 5.2]). This remark might be useful in constructing the pointwise extensions needed to apply Corollary 2.

Proof of the Theorem. By an easy partition of unity argument, it suffices to assume $X = K$, a compact subset of \mathbf{R}^n . Let $\{\Phi_i \mid i \in I\}$ be a Whitney partition of unity on $\mathbf{R}^n - K$ (as in [9, Chapter IV, Lemma 2.1]); i.e. a family of functions $\Phi_i \in \mathcal{E}(\mathbf{R}^n - K)$ satisfying the following conditions:

i) $\{\text{supp } \Phi_i \mid i \in I\}$ is a locally finite family. If $N(x)$ is the number of $\text{supp } \Phi_i$ to which x belongs, then $N(x) \leq 4^n$.

ii) $\Phi_i \geq 0$ for all $i \in I$. $\sum_{i \in I} \Phi_i(x) = 1$ for all $x \in \mathbf{R}^n - K$.

iii) $2d(\text{supp } \Phi_i, K) \geq \text{diam}(\text{supp } \Phi_i)$ for all $i \in I$.

iv) There exists a constant C_k , depending only on k and n , such that for all $x \in \mathbf{R}^n - K$,

$$|D^k \Phi_i(x)| \leq C_k \left(1 + \frac{1}{d(x, K)^{|k|}} \right).$$

Let $F = G(\xi) \in \mathcal{E}(K)$. For each $i \in I$, choose a point $a_i \in K$ such that $d(\text{supp } \Phi_i, K) = d(\text{supp } \Phi_i, a_i)$. Define $f = \tilde{G}(\xi) \in \mathcal{E}(\mathbf{R}^n)$ by

$$\begin{aligned} f(x) &= F^0(x), & x \in K, \\ f(x) &= \sum_{i \in I} \Phi_i(x) G_{a_i}(\xi)(x), & x \notin K. \end{aligned}$$

Then $f = \tilde{G}(\xi)$ clearly depends linearly on ξ , and is \mathcal{C}^∞ on $\mathbf{R}^n - K$. We must show that f is \mathcal{C}^∞ , $D^k f|_K = F^k$, and that \tilde{G} is continuous. We write

$$\begin{aligned} f^k(x) &= F^k(x), & x \in K, \\ f^k(x) &= D^k f(x), & x \notin K. \end{aligned}$$

Let $m \in \mathbf{N}$, and L be a cube in \mathbf{R}^n such that $K \subset \text{Int } L$. There is a constant $c_1 = c_1(m, L)$ such that if $g \in \mathcal{E}(L)$, $|k| \leq m$, then

$$(4) \quad |(R_a^m g)^k(x)| \leq c_1 |g|_m^L \cdot |x - a|^{m-|k|}$$

for all $a, x \in L$ (for example by [9, Chapter IV, (1.5.2)] and Remark 2 above).

Recall that a *modulus of continuity* is a continuous increasing function $\alpha: [0, \infty[\rightarrow [0, \infty[$ such that α is concave downwards and $\alpha(0) = 0$. By [9, Chapter IV, Remark 1.8] there exists a modulus of continuity α such that

$$(5) \quad |(R_a^m F)^k(x)| \leq \alpha(|x - a|) \cdot |x - a|^{m-|k|}$$

if $a, x \in K, |k| \leq m$; and

$$(6) \quad \begin{aligned} \alpha(t) &= \alpha(\text{diam } K) \quad \text{if } t \geq \text{diam } K, \\ \|F\|_m^K &= |F|_m^K + \alpha(\text{diam } K). \end{aligned}$$

It follows from (5) that if $a, b \in K, |k| \leq m$, then

$$(7) \quad \begin{aligned} &|D^k(T_a^m F)(x) - D^k(T_b^m F)(x)| \\ &\leq 2^{m-|k|} e^{n/2} \alpha(|a - b|) \cdot (|x - a|^{m-|k|} + |x - b|^{m-|k|}) \end{aligned}$$

for all $x \in \mathbf{R}^n$ [9, Chapter IV, Remark 1.7].

Claim. There exists a constant $c_2 = c_2(m, L)$ such that if $|k| \leq m, a \in K, x \in L$, then

$$(8) \quad \begin{aligned} &|f^k(x) - D^k \circ G_a(\xi)(x)| \\ &\leq c_2 \cdot (\|\xi\|_{\lambda(m, L)} + \alpha(|x - a|)) \cdot |x - a|^{m-|k|}. \end{aligned}$$

Once the claim is established, the proof of the theorem may be completed as follows. Let (j) be the multiindex whose j 'th component is 1 and whose other components are 0. Let $k \in \mathbf{N}^n, a \in K, x \notin K$. Then

$$\begin{aligned} &|f^k(x) - f^k(a) - \sum_{j=1}^n (x_j - a_j) \cdot f^{k+(j)}(a)| \\ &\leq |f^k(x) - D^k \circ G_a(\xi)(x)| \\ &+ |D^k \circ G_a(\xi)(x) - D^k \circ G_a(\xi)(a) - \sum_{j=1}^n (x_j - a_j) \cdot D^{k+(j)} \circ G_a(\xi)(a)|. \end{aligned}$$

The second term in the right hand side is $o(|x - a|)$ since $G_a(\xi) \in \mathcal{C}(\mathbf{R}^n)$, while the first is $o(|x - a|)$ by the claim. Hence f^k is continuously differentiable, and $\frac{\partial f^k}{\partial x_j} = f^{k+(j)}$.

Let $\mu = \sup_{x \in L} d(x, K), m \in \mathbf{N}, |k| \leq m$. Applying the claim to a point $x \in L$ and a point $a \in K$ such that $d(x, K) = d(x, a)$, we have

$$\begin{aligned} |D^k f(x)| &\leq |D^k \circ G_a(\xi)(x)| + c_2 \cdot (\|\xi\|_{\lambda(m,L)} + \alpha(\mu)) \cdot \mu^{m-|k|} \\ &\leq c \|\xi\|_{\lambda(m,L)} + c_2 \mu^{m-|k|} \cdot (\|\xi\|_{\lambda(m,L)} + \|G(\xi)\|_m^K) \end{aligned}$$

by (8), (6). Hence there is a constant $c_3 = c_3(m, L)$ such that

$$|\tilde{G}(\xi)|_m^L \leq c_3 \cdot (\|\xi\|_{\lambda(m,L)} + \|G(\xi)\|_m^K).$$

It follows that \tilde{G} is continuous.

Proof of claim. We may assume $x \notin K$. Then

$$f(x) - G_a(\xi)(x) = \sum_{i \in I} \Phi_i(x) \cdot (G_{a_i}(\xi)(x) - G_a(\xi)(x)).$$

Hence

$$f^k(x) - D^k \circ G_a(\xi)(x) = \sum_{l \leq k} \binom{k}{l} S_l(x),$$

where

$$S_l(x) = \sum_{i \in I} D^l \Phi_i(x) \cdot D^{k-l}(G_{a_i}(\xi)(x) - G_a(\xi)(x)).$$

If $a, b \in K$, $|j| \leq m$, write

$$\begin{aligned} G_b(\xi)^j(x) - G_a(\xi)^j(x) &= G_b(\xi)^j(x) - (T_b^m \circ G_b(\xi))^j(x) \\ &+ (T_a^m \circ G_a(\xi))^j(x) - G_a(\xi)^j(x) + (T_b^m \circ G_b(\xi))^j(x) - (T_a^m \circ G_a(\xi))^j(x). \end{aligned}$$

Since $G_a(\xi)^j(a) = F^j(a)$, then

$$\begin{aligned} (9) \quad &|G_b(\xi)^j(x) - G_a(\xi)^j(x)| \\ &\leq c_1 |G_b(\xi)|_m^L \cdot |x - b|^{m-j} + c_1 |G_a(\xi)|_m^L \cdot |x - a|^{m-|j|} \\ &\quad + 2^{m-|j|} e^{n/2} \alpha(|a - b|) \cdot (|x - a|^{m-|j|} + |x - b|^{m-|j|}) \\ &\hspace{15em} \text{by (4), (7)} \\ &\leq (cc_1 \|\xi\|_{\lambda(m,L)} + 2^{m-|j|} e^{n/2} \alpha(|a - b|)) \cdot (|x - a|^{m-|j|} + |x - b|^{m-|j|}) \\ &\hspace{15em} \text{by (2).} \end{aligned}$$

To estimate $|S_0(x)|$, note that if $x \in \text{supp } \Phi_i$, then $|x - a_i| \leq 3|x - a|$ by iii), so that $|a - a_i| \leq 4|x - a|$ and $\alpha(|a - a_i|) \leq 4\alpha(|x - a|)$. Hence

$$\begin{aligned} |S_0(x)| &\leq 4^n (3^{m-|k|} + 1) \cdot (cc_1 \|\xi\|_{\lambda(m,L)} + 2^{m-|k|+2} e^{n/2} \alpha(|x - a|)) \\ &\quad \cdot |x - a|^{m-|k|} \end{aligned}$$

by i), ii).

Now consider $|S_l(x)|$, $l \neq 0$. For all $b \in K$,

$$S_l(x) = \sum_{i \in I} D^l \Phi_i(x) \cdot D^{k-l}(G_{a_i}(\xi)(x) - G_b(\xi)(x)),$$

since $\sum_{i \in I} D^l \Phi_i(x) = 0$. Choose b so that $|x - b| = d(x, K)$. As before, then $|x - a_i| \leq 3|x - b| \leq 3d(x, K)$, $|b - a_i| \leq 4d(x, K)$, $\alpha(|b - a_i|) \leq 4\alpha(d(x, K))$. By (9) and iv), there exist constants c' , c'' depending only on m, L , such that

$$\begin{aligned} |S_l(x)| &\leq [c' \|\xi\|_{\lambda(m,L)} + c'' \alpha(d(x, K))] \cdot d(x, K)^{m-|k|} \\ &\leq (c' \|\xi\|_{\lambda(m,L)} + c'' \alpha(|x - a|)) \cdot |x - a|^{m-|k|}. \end{aligned}$$

This completes the proof of the claim, and the theorem.

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