$$C_\eta \xleftarrow{\;\approx\;} D_\eta \xrightarrow{\;\Phi_\eta\;} \mathbf{P}^N \times \operatorname{Spec} k((t))$$

$$\cap \qquad\qquad \cap \qquad\qquad\qquad \cap$$

$$\mathscr{C} \xleftarrow{\qquad\quad} \mathscr{D} \xrightarrow[\;\Phi\;]{\qquad\quad} \mathbf{P}^N \times \operatorname{Spec} k[[t]]$$

$$\Phi_\eta^*\!\left(\mathcal{O}_{\mathbf{P}^N}(1)\right) = \omega_{D_\eta \,/\, k((t))}^{\otimes n} \,.$$

Let $L = \mathcal{O}_{\mathscr{D}}(1)$. It follows that $L \cong \omega_{\mathscr{D}/k[[t]]}^{\otimes n}\left(-\sum r_i D_i\right)$, where $D_i$ are the components of $D_0$. Multiplying the isomorphism by $t^{\min\,(r_i)}$, we can assume $r_i \geqq 0$, $\min r_i = 0$. Let $D_1 = \bigcup_{r_i = 0} D_i$, $D_2 = \bigcup_{r_i > 0} D_i$. If $f$ is a local equation of $\sum r_i D_i$, then $f \not\equiv 0$ in any component of $D_1$ since $r_i = 0$ on all these while $f(x) = 0$, all $x \in D_1 \cap D_2$, so

$$\#\,(D_1 \cap D_2) \leqq \deg_{D_1}\!\left(\mathcal{O}_{\mathscr{D}_0}\!\left(\textstyle\sum r_i D_i\right)\right)\,.$$

But this last degree equals $\left(\deg D_1 - n \deg_{D_1}(\omega_{D_0})\right)$ which contradicts iii) of Proposition 5.5 unless all $r_i$ are zero. Hence $L = \omega_C^{\otimes n}$ which shows $\mathscr{D} = \mathscr{C}$.

## LINE BUNDLES ON THE MODULI SPACE

For the remainder of this section we examine $\operatorname{Pic}(\overline{\mathcal{M}}_g)$. We fix a genus $g \geqq 2$ and an $e \geqq 3$. Then for all stable $C$, $\omega_C^{\otimes e}$ is very ample and in this embedding $C$ has degree $d = 2e(g-1)$, the ambient space has dimension $v - 1$ where $v = (2e-1)(g-1)$ and $C$ has Hilbert polynomial $P(X) = dX - (g-1)$. Let $H \subset \operatorname{Hilb}_{\mathbf{P}^{v-1}}^P$ be the locally closed smooth subscheme of $e$-canonical stable curves $C$, let $C \subset H \times \mathbf{P}^{v-1}$ be the universal curve and let

$$\text{ch} : H \to \operatorname{Div} = \operatorname{Div}^{d,d} = \left\{ \begin{array}{l} \text{projective space of bihomogeneous forms} \\ \text{of bidegree } (d,d) \text{ in dual coordinates} \\ u,\, v \ (\text{cf. } \S\,1). \end{array} \right\}$$

be the Chow map. These are related by the diagram

$$
\begin{array}{c}
C \\
\big\downarrow{\scriptstyle \pi} \\
\operatorname{Div} \xleftarrow{\;\;\text{ch}\;\;} H \xrightarrow{\;\;\rho\;\;} \overline{\mathcal{M}}_g = H/PGL(v)
\end{array}
$$

If $\operatorname{Pic}(H, PGL(v))$ is the Picard group of invertible sheaves on $H$ with $PGL(v)$-action, we have a diagram

$$\text{Pic}\,(\bar{\mathcal{M}}_g) \xrightarrow{\ \ \rho^*\ \ } \text{Pic}\,(H, PGL(v)) \xrightarrow{\ \ \alpha\ \ } \text{Pic}\,(H)^{PGL(v)} \subset \text{Pic}\,(H)\,.$$

In this situation, we have:

LEMMA 5.8. *In the sequence above, $\rho^*$ is injective with torsion cokernel and $\alpha$ is an isomorphism.*

*Proof.* $\alpha$ is an isomorphism by Prop. 1.4 [14]; $\rho^*$ injective is easy; coker $\rho^*$ torsion can be proved, for instance, using Seshadri's construction, Th. 6.1 [19].

This lemma allows us to examine $\text{Pic}\,(\bar{\mathcal{M}}_g)$ by looking inside $\text{Pic}\,(H)^{PGL(v)}$ which is a much easier group to come to grips with.

DEFINITION 5.9. *Let $\Delta \subset H$ be the divisor of singular curves, $\delta = \mathcal{O}_H(\Delta)$ and $\lambda_n = \Lambda^{\max}\left(\pi_* \left(\omega_{C/H}^{\otimes n}\right)\right), (n \geq 1)$. We write $\lambda$ for $\lambda_1$.*

The sheaves $\lambda_n$ and $\delta$ are the most obviously interesting invertible sheaves on $H$ from a moduli point of view. The next theorem expresses all of these in terms just involving $\lambda$ and $\delta$.

THEOREM 5.10. $\lambda_n = \mu^{\binom{n}{2}} \otimes \lambda$ *where* $\mu = \lambda^{12} \otimes \delta^{-1}$.

*Proof.* The proof is based on Grothendieck's relative Riemann-Roch theorem (see Borel-Serre [4]), which we will briefly recall.

Let $X$ and $Y$ be complete smooth varieties over $k$, $A(X)$ be the Chow ring of $X$ and $\mathcal{F}$ be a coherent sheaf on $X$. Let $c_i(\mathcal{F}) \in A(X)$ denote the $i^{\text{th}}$ Chern class of $\mathcal{F}$, Chern $(\mathcal{F}) \in A(X) \otimes \mathbf{Q}$ its Chern character and $\mathcal{T}(\mathcal{F}) \in A(X) \otimes \mathbf{Q}$ its Todd genus. These are related by:

$$(5.11) \quad \text{Chern}\,(\mathcal{F}) = rk\,\mathcal{F} + c_1(\mathcal{F}) + \frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F})$$

$$+ \text{ terms of higher codimension,}$$

$$\mathcal{T}(\mathcal{F}) = 1 - \frac{c_1(\mathcal{F})}{2} + \frac{c_1(\mathcal{F})^2 + c_2(\mathcal{F})}{12}$$

$$+ \text{ terms of higher codimension.}$$

Let $K(Y)$ be the Grothendieck group of $Y$, $f : X \to Y$ be a proper map, and $f_!(\mathcal{F}) = \sum (-1)^i [\mathbf{R}^i f_* \mathcal{F}] \in K(Y)$. The relative Riemann-Roch theorem expresses the Chern character of $f_!(\mathcal{F})$, *modulo torsion* as

$$\text{Chern}\,(f_! \mathcal{F}) = f_*\left(\text{Chern}\,\mathcal{F} \cdot \mathcal{T}\,(\Omega^1_{X/Y})\right)$$

which using (5.11) gives:

(5.12) $\quad rk\, f_!\, \mathscr{F} + c_1(f_!\, \mathscr{F}) + \ldots\ldots\ldots$

$$= f_*\left[\left(rk(\mathscr{F}) + c_1(\mathscr{F}) + \frac{c_1(\mathscr{F})^2}{2} - c_2(\mathscr{F})\right) \cdot \right.$$

$$\left. \left(1 - \frac{c_1(\Omega^1_{X/Y})}{2} + \frac{c_1(\Omega^1_{X/Y})^2 + c_2(\Omega^1_{X/Y})}{12}\right)\right]$$

For the time being, we work implicitly modulo torsion.

Now suppose $\mathscr{F}$ is a line bundle such that $R^i f_*(\mathscr{F}) = 0$, $i > 0$ and suppose dim $X$ = dim $Y + 1$. Then the codimension 1 term on the left of (5.12) (i.e. on $Y$) corresponds to the codimension two term on the right (i.e. on $X$). Since $c_2(\mathscr{F}) = 0$, this gives

(5.13) $\quad c_1(f_*\, \mathscr{F}) = c_1(f_!\, \mathscr{F})$

$$= f_*\left[\frac{c_1(\Omega^1_{X/Y})^2 + c_2(\Omega^1_{X/Y})}{12} - \frac{c_1(\mathscr{F})c_1(\Omega^1_{X/Y})}{2} + \frac{c_1(\mathscr{F})^2}{2}\right]$$

In case $f: C \to S$ is a moduli-stable curve over $S$, $X = C$ and $Y = S$, we can simplify this. Indeed I claim that if Sing $C$ is the singular set on $C$ and $I_{\mathrm{sing}}$ is its ideal, then

i) codim Sing $C = 2$

ii) the canonical homomorphism $\Omega^1_{C/S} \to \omega_{C/S}$ induces an isomorphism $\Omega^1_{C/S} = I_{\mathrm{sing}} \cdot \omega_{C/S}$.

We certainly have the isomorphism of ii) off Sing $C$. At a singular point $C$ has a local equation of the form $xy = t^n$, where $t$ is a parameter on $S$, $x$ and $y$ are affine coordinates on the fibre. Moreover locally $C$ is singular only at the points $(0, 0)$ in the fibres where $t = 0$, so Sing $C$ has codimension 2. Near the singular point

$$\Omega^1_{C/S} = (\mathscr{O}_C dx + \mathscr{O}_C dy)/(xdy + ydx)\,\mathscr{O}_C$$

while $\omega_{C/S}$ is the invertible sheaf generated by the differential $\zeta$ which is given by $dx/x$ outside $x = 0$ and by $-dy/y$ outside $y = 0$. Thus

$$\Omega^1_{C/S} = \mathscr{M}_{(0,0),C} \cdot \zeta = \mathscr{M}_{(0,0),C} \cdot \omega_{C/S} \, .$$

Recall the following corollary to Riemann-Roch: if $X$ is a smooth variety, $Y \subset X$ a subvariety of codim $r$ and $\mathscr{F}$ is coherent on $Y$, then considering $\mathscr{F}$ as a sheaf on $X$

$$c_i(\mathscr{F}) = \begin{cases} 0, 1 \le i \le r-1 \\ ((-1)^{r-1}(r-1)! \, rk \, \mathscr{F}) \, Y, i = r \end{cases}$$

Set $X = C$, $Y = \mathrm{Sing}\, C$ and $\mathscr{F} = \Omega^1_{C/S}$. The Whitney product formula applied to the chern classes of the exact sequence

$$0 \to \Omega^1_{C/S} \to \omega_{C/S} \to \omega_{C/S} \otimes \mathcal{O}_{\mathrm{Sing}\, C} \to 0$$

gives, taking account of the corollary

$1 + c_1(\omega_{C/S})$
$= (1 + c_1(\Omega^1_{C/S}) + c_2(\Omega^1_{C/S}) + \ldots) \cdot (1 + 0 - [\mathrm{Sing}\, C] + \ldots)$

Equating terms of equal codimension, we see that $c_1(\Omega^1_{C/S}) = c_1(\omega)$ and $c_2(\Omega^1_{C/S}) = [\mathrm{Sing}\, C]$ so that (5.13) becomes

$$c_1(f_* \mathscr{F}) = f_* \left[ \frac{c_1(\omega_{C/S})^2 + [\mathrm{Sing}\, C]}{12} - \frac{c_1(\mathscr{F}) c_1(\omega_{C/S})}{2} + \frac{c_1(\mathscr{F})^2}{2} \right]$$

Applying this to the map $\pi : C \to H$, when $\mathscr{F} = \omega^{\otimes n}_{C/H}$ gives

$$\lambda_n = \Lambda^{\max}(\pi_* \omega^{\otimes n}_{C/H}) = c_1(\pi_* \omega^{\otimes n}_{C/H})$$

$$= \pi_* \left[ \frac{c_1(\omega_{C/H})^2 + [\mathrm{Sing}\, C]}{12} - \frac{c_1(\omega^{\otimes n}_{C/H}) c_1(\omega_{C/H})}{2} + \frac{c_1(\omega^{\otimes n}_{C/H})^2}{2} \right]$$

$$= \binom{n}{2} \pi_* (c_1(\omega_{C/H})^2) + \frac{\pi_* (c_1(\omega_{C/H})^2) + [\Delta]}{12}$$

Setting[1] $n = 1$, we see that $\lambda = \left[ \dfrac{\pi_* (c_1(\omega_{C/H})^2) + [\Delta]}{12} \right]$ and $\pi_* (c_1(\omega_{C/H})^2)$

$= 12\,\lambda - [\Delta]$. Plugging these values back in gives us the theorem up to torsion. But in fact:

LEMMA 5.14. *Over* **C**, *Pic*$(H, PGL(v))$ *is torsion free.*

Note that this will prove what we want because the invertible sheaves that we are trying to show are isomorphic all "live" on the full scheme $H_Z$ over Spec **Z** of stable $\mathscr{C}$-canonical curves. If they are isomorphic on $H_Z$, they are isomorphic after any base change. But on the other hand, I claim that Pic $(H, PGL(v))$ injects into Pic $(H_C, PGL_C(v))$:

---

[1] For $n = 1$, $R^1 \pi_*(\omega_{C/H})$ is not zero, but it is the trivial line bundle, hence doesn't affect $\pi_!$.

If $L$ is a line bundle on $H$ with $PGL(v)$ action such that $L \otimes \mathbf{C}$ is trivial over $H_\mathbf{C}$, then

$$H^0(H, L)^{PGL(v)} \otimes \mathbf{C} = H^0(H_\mathbf{C}, L \otimes \mathbf{C})^{PGL(v)}$$

$$\wr\wr \Big\downarrow \alpha$$

$$H^0(H_\mathbf{C}, \mathcal{O}_{H\mathbf{C}})^{PGL(v)} = \mathbf{C}$$

since $H_\mathbf{C}/_{PGL(v)}$ is compact. Thus we can find a non-zero section $s \in H^0(H, L)^{PGL(v)}$, which over $\mathbf{C}$ can be used to give the trivialization $\alpha$. Over $\mathbf{C}$, $s$ has no zeros so the divisor $(s)_0$ of the zeros of $s$ on $H$, has support only over the closed fibres of Spec $(\mathbf{Z})$. Mumford and Deligne [6] have shown that $H \to \mathrm{Spec}\ \mathbf{Z}$ is smooth with irreducible fibres, hence $(s)_0 = \sum r_i \pi^{-1}(p)$, $r_i \geqq 0$ i.e. $(s)_0 = (n)$ for some integer $n$. Then $\left(\dfrac{s}{n}\right)$ is a global section of $L$ with no zeros so $L$ is trivial.

*Proof of Lemma.* Over $\mathbf{C}$, we have Teichmüller theory at our disposal. Let $\Pi$ be a standard model of a group with generators $\{a_i, b_i \mid 1 \leqq i \leqq g\}$ mod the relation $\prod\limits_{i=1}^{g} (a_i b_i a_i^{-1} b_i^{-1}) = 1$. Then the Teichmüller modular group $\Gamma$ is

$$\Gamma = \{\alpha \mid \alpha : \Pi \to \Pi \text{ is an orientation preserving}\}/\text{inner}$$
$$\text{isomorphism} \quad \text{automorphisms}$$

The Teichmüller space $\mathcal{T}_g$ is given by

$$\mathcal{T}_g = \left\{ (C, \alpha) \ \middle| \ \begin{array}{l} C \text{ a smooth curve of genus } g \text{ and } \alpha : \pi_1(C) \to \Pi \text{ an} \\ \text{orientation preserving isomorphism given up to inner} \\ \text{automorphism} \end{array} \right\}$$

Fix a model $M_g$ of the real surface of genus $g$, and identify $\pi_1(M_g)$ and $\Pi$. Then $\Gamma$ is generated by the maps which are induced by certain automorphisms of $M_g$, called Dehn twists. The Dehn twist $h_\gamma$ corresponding to a loop $\gamma : [0, 1] \to M_g$ on $M_g$ is given by taking an $\varepsilon$-collar $\gamma \times [-\varepsilon, \varepsilon]$ about $\gamma$, letting $h =$ identify off the collar and letting $h(\gamma(t), \eta - \varepsilon) = \left(\gamma\left(t + \dfrac{\eta}{2\varepsilon}\right), \eta - \varepsilon\right)$ as shown below.
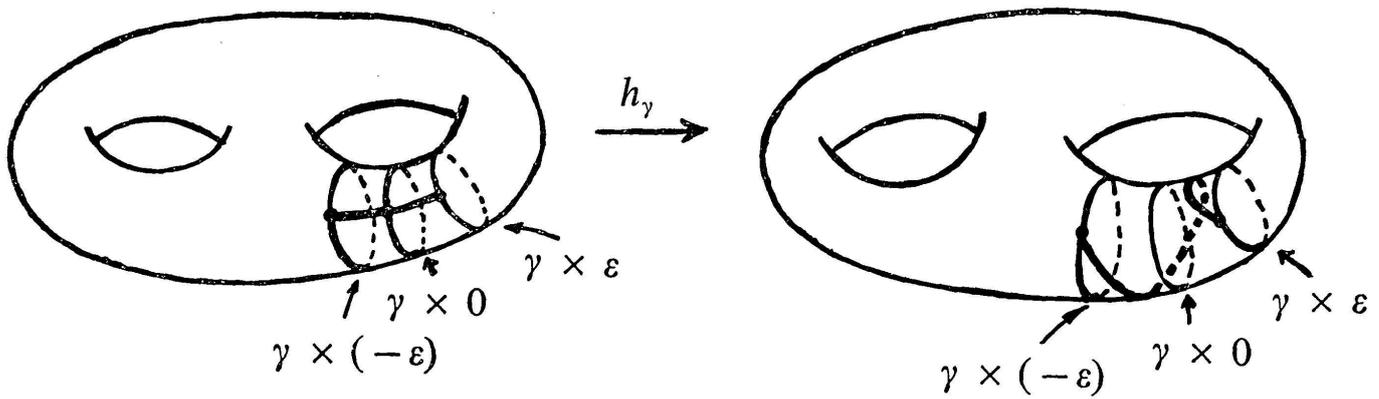
Up to inner automorphism $h_\gamma$ is determined by which of the pictures below results from cutting open $M_g$ along $\gamma$. We have name these elements of $\Gamma$ in the diagrams:
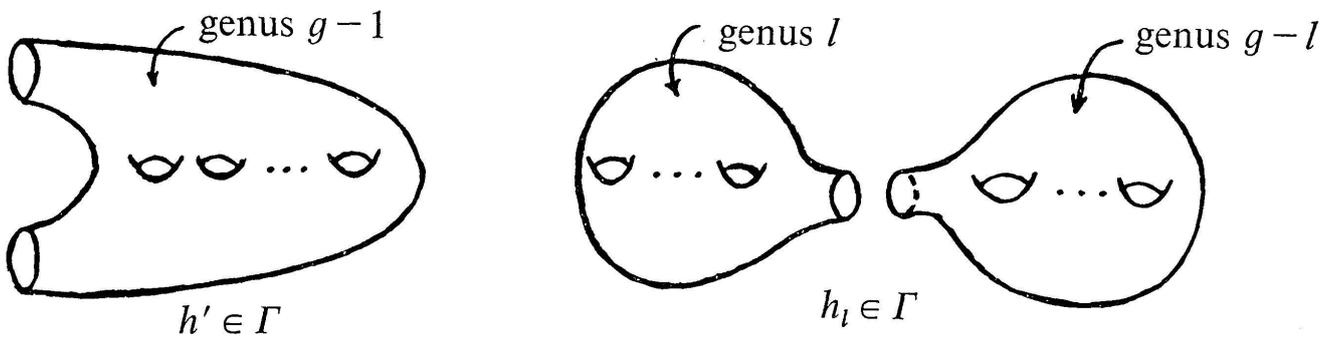


The Dehn twist $h_\gamma$ can also be described as the monodromy map obtained by going around a curve $C_0$ with one double point for which $\gamma$ is the vanishing cycle.

The components of $\Delta \subset H$ correspond to the different ways of putting a stable double point on a smooth moduli stable curve $C$. They are the closures of the sets of curves of the forms shown below: again, we name these components in the diagram:
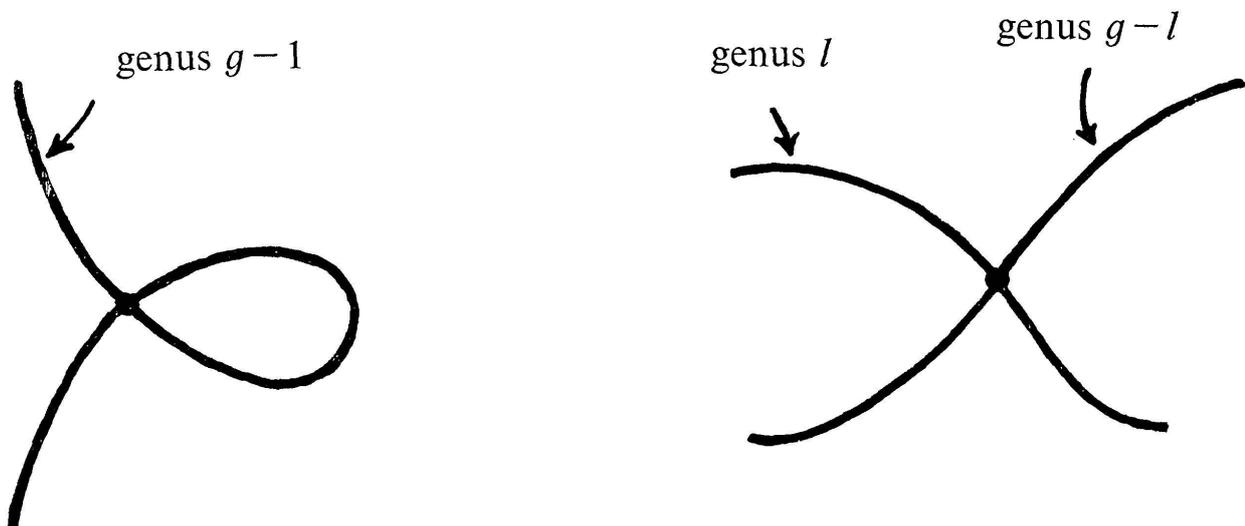
Let $\tilde{H} = \left\{ (C, \alpha, B) \,\middle|\, \begin{array}{l} (C, \alpha) \in \mathscr{T}_g, \ B \text{ a basis of the } e\text{-tuple dif-} \\ \text{ferentials on } C \text{ given up to a scalar} \end{array} \right\}$

Suppose we are given a line bundle $L$ on $H$ with $PGL\,(v)$-action such that $L^n \cong \mathcal{O}_H$. $L$ induces a cyclic covering $H'$ of $H$ plus a lifting of the $PGL\,(v)$-action to $H'$. If we choose $n$ minimal this covering is not split: we denote its structure group by $\Gamma_L$. Let $\tilde{H}'$ be the pullback of covering over $\tilde{H}$, and let $\mathscr{T}'_g$ denote the quotient of $\tilde{H}'$ by $PGL\,(v)$—this is a covering of $\mathscr{T}_g$. These coverings are related by
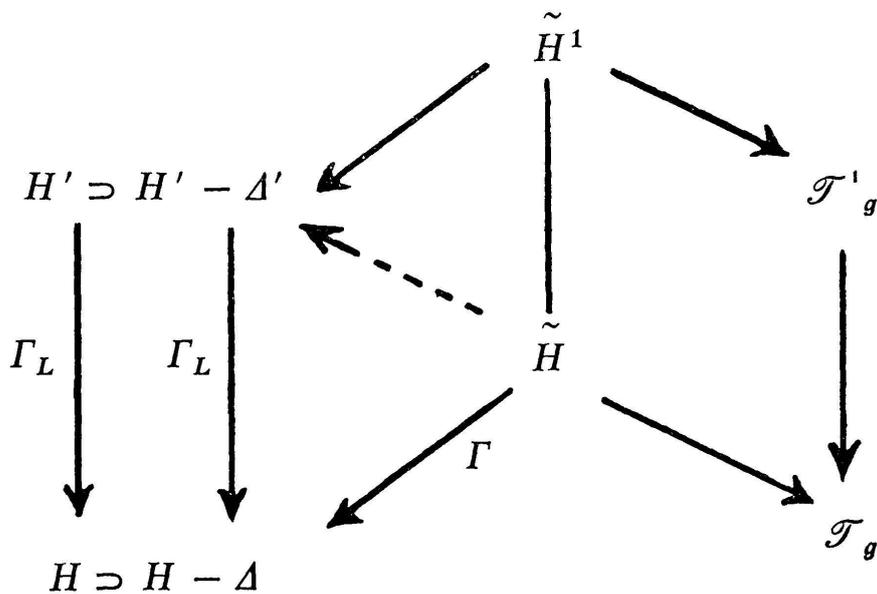


$\mathscr{T}_g$ is simply connected so the cover $\mathscr{T}'_g \to \mathscr{T}_g$ splits, hence so does $\tilde{H}' \to \tilde{H}$. A section of this last cover gives a map from $\tilde{H}$ to $H' - \Delta'$ (shown dashed in the diagram), so $\Gamma_L$ is a quotient of $\Gamma$, of finite order.

Let $\gamma'$ [resp. $\gamma_e$] be a loop at a fixed base point $P_0 \in H - \Delta$ going around $\Delta'$ [resp.: $\Delta_e$] but homotopic to 0 in $\tilde{H}$. Fix a point $\tilde{P}_0 \in \tilde{H}$ over $P_0$. The monodromy characterization of the Dehn twists implies that $\gamma'$ [resp.: $\gamma_e$] lifted to $\tilde{H}$ goes from $\tilde{P}_0$ to $h'\,(\tilde{P}_0)$ [resp.: to $h_e\,(\tilde{P}_0)$]. Since $\gamma'$ [resp.: $\gamma_e$] are homotopic to 0 in $H$, and the covering $H' - \Delta'$ extends over $H$, this implies that the image of $h'$ [resp.: $h_l$] in $\Gamma_L$ is 0. But these elements and their conjugates generate $\Gamma_L$, so $\Gamma_L = \{\,1\,\}$, hence $L \cong \mathcal{O}_H$, proving the lemma and the theorem.

In order to describe the ample cone on $\mathrm{Pic}\,(\overline{\mathcal{M}}_g)$ we prove:

THEOREM 5.15. $\mathrm{Ch}^*\big(\mathcal{O}_{\mathrm{Div}}\,(v)\big) = (\mu^e \otimes \lambda^{-4})^{e(g-1)}$

*Proof.* The proof depends on a result which we simply quote from Fogarty [8] or Knudsen [12]:

PROPOSITION 5.16. *Let $S$ be a locally closed subscheme of a Hilbert scheme $\mathrm{Hilb}^P_{\mathbf{P}^\nu-1}$, Ch be the associated Chow map $\mathrm{Ch}\colon S \to \mathrm{Div}$ and $Z \subset \mathbf{P}^\nu \times S$ have relative dimension $r$ over $S$. Then if $n \gg 0$, $\Lambda^{\max} p_{2,*}\,(\mathcal{O}_Z(n))$*

$$= \overset{r+1}{\underset{i=0}{\otimes}}\; \mu_i^{\binom{n}{i}} \; and \; \mathrm{Ch}^*\big(\mathcal{O}_{\mathrm{Div}}\,(1)\big) = \mu_{r+1},\; where \; \mu_i \; are\; suitable\; invertible$$

*sheaves on $S$.*

In the situation of our theorem, with $S = H$ and $Z = C$, $\mathcal{O}_C(1) = \omega^{\otimes e}_{C/H} \otimes \pi^* Q$ where $Q$ is the invertible sheaf determined by $(\pi_* \omega^{\otimes e}_{C/H}) \otimes Q = \pi_* \mathcal{O}_C(1) = \pi_* \mathcal{O}_{\mathbf{P}^\nu-1}(1) = \mathcal{O}^\nu_H$, hence

$$(5.17) \qquad \mathcal{O}_H = \big[\Lambda^{\max} \pi_*\,(\omega^{\otimes e}_{C/H})\big] \otimes Q^\nu = \mu^{\binom{e}{2}} \otimes \lambda \otimes Q^\nu.$$

On the other hand,

$$\Lambda^{\max}\big(\pi_*\,\mathcal{O}_C(n)\big) = \Lambda^{\max}\big[\pi_*\,(\omega^{\otimes ne}_{C/H}) \otimes Q^n\big] = \mu^{\binom{ne}{2}} \otimes \lambda \otimes Q^{P(n).n}.$$

This has leading term in $n$ of $\mu^{n^2 e^2/2} \otimes Q^{2e(g-1)n^2}$ so

$$Ch^*\big(\mathcal{O}_{\mathrm{Div}}(\nu)\big) = \mu^{\nu e^2} \otimes Q^{4e(g-1)\nu}$$

$$= \mu^{\nu e^2 - \binom{e}{2}.4e(g-1)} \otimes \lambda^{-4e(g-1)} \qquad\qquad \text{using (5.17)}.$$

Finally, therefore, $Ch^*\big(\mathcal{O}_{\mathrm{Div}}(\nu)\big) = \mu^{e^2(g-1)} \otimes \lambda^{-4e(g-1)}$ as required.

COROLLARY 5.18. *If $e \geqq 5$, $\mu^e \otimes \lambda^{-4}\; (= \lambda^{12e-4} \otimes \delta^{-e})$ is "ample on $\overline{\mathcal{M}}_g$", i.e. those positive powers of this bundle which are pull-backs of bundles on $\mathcal{M}_g$ are ample on $\mathcal{M}_g$.*

*Proof.* This is an immediate consequence of the Theorem and our main result: that $PGL\,(\nu)$-invariant sections of $\mathrm{Ch}^*\,(\mathcal{O}_{\mathrm{Div}}\,(1))$ define a projective embedding of $\overline{\mathcal{M}}_g$.

REMARK 5.19. A similar argument using the facts that

(1) $\omega^{\otimes e}$ is base point free for all canonical curves when $e \geqq 2$,

(2) smooth curves are stable if $d > 2g$,

shows that if $e \geqq 2$, the sections of $\lambda^{12e-4} \otimes \delta^{-e}$ on $\overline{\mathcal{M}}_g$ separate points on $\mathcal{M}_g$.

To get a good picture of the ample cone on $\overline{\mathcal{M}}_g$ we need to use the realization via $\Theta$ functions $\mathscr{A}_{g,1} \overset{\Theta}{\longrightarrow} \mathbf{P}^N$ of the moduli scheme $\mathscr{A}_{g,1}$ of

principally polarized abelian varieties. More precisely, let $J : \mathscr{M}_g \to \mathscr{A}_{g,1}$ be the map taking a curve $C$ to its Jacobian. Then we have:

**THEOREM 5.20.** *In characteristic 0, the morphism* $\mathscr{M}_g \xrightarrow{\ J\ } \mathscr{A}_{g,1} \xrightarrow{\ \Theta\ } \mathbf{P}^N$ *extends to a morphism* $\overline{\mathscr{M}}_g \xrightarrow{\ \theta\ } \mathbf{P}^N$ *so that for some* $m$, $\theta^*\big(\mathcal{O}_{\mathbf{P}^N}(1)\big) = \lambda^m$.

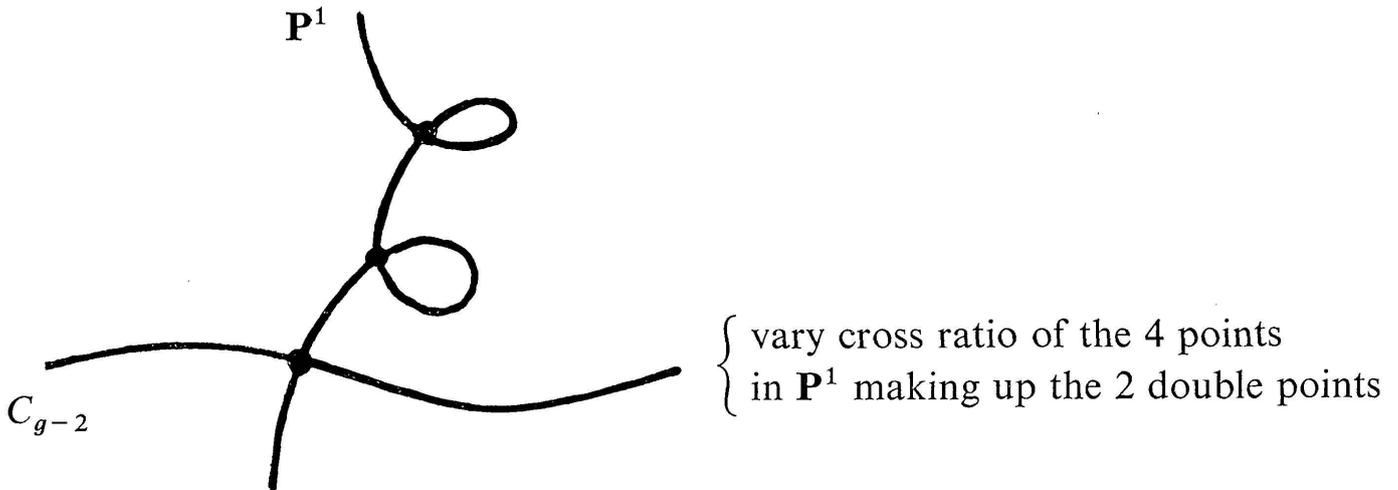*Proof.* See Arakelov [1] or Knudsen [12].

**REMARK.** This should also hold in characteristic $p$, but it seems to be a rather messy problem there.

Putting together 5.18 and 5.20, we get a whole sector in the $(a, b)$-plane such that $\lambda^b \otimes \delta^{-a}$ is ample for $(a, b)$ in this sector. This is depicted in the diagram below:
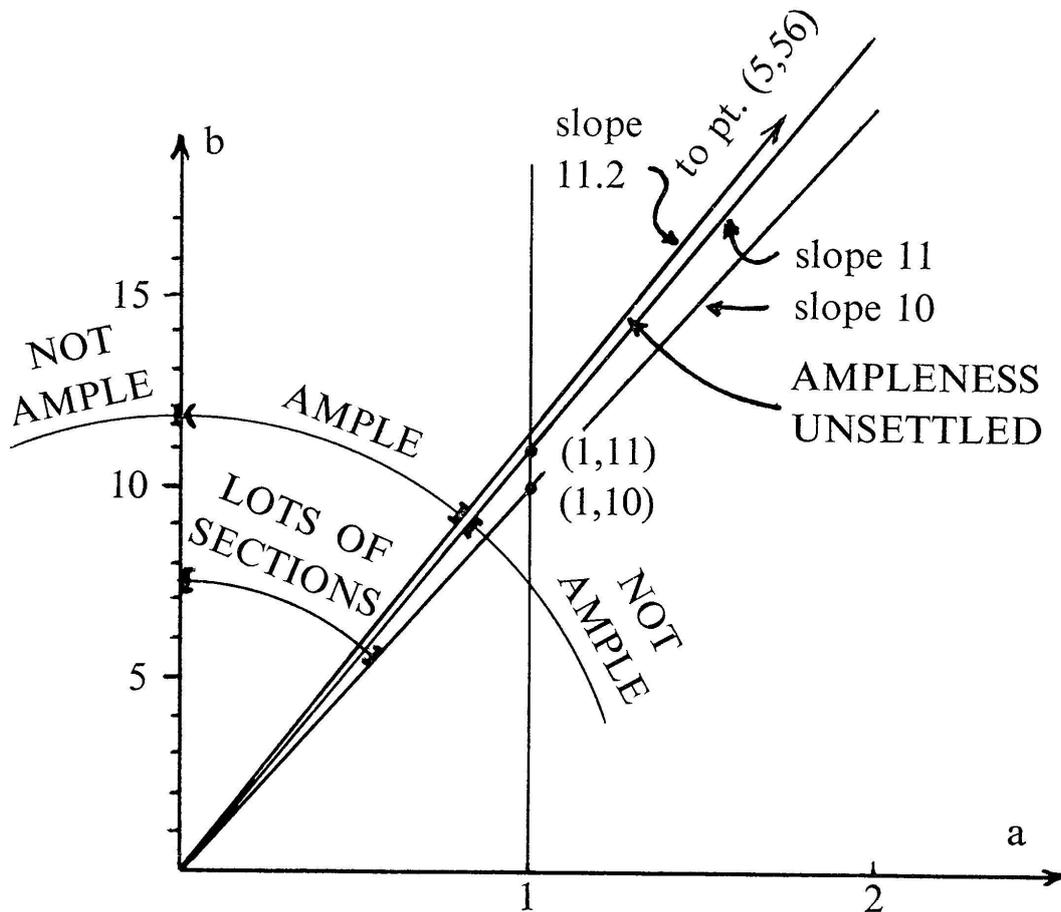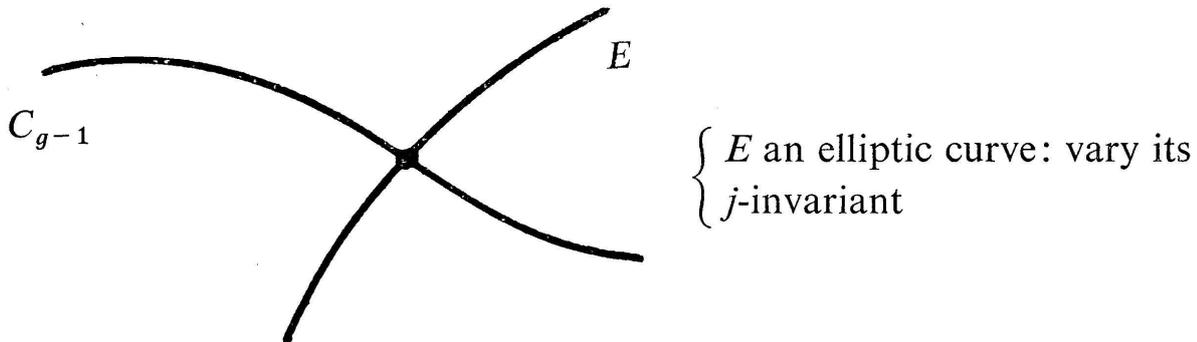


The fact that $\lambda$ and $\lambda^{11} \otimes \delta^{-1}$ are not ample can be seen by examining the following 2 curves in $\overline{\mathscr{M}}_g$:

(1)  If $S_1$ is a curve in $\overline{\mathscr{M}}_g$ composed of curves of the form:

$\mathbf{P}^1$

$C_{g-2}$

$\left\{\begin{array}{l}\text{vary cross ratio of the 4 points}\\\text{in } \mathbf{P}^1 \text{ making up the 2 double points}\end{array}\right.$

where $C_{g-2}$ is a fixed genus $(g-2)$ component, then $\lambda \mid_{S_1} = \mathcal{O}_{S_1}$, hence sections of $\lambda$ always collapse such families.

(2) If $S_2$ is a curve in $\overline{\mathscr{M}_g}$ composed of curves of the form:



$C_{g-1}$

$E$

$\left\{\begin{array}{l}E \text{ an elliptic curve: vary its}\\j\text{-invariant}\end{array}\right.$

where $C_{g-1}$ is a fixed genus $(g-1)$ component, then $\lambda^{11} \otimes \delta^{-1} \mid_{S_2} = \mathcal{O}_{S_2}$ i.e. $\lambda^{11} \otimes \delta^{-1}$ collapses these families.

We omit the details.

## APPENDIX

We wish to fill in the gap in the proof of Proposition 5.5 on page 95. The difficulty occurs if the support of $\mathscr{I}$, i.e. $(0) \times L_1$, contains some of the components of $C_2$ meeting $C_1$. In this case, the inequality

$$e_L(\mathscr{I}_2) \geqslant w$$

is not clear. Indeed, if $D_1, ..., D_k$ are the components of $C_2$ meeting $C_1$, $w_i = \#(D_i \cap C_1)$, and $\mathscr{K}_i$ is the pull-back of $\mathscr{I}_2$ to $D_i$, then