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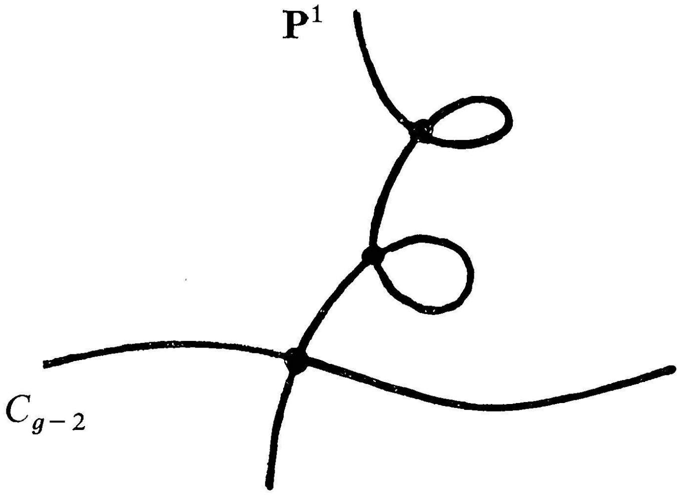
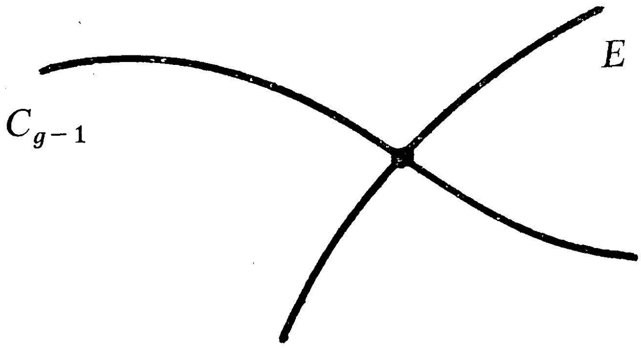



{ vary cross ratio of the 4 points
in $\mathbf{P}^1$ making up the 2 double points

where $C_{g-2}$ is a fixed genus $(g-2)$ component, then $\lambda \mid_{S_1} = \mathcal{O}_{S_1}$, hence sections of $\lambda$ always collapse such families.

(2) If $S_2$ is a curve in $\overline{\mathcal{M}}_g$ composed of curves of the form:




{ $E$ an elliptic curve: vary its
$j$-invariant

where $C_{g-1}$ is a fixed genus $(g-1)$ component, then $\lambda^{11} \otimes \delta^{-1} \mid_{S_2} = \mathcal{O}_{S_2}$ i.e. $\lambda^{11} \otimes \delta^{-1}$ collapses these families.

We omit the details.

## APPENDIX

We wish to fill in the gap in the proof of Proposition 5.5 on page 95. The difficulty occurs if the support of $\mathcal{I}$, i.e. $(0) \times L_1$, contains some of the components of $C_2$ meeting $C_1$. In this case, the inequality

$$e_L(\mathcal{I}_2) \geqslant w$$

is not clear. Indeed, if $D_1, \ldots, D_k$ are the components of $C_2$ meeting $C_1$, $w_i = \#(D_i \cap C_1)$, and $\mathcal{K}_i$ is the pull-back of $\mathcal{I}_2$ to $D_i$, then

$$e_L(\mathscr{I}_2) = \sum e_L(\mathscr{K}_i)$$
$$e_L(\mathscr{K}_i) \geqslant w_i \text{ if } L_1 \not\supseteq D_i$$
$$= 2 \deg D_i \text{ if } L_1 \supseteq D_i\,.$$

Now suppose $C_1$ is *irreducible* and $D_i \subseteq L_1$. Then (5.7) is modified to:

$$2 \deg D_i + 2 \deg C_1 \leqslant \frac{16}{7}(n_1+1)\,.$$

Since $C_1$ spans $L_1$, $n_1 \leqslant \deg C_1$. Substituting this, we find

$$\deg D_i \leqslant \frac{\deg C_1}{7} + \frac{8}{7}$$

hence $\deg D_i \leqslant \deg C_1$ (except in the lowest case $\deg C_1 = 1$; in this case, $C_1$ is a line, so $C_1 = L_1$ and $\operatorname{Supp}\mathscr{K}_i = D_i \cap \mathrm{L}_1 \subsetneq D_i$). Now the reverse of this inequality cannot be true too. This means that if we apply the same arguement to

$$C_{\text{red}} = D_i \cup \overline{(C-D_i)}$$

then the linear span $M$ of $D_i$ cannot contain $C_1$. Therefore

$$w_i + 2 \deg D_i \leqslant \frac{16}{7}(\dim M_i + 1) \leqslant \frac{16}{7}(\deg D_i + 1)$$

$$\therefore\ w_i \leqslant 2 \deg D_i$$
$$\therefore\ e_L(\mathscr{K}_i) \geqslant w_i \quad \text{in all cases}$$
$$\therefore\ e_L(\mathscr{I}_2) \geqslant w \quad \text{as required.}$$

This proves (5.7) if $C_1$ is irreducible, hence (a) and (b) that follow are correct. In particular, (b) shows that $\mathcal{O}_{C_1}(1)(-\mathscr{W})$ always has sections, unless $C_1$ is a line and $\# \mathscr{W} = 2$. The next paragraph shows that $C$ is embedded by a complete linear system. So when $\Gamma(\mathcal{O}_{C_1}(1)(-\mathscr{W})) \neq (0)$, there is a hyperplane containing all components of $C$ except $C_1$. Returning to the general case of (5.7) where $C_1$ is any subset of the components of $C$, it follows that the linear span $L_1$ of $C_1$ contains only $C_1$ and the *lines* $D_i$ which meet $C_1$ in 2 points. For these, $\#(D_i \cap C_1) = 2 \deg D_i$, so in all cases it is true that $e_L(\mathscr{I}_2) \geqslant w$ as required.