

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 23 (1977)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** STABILITY OF PROJECTIVE VARIETIES  
**Autor:** Mumford, David  
**Kapitel:** LINE BUNDLES ON THE MODULI SPACE  
**DOI:** <https://doi.org/10.5169/seals-48919>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 24.12.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

$$\begin{array}{ccccc}
 C_\eta & \xleftarrow{\approx} & D_\eta & \xrightarrow{\Phi_\eta} & \mathbf{P}^N \times \operatorname{Spec} k((t)) \\
 \cap & & \cap & & \cap \\
 \mathcal{C} & \xleftarrow{\quad} & \mathcal{D} & \xrightarrow{\phi} & \mathbf{P}^N \times \operatorname{Spec} k[[t]] \\
 \Phi_\eta^*(\mathcal{O}_{\mathbf{P}^N}(1)) & = & \omega_{D_\eta/k((t))}^{\otimes n} & & 
 \end{array}$$

Let  $L = \mathcal{O}_{\mathcal{D}}(1)$ . It follows that  $L \cong \omega_{\mathcal{D}/k[[t]]}^{\otimes n} (-\sum r_i D_i)$ , where  $D_i$  are the components of  $D_0$ . Multiplying the isomorphism by  $t^{\min(r_i)}$ , we can assume  $r_i \geq 0$ ,  $\min r_i = 0$ . Let  $D_1 = \bigcup_{r_i=0} D_i$ ,  $D_2 = \bigcup_{r_i>0} D_i$ . If  $f$  is a local equation of  $\sum r_i D_i$ , then  $f \not\equiv 0$  in any component of  $D_1$  since  $r_i = 0$  on all these while  $f(x) = 0$ , all  $x \in D_1 \cap D_2$ , so

$$\#(D_1 \cap D_2) \leq \deg_{D_1}(\mathcal{O}_{\mathcal{D}}(\sum r_i D_i)).$$

But this last degree equals  $(\deg D_1 - n \deg_{D_1}(\omega_{D_0}))$  which contradicts iii) of Proposition 5.5 unless all  $r_i$  are zero. Hence  $L = \omega_{\mathcal{C}}^{\otimes n}$  which shows  $\mathcal{D} = \mathcal{C}$ .

### LINE BUNDLES ON THE MODULI SPACE

For the remainder of this section we examine  $\operatorname{Pic}(\bar{\mathcal{M}}_g)$ . We fix a genus  $g \geq 2$  and an  $e \geq 3$ . Then for all stable  $C$ ,  $\omega_C^{\otimes e}$  is very ample and in this embedding  $C$  has degree  $d = 2e(g-1)$ , the ambient space has dimension  $v-1$  where  $v = (2e-1)(g-1)$  and  $C$  has Hilbert polynomial  $P(X) = dX - (g-1)$ . Let  $H \subset \operatorname{Hilb}_{\mathbf{P}^{v-1}}^P$  be the locally closed smooth subscheme of  $e$ -canonical stable curves  $C$ , let  $C \subset H \times \mathbf{P}^{v-1}$  be the universal curve and let

$$\operatorname{ch} : H \rightarrow \operatorname{Div} = \operatorname{Div}^{d,d} = \left\{ \begin{array}{l} \text{projective space of bihomogeneous forms} \\ \text{of bidegree } (d, d) \text{ in dual coordinates} \\ u, v \text{ (cf. § 1).} \end{array} \right\}$$

be the Chow map. These are related by the diagram

$$\begin{array}{ccccc}
 & & C & & \\
 & & \downarrow \pi & & \\
 \operatorname{Div} & \xleftarrow{\operatorname{ch}} & H & \xrightarrow{\rho} & \bar{\mathcal{M}}_g = H/PGL(v)
 \end{array}$$

If  $\operatorname{Pic}(H, PGL(v))$  is the Picard group of invertible sheaves on  $H$  with  $PGL(v)$ -action, we have a diagram

$$\mathrm{Pic}(\bar{\mathcal{M}}_g) \xrightarrow{\rho^*} \mathrm{Pic}(H, \mathrm{PGL}(v)) \xrightarrow{\alpha} \mathrm{Pic}(H)^{\mathrm{PGL}(v)} \subset \mathrm{Pic}(H).$$

In this situation, we have:

LEMMA 5.8. *In the sequence above,  $\rho^*$  is injective with torsion cokernel and  $\alpha$  is an isomorphism.*

*Proof.*  $\alpha$  is an isomorphism by Prop. 1.4 [14];  $\rho^*$  injective is easy; coker  $\rho^*$  torsion can be proved, for instance, using Seshadri's construction, Th. 6.1 [19].

This lemma allows us to examine  $\mathrm{Pic}(\bar{\mathcal{M}}_g)$  by looking inside  $\mathrm{Pic}(H)^{\mathrm{PGL}(v)}$  which is a much easier group to come to grips with.

DEFINITION 5.9. *Let  $\Delta \subset H$  be the divisor of singular curves,  $\delta = \mathcal{O}_H(\Delta)$  and  $\lambda_n = \Lambda^{\max}(\pi_*(\omega_{C/H}^{\otimes n}))$ , ( $n \geq 1$ ). We write  $\lambda$  for  $\lambda_1$ .*

The sheaves  $\lambda_n$  and  $\delta$  are the most obviously interesting invertible sheaves on  $H$  from a moduli point of view. The next theorem expresses all of these in terms just involving  $\lambda$  and  $\delta$ .

THEOREM 5.10.  $\lambda_n = \mu^{\binom{n}{2}} \otimes \lambda$  where  $\mu = \lambda^{12} \otimes \delta^{-1}$ .

*Proof.* The proof is based on Grothendieck's relative Riemann-Roch theorem (see Borel-Serre [4]), which we will briefly recall.

Let  $X$  and  $Y$  be complete smooth varieties over  $k$ ,  $A(X)$  be the Chow ring of  $X$  and  $\mathcal{F}$  be a coherent sheaf on  $X$ . Let  $c_i(\mathcal{F}) \in A(X)$  denote the  $i^{\mathrm{th}}$  Chern class of  $\mathcal{F}$ ,  $\mathrm{Chern}(\mathcal{F}) \in A(X) \otimes \mathbf{Q}$  its Chern character and  $\mathcal{T}(\mathcal{F}) \in A(X) \otimes \mathbf{Q}$  its Todd genus. These are related by:

$$(5.11) \quad \mathrm{Chern}(\mathcal{F}) = rk \mathcal{F} + c_1(\mathcal{F}) + \frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F}) \\ + \text{terms of higher codimension,}$$

$$\mathcal{T}(\mathcal{F}) = 1 - \frac{c_1(\mathcal{F})}{2} + \frac{c_1(\mathcal{F})^2 + c_2(\mathcal{F})}{12} \\ + \text{terms of higher codimension.}$$

Let  $K(Y)$  be the Grothendieck group of  $Y$ ,  $f: X \rightarrow Y$  be a proper map, and  $f_!(\mathcal{F}) = \sum (-1)^i [\mathbf{R}^i f_* \mathcal{F}] \in K(Y)$ . The relative Riemann-Roch theorem expresses the Chern character of  $f_!(\mathcal{F})$ , modulo torsion as

$$\mathrm{Chern}(f_! \mathcal{F}) = f_*(\mathrm{Chern} \mathcal{F} \cdot \mathcal{T}(\Omega_{X/Y}^1))$$

which using (5.11) gives:

$$(5.12) \quad rk f_! \mathcal{F} + c_1(f_! \mathcal{F}) + \dots \\ = f_* \left[ \left( rk(\mathcal{F}) + c_1(\mathcal{F}) + \frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F}) \right) \cdot \left( 1 - \frac{c_1(\Omega_{X/Y}^1)}{2} + \frac{c_1(\Omega_{X/Y}^1)^2 + c_2(\Omega_{X/Y}^1)}{12} \right) \right]$$

For the time being, we work implicitly modulo torsion.

Now suppose  $\mathcal{F}$  is a line bundle such that  $R^i f_*(\mathcal{F}) = 0$ ,  $i > 0$  and suppose  $\dim X = \dim Y + 1$ . Then the codimension 1 term on the left of (5.12) (i.e. on  $Y$ ) corresponds to the codimension two term on the right (i.e. on  $X$ ). Since  $c_2(\mathcal{F}) = 0$ , this gives

$$(5.13) \quad c_1(f_* \mathcal{F}) = c_1(f_! \mathcal{F}) \\ = f_* \left[ \frac{c_1(\Omega_{X/Y}^1)^2 + c_2(\Omega_{X/Y}^1)}{12} - \frac{c_1(\mathcal{F})c_1(\Omega_{X/Y}^1)}{2} + \frac{c_1(\mathcal{F})^2}{2} \right]$$

In case  $f: C \rightarrow S$  is a moduli-stable curve over  $S$ ,  $X = C$  and  $Y = S$ , we can simplify this. Indeed I claim that if  $\text{Sing } C$  is the singular set on  $C$  and  $I_{\text{sing}}$  is its ideal, then

- i)  $\text{codim } \text{Sing } C = 2$
- ii) the canonical homomorphism  $\Omega_{C/S}^1 \rightarrow \omega_{C/S}$  induces an isomorphism  $\Omega_{C/S}^1 = I_{\text{sing}} \cdot \omega_{C/S}$ .

We certainly have the isomorphism of ii) off  $\text{Sing } C$ . At a singular point  $C$  has a local equation of the form  $xy = t^n$ , where  $t$  is a parameter on  $S$ ,  $x$  and  $y$  are affine coordinates on the fibre. Moreover locally  $C$  is singular only at the points  $(0, 0)$  in the fibres where  $t = 0$ , so  $\text{Sing } C$  has codimension 2. Near the singular point

$$\Omega_{C/S}^1 = (\mathcal{O}_C dx + \mathcal{O}_C dy) / (x dy + y dx) \mathcal{O}_C$$

while  $\omega_{C/S}$  is the invertible sheaf generated by the differential  $\zeta$  which is given by  $dx/x$  outside  $x = 0$  and by  $-dy/y$  outside  $y = 0$ . Thus

$$\Omega_{C/S}^1 = \mathcal{M}_{(0,0),C} \cdot \zeta = \mathcal{M}_{(0,0),C} \cdot \omega_{C/S}.$$

Recall the following corollary to Riemann-Roch: if  $X$  is a smooth variety,  $Y \subset X$  a subvariety of  $\text{codim } r$  and  $\mathcal{F}$  is coherent on  $Y$ , then considering  $\mathcal{F}$  as a sheaf on  $X$



$$c_i(\mathcal{F}) = \begin{cases} 0, & 1 \leq i \leq r-1 \\ ((-1)^{r-1} (r-1)! rk \mathcal{F}) Y, & i = r \end{cases}$$

Set  $X = C$ ,  $Y = \text{Sing } C$  and  $\mathcal{F} = \Omega_{C/S}^1$ . The Whitney product formula applied to the chern classes of the exact sequence

$$0 \rightarrow \Omega_{C/S}^1 \rightarrow \omega_{C/S} \rightarrow \omega_{C/S} \otimes \mathcal{O}_{\text{Sing } C} \rightarrow 0$$

gives, taking account of the corollary

$$\begin{aligned} 1 + c_1(\omega_{C/S}) \\ = (1 + c_1(\Omega_{C/S}^1) + c_2(\Omega_{C/S}^1) + \dots) \cdot (1 + 0 - [\text{Sing } C] + \dots) \end{aligned}$$

Equating terms of equal codimension, we see that  $c_1(\Omega_{C/S}^1) = c_1(\omega)$  and  $c_2(\Omega_{C/S}^1) = [\text{Sing } C]$  so that (5.13) becomes

$$c_1(f_* \mathcal{F}) = f_* \left[ \frac{c_1(\omega_{C/S})^2 + [\text{Sing } C]}{12} - \frac{c_1(\mathcal{F}) c_1(\omega_{C/S})}{2} + \frac{c_1(\mathcal{F})^2}{2} \right]$$

Applying this to the map  $\pi: C \rightarrow H$ , when  $\mathcal{F} = \omega_{C/H}^{\otimes n}$  gives

$$\begin{aligned} \lambda_n &= A^{\max}(\pi_* \omega_{C/H}^{\otimes n}) = c_1(\pi_* \omega_{C/H}^{\otimes n}) \\ &= \pi_* \left[ \frac{c_1(\omega_{C/H})^2 + [\text{Sing } C]}{12} - \frac{c_1(\omega_{C/H}^{\otimes n}) c_1(\omega_{C/H})}{2} + \frac{c_1(\omega_{C/H}^{\otimes n})^2}{2} \right] \\ &= \binom{n}{2} \pi_*(c_1(\omega_{C/H})^2) + \frac{\pi_*(c_1(\omega_{C/H})^2) + [\Delta]}{12} \end{aligned}$$

Setting<sup>1)</sup>  $n = 1$ , we see that  $\lambda = \left[ \frac{\pi_*(c_1(\omega_{C/H})^2) + [\Delta]}{12} \right]$  and  $\pi_*(c_1(\omega_{C/H})^2) = 12\lambda - [\Delta]$ . Plugging these values back in gives us the theorem up to torsion. But in fact:

LEMMA 5.14. *Over  $\mathbf{C}$ ,  $\text{Pic}(H, \text{PGL}(v))$  is torsion free.*

Note that this will prove what we want because the invertible sheaves that we are trying to show are isomorphic all “live” on the full scheme  $H_{\mathbf{Z}}$  over  $\text{Spec } \mathbf{Z}$  of stable  $\mathcal{C}$ -canonical curves. If they are isomorphic on  $H_{\mathbf{Z}}$ , they are isomorphic after any base change. But on the other hand, I claim that  $\text{Pic}(H, \text{PGL}(v))$  injects into  $\text{Pic}(H_{\mathbf{C}}, \text{PGL}_{\mathbf{C}}(v))$ :

<sup>1)</sup> For  $n = 1$ ,  $R^1 \pi_*(\omega_{C/H})$  is not zero, but it is the trivial line bundle, hence doesn't affect  $\pi_*$ .

If  $L$  is a line bundle on  $H$  with  $PGL(v)$  action such that  $L \otimes \mathbf{C}$  is trivial over  $H_{\mathbf{C}}$ , then

$$\begin{array}{c} H^0(H, L)^{PGL(v)} \otimes \mathbf{C} = H^0(H_{\mathbf{C}}, L \otimes \mathbf{C})^{PGL(v)} \\ \Downarrow \alpha \\ H^0(H_{\mathbf{C}}, \mathcal{O}_{H_{\mathbf{C}}})^{PGL(v)} = \mathbf{C} \end{array}$$

since  $H_{\mathbf{C}}/PGL(v)$  is compact. Thus we can find a non-zero section  $s \in H^0(H, L)^{PGL(v)}$ , which over  $\mathbf{C}$  can be used to give the trivialization  $\alpha$ . Over  $\mathbf{C}$ ,  $s$  has no zeros so the divisor  $(s)_0$  of the zeros of  $s$  on  $H$ , has support only over the closed fibres of  $\text{Spec}(\mathbf{Z})$ . Mumford and Deligne [6] have shown that  $H \rightarrow \text{Spec} \mathbf{Z}$  is smooth with irreducible fibres, hence  $(s)_0 = \sum r_i \pi^{-1}(p)$ ,  $r_i \geq 0$  i.e.  $(s)_0 = (n)$  for some integer  $n$ . Then  $\left(\frac{s}{n}\right)$  is a global section of  $L$  with no zeros so  $L$  is trivial.

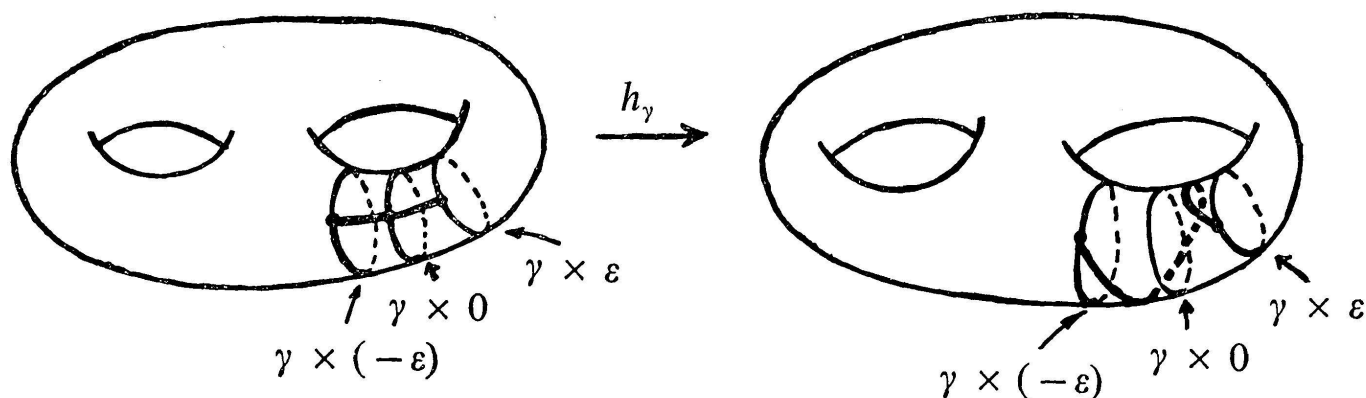
*Proof of Lemma.* Over  $\mathbf{C}$ , we have Teichmüller theory at our disposal. Let  $\Pi$  be a standard model of a group with generators  $\{a_i, b_i \mid 1 \leq i \leq g\}$  mod the relation  $\prod_{i=1}^g (a_i b_i a_i^{-1} b_i^{-1}) = 1$ . Then the Teichmüller modular group  $\Gamma$  is

$$\Gamma = \{ \alpha \mid \alpha : \Pi \rightarrow \Pi \text{ is an orientation preserving } \} / \begin{array}{l} \text{inner} \\ \text{isomorphism} \quad \text{automorphisms} \end{array}$$

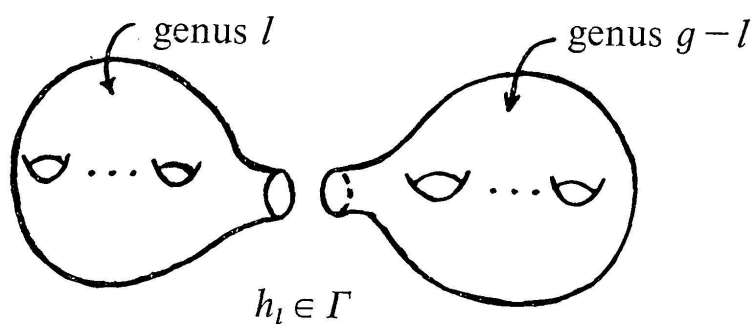
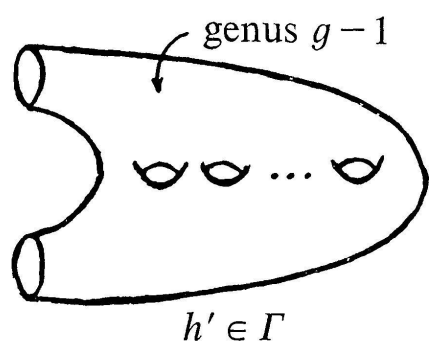
The Teichmüller space  $\mathcal{T}_g$  is given by

$$\mathcal{T}_g = \left\{ (C, \alpha) \left| \begin{array}{l} C \text{ a smooth curve of genus } g \text{ and } \alpha : \pi_1(C) \rightarrow \Pi \text{ an} \\ \text{orientation preserving isomorphism given up to inner} \\ \text{automorphism} \end{array} \right. \right\}$$

Fix a model  $M_g$  of the real surface of genus  $g$ , and identify  $\pi_1(M_g)$  and  $\Pi$ . Then  $\Gamma$  is generated by the maps which are induced by certain automorphisms of  $M_g$ , called Dehn twists. The Dehn twist  $h_\gamma$  corresponding to a loop  $\gamma : [0, 1] \rightarrow M_g$  on  $M_g$  is given by taking an  $\varepsilon$ -collar  $\gamma \times [-\varepsilon, \varepsilon]$  about  $\gamma$ , letting  $h$  = identify off the collar and letting  $h(\gamma(t), \eta - \varepsilon) = \left( \gamma\left(t + \frac{\eta}{2\varepsilon}\right), \eta - \varepsilon \right)$  as shown below.

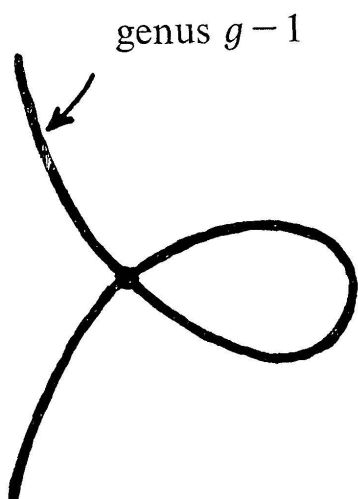


Up to inner automorphism  $h_\gamma$  is determined by which of the pictures below results from cutting open  $M_g$  along  $\gamma$ . We have name these elements of  $\Gamma$  in the diagrams:

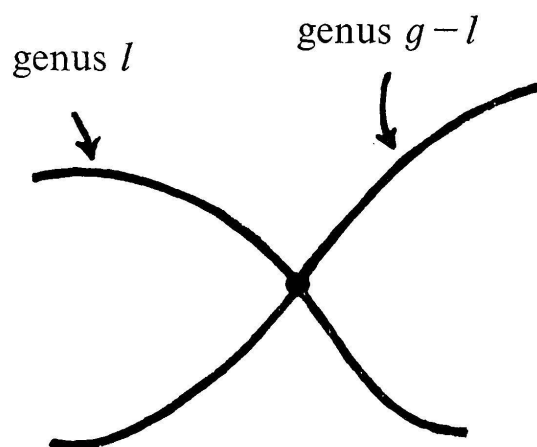


The Dehn twist  $h_\gamma$  can also be described as the monodromy map obtained by going around a curve  $C_0$  with one double point for which  $\gamma$  is the vanishing cycle.

The components of  $\Delta \subset H$  correspond to the different ways of putting a stable double point on a smooth moduli stable curve  $C$ . They are the closures of the sets of curves of the forms shown below: again, we name these components in the diagram:



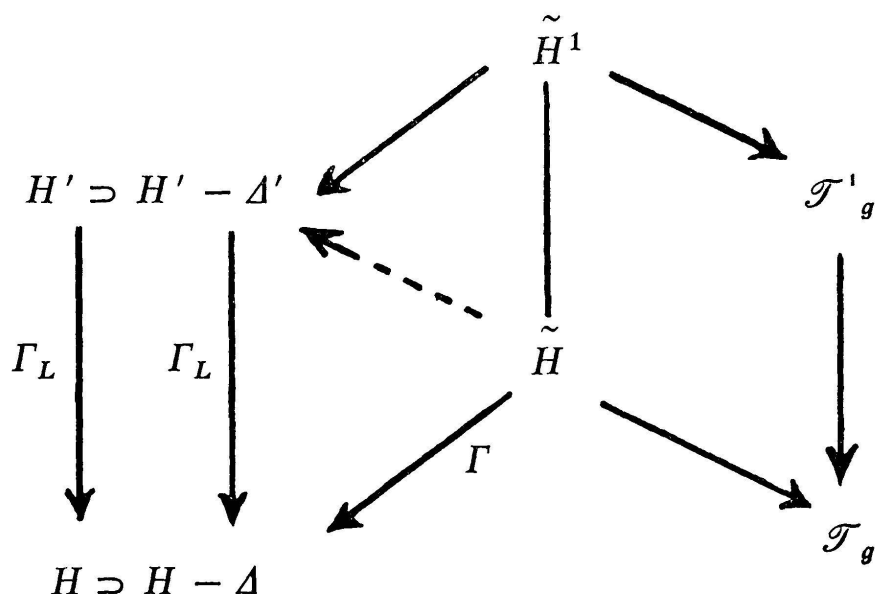
$$\Delta' = \left( \begin{array}{l} \text{closure of set} \\ \text{formed by curves} \\ \text{like this} \end{array} \right)$$



$$\Delta_l = \left( \begin{array}{l} \text{closure of set} \\ \text{formed by curves} \\ \text{like this} \end{array} \right)$$

Let  $\tilde{H} = \left\{ (C, \alpha, B) \mid \begin{array}{l} (C, \alpha) \in \mathcal{T}_g, B \text{ a basis of the } e\text{-tuple dif-} \\ \text{ferentials on } C \text{ given up to a scalar} \end{array} \right\}$

Suppose we are given a line bundle  $L$  on  $H$  with  $PGL(v)$ -action such that  $L^n \cong \mathcal{O}_H$ .  $L$  induces a cyclic covering  $H'$  of  $H$  plus a lifting of the  $PGL(v)$ -action to  $H'$ . If we choose  $n$  minimal this covering is not split: we denote its structure group by  $\Gamma_L$ . Let  $\tilde{H}'$  be the pullback of covering over  $\tilde{H}$ , and let  $\mathcal{T}'_g$  denote the quotient of  $\tilde{H}'$  by  $PGL(v)$ —this is a covering of  $\mathcal{T}_g$ . These coverings are related by



$\mathcal{T}_g$  is simply connected so the cover  $\mathcal{T}'_g \rightarrow \mathcal{T}_g$  splits, hence so does  $\tilde{H}' \rightarrow \tilde{H}$ . A section of this last cover gives a map from  $\tilde{H}$  to  $H' - \Delta'$  (shown dashed in the diagram), so  $\Gamma_L$  is a quotient of  $\Gamma$ , of finite order.

Let  $\gamma'$  [resp.  $\gamma_e$ ] be a loop at a fixed base point  $P_0 \in H - \Delta$  going around  $\Delta'$  [resp.:  $\Delta_e$ ] but homotopic to 0 in  $H$ . Fix a point  $\tilde{P}_0 \in \tilde{H}$  over  $P_0$ . The monodromy characterization of the Dehn twists implies that  $\gamma'$  [resp.:  $\gamma_e$ ] lifted to  $\tilde{H}$  goes from  $\tilde{P}_0$  to  $h'(\tilde{P}_0)$  [resp.: to  $h_e(\tilde{P}_0)$ ]. Since  $\gamma'$  [resp.:  $\gamma_e$ ] are homotopic to 0 in  $H$ , and the covering  $H' - \Delta'$  extends over  $H$ , this implies that the image of  $h'$  [resp.:  $h_e$ ] in  $\Gamma_L$  is 0. But these elements and their conjugates generate  $\Gamma_L$ , so  $\Gamma_L = \{1\}$ , hence  $L \cong \mathcal{O}_H$ , proving the lemma and the theorem.

In order to describe the ample cone on  $\text{Pic}(\overline{\mathcal{M}}_g)$  we prove:

THEOREM 5.15.  $\text{Ch}^*(\mathcal{O}_{\text{Div}}(v)) = (\mu^e \otimes \lambda^{-4})^{e(g-1)}$

*Proof.* The proof depends on a result which we simply quote from Fogarty [8] or Knudsen [12]:

PROPOSITION 5.16. *Let  $S$  be a locally closed subscheme of a Hilbert scheme  $\text{Hilb}_{\mathbf{P}^v-1}^P$ ,  $\text{Ch}$  be the associated Chow map  $\text{Ch}: S \rightarrow \text{Div}$  and  $Z \subset \mathbf{P}^v \times S$  have relative dimension  $r$  over  $S$ . Then if  $n \geq 0$ ,  $\Lambda^{\max} p_{2,*}(\mathcal{O}_Z(n)) = \bigotimes_{i=0}^{r+1} \mu_i^{\binom{n}{i}}$  and  $\text{Ch}^*(\mathcal{O}_{\text{Div}}(1)) = \mu_{r+1}$ , where  $\mu_i$  are suitable invertible sheaves on  $S$ .*

In the situation of our theorem, with  $S = H$  and  $Z = C$ ,  $\mathcal{O}_C(1) = \omega_{C/H}^{\otimes e} \otimes \pi^*Q$  where  $Q$  is the invertible sheaf determined by  $(\pi_*\omega_{C/H}^{\otimes e}) \otimes Q = \pi_*\mathcal{O}_C(1) = \pi_*\mathcal{O}_{\mathbf{P}^v-1}(1) = \mathcal{O}_H^v$ , hence

$$(5.17) \quad \mathcal{O}_H = [\Lambda^{\max} \pi_*(\omega_{C/H}^{\otimes e})] \otimes Q^v = \mu^{\binom{e}{2}} \otimes \lambda \otimes Q^v.$$

On the other hand,

$$\Lambda^{\max}(\pi_*\mathcal{O}_C(n)) = \Lambda^{\max}[\pi_*(\omega_{C/H}^{\otimes ne} \otimes Q^n)] = \mu^{\binom{ne}{2}} \otimes \lambda \otimes Q^{P(n).n}.$$

This has leading term in  $n$  of  $\mu^{n^2e^2/2} \otimes Q^{2e(g-1)n^2}$  so

$$\begin{aligned} \text{Ch}^*(\mathcal{O}_{\text{Div}}(v)) &= \mu^{ve^2} \otimes Q^{4e(g-1)v} \\ &= \mu^{ve^2 - \binom{e}{2}.4e(g-1)} \otimes \lambda^{-4e(g-1)} \quad \text{using (5.17)}. \end{aligned}$$

Finally, therefore,  $\text{Ch}^*(\mathcal{O}_{\text{Div}}(v)) = \mu^{e^2(g-1)} \otimes \lambda^{-4e(g-1)}$  as required.

COROLLARY 5.18. *If  $e \geq 5$ ,  $\mu^e \otimes \lambda^{-4} (= \lambda^{12e-4} \otimes \delta^{-e})$  is “ample on  $\overline{\mathcal{M}}_g$ ”, i.e. those positive powers of this bundle which are pull-backs of bundles on  $\overline{\mathcal{M}}_g$  are ample on  $\mathcal{M}_g$ .*

*Proof.* This is an immediate consequence of the Theorem and our main result: that  $PGL(v)$ -invariant sections of  $\text{Ch}^*(\mathcal{O}_{\text{Div}}(1))$  define a projective embedding of  $\overline{\mathcal{M}}_g$ .

REMARK 5.19. A similar argument using the facts that

- (1)  $\omega^{\otimes e}$  is base point free for all canonical curves when  $e \geq 2$ ,
- (2) smooth curves are stable if  $d > 2g$ ,

shows that if  $e \geq 2$ , the sections of  $\lambda^{12e-4} \otimes \delta^{-e}$  on  $\overline{\mathcal{M}}_g$  separate points on  $\mathcal{M}_g$ .

To get a good picture of the ample cone on  $\overline{\mathcal{M}}_g$  we need to use the realization via  $\Theta$  functions  $\mathcal{A}_{g,1} \xrightarrow{\Theta} \mathbf{P}^N$  of the moduli scheme  $\mathcal{A}_{g,1}$  of

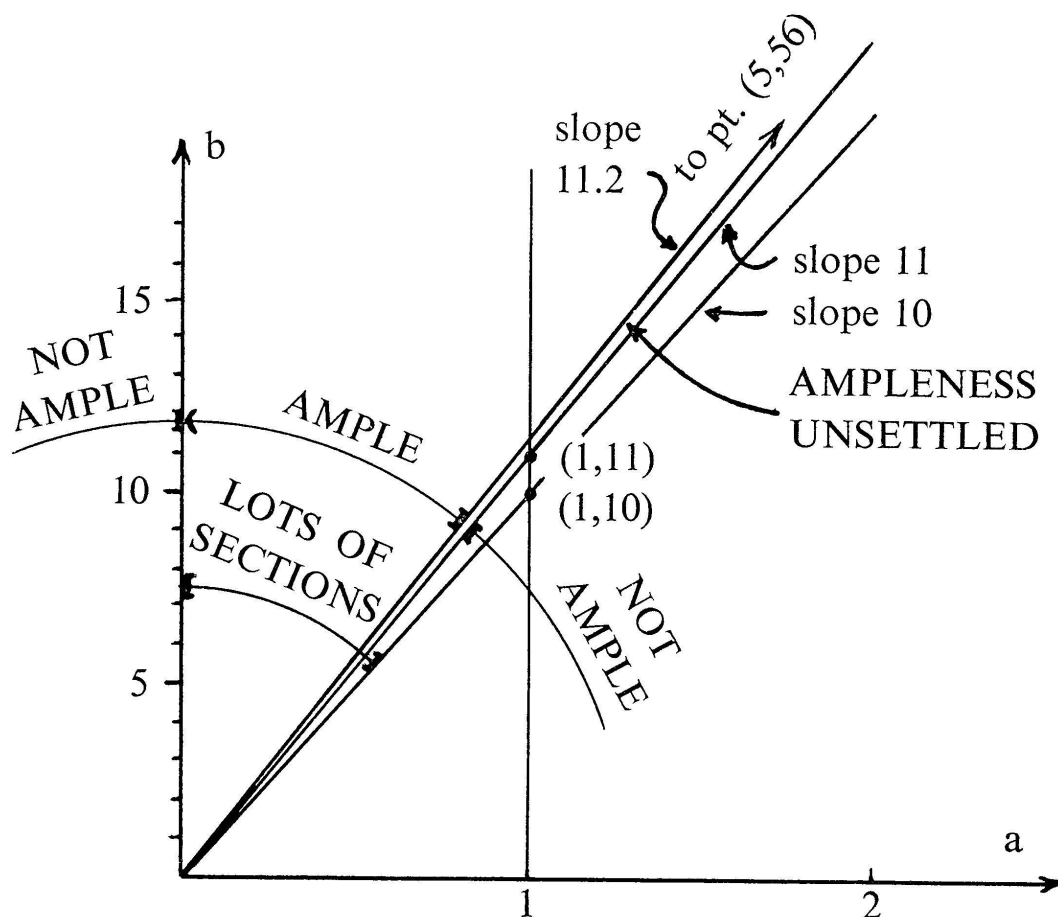
principally polarized abelian varieties. More precisely, let  $J : \mathcal{M}_g \rightarrow \mathcal{A}_{g,1}$  be the map taking a curve  $C$  to its Jacobian. Then we have:

**THEOREM 5.20.** *In characteristic 0, the morphism  $\mathcal{M}_g \xrightarrow{J} \mathcal{A}_{g,1} \xrightarrow{\theta} \mathbf{P}^N$  extends to a morphism  $\overline{\mathcal{M}}_g \xrightarrow{\theta} \mathbf{P}^N$  so that for some  $m$ ,  $\theta^*(\mathcal{O}_{\mathbf{P}^N}(1)) = \lambda^m$ .*

*Proof.* See Arakelov [1] or Knudsen [12].

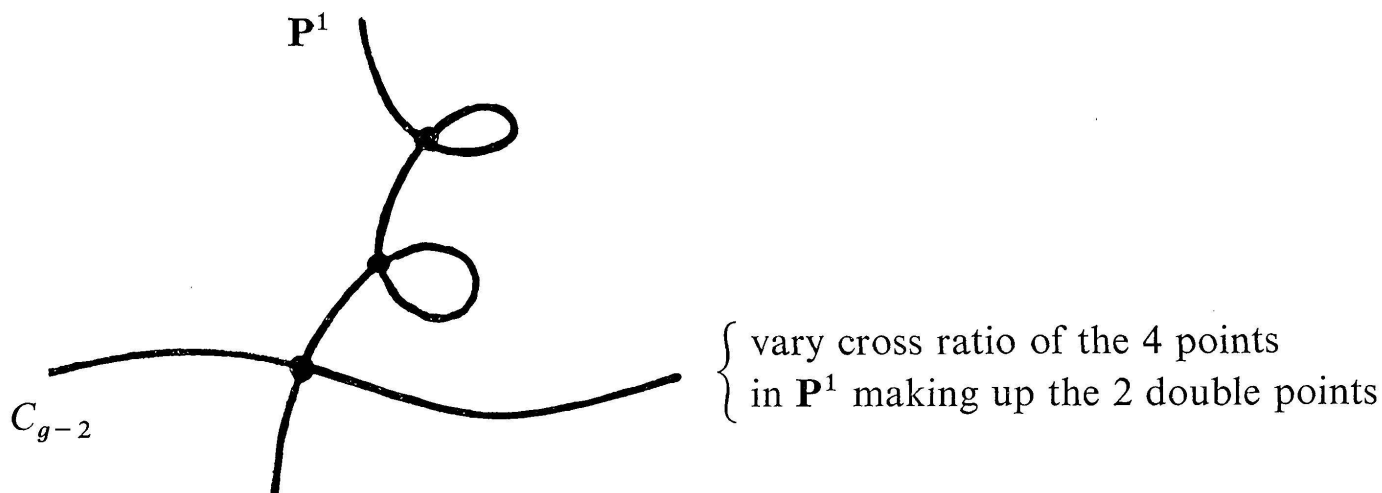
**REMARK.** This should also hold in characteristic  $p$ , but it seems to be a rather messy problem there.

Putting together 5.18 and 5.20, we get a whole sector in the  $(a, b)$ -plane such that  $\lambda^b \otimes \delta^{-a}$  is ample for  $(a, b)$  in this sector. This is depicted in the diagram below:



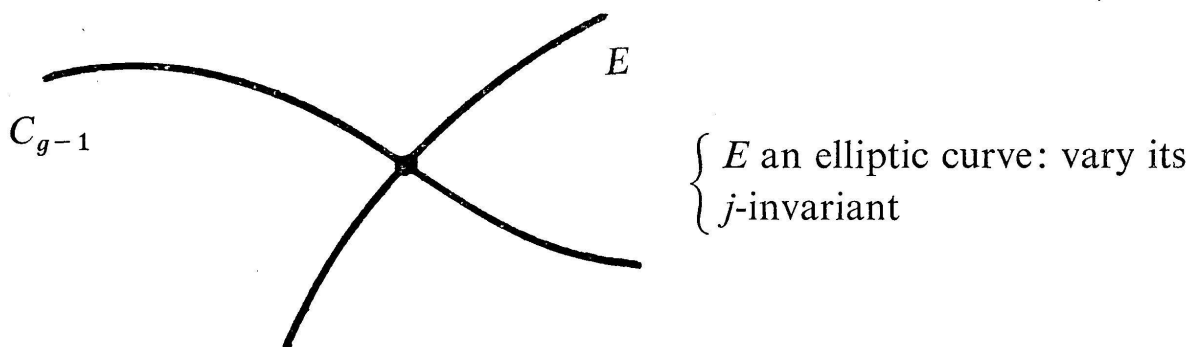
The fact that  $\lambda$  and  $\lambda^{11} \otimes \delta^{-1}$  are not ample can be seen by examining the following 2 curves in  $\overline{\mathcal{M}}_g$ :

- (1) If  $S_1$  is a curve in  $\overline{\mathcal{M}}_g$  composed of curves of the form:



where  $C_{g-2}$  is a fixed genus  $(g-2)$  component, then  $\lambda|_{S_1} = \mathcal{O}_{S_1}$ , hence sections of  $\lambda$  always collapse such families.

(2) If  $S_2$  is a curve in  $\overline{\mathcal{M}}_g$  composed of curves of the form:



where  $C_{g-1}$  is a fixed genus  $(g-1)$  component, then  $\lambda^{11} \otimes \delta^{-1}|_{S_2} = \mathcal{O}_{S_2}$  i.e.  $\lambda^{11} \otimes \delta^{-1}$  collapses these families.

We omit the details.

## APPENDIX

We wish to fill in the gap in the proof of Proposition 5.5 on page 95. The difficulty occurs if the support of  $\mathcal{I}$ , i.e.  $(0) \times L_1$ , contains some of the components of  $C_2$  meeting  $C_1$ . In this case, the inequality

$$e_L(\mathcal{I}_2) \geq w$$

is not clear. Indeed, if  $D_1, \dots, D_k$  are the components of  $C_2$  meeting  $C_1$ ,  $w_i = \#(D_i \cap C_1)$ , and  $\mathcal{K}_i$  is the pull-back of  $\mathcal{I}_2$  to  $D_i$ , then