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§ 5. The Moduli Space of Stable Curves

Our main result is:

THEOREM 5.1. *Fix* $n \geq 5$, *and for any curve* C *of genus* g *let* $\Phi_n(C) \subset \mathbf{P}^{(2n-1)(g-1)-1}$ *be the image of* C *embedded by a basis of* $\Gamma(C, \omega_c^{\otimes n})$. *Then if* C *is moduli-stable,* $\Phi_n(C)$ *is Chow stable.*

In view of the basic results of § 1, and those of [20], this shows:

COROLLARY 5.2. *(F. Knudsen)* $\overline{\mathcal{M}}_g$ *is a projective variety.*

Recall that C moduli-stable means

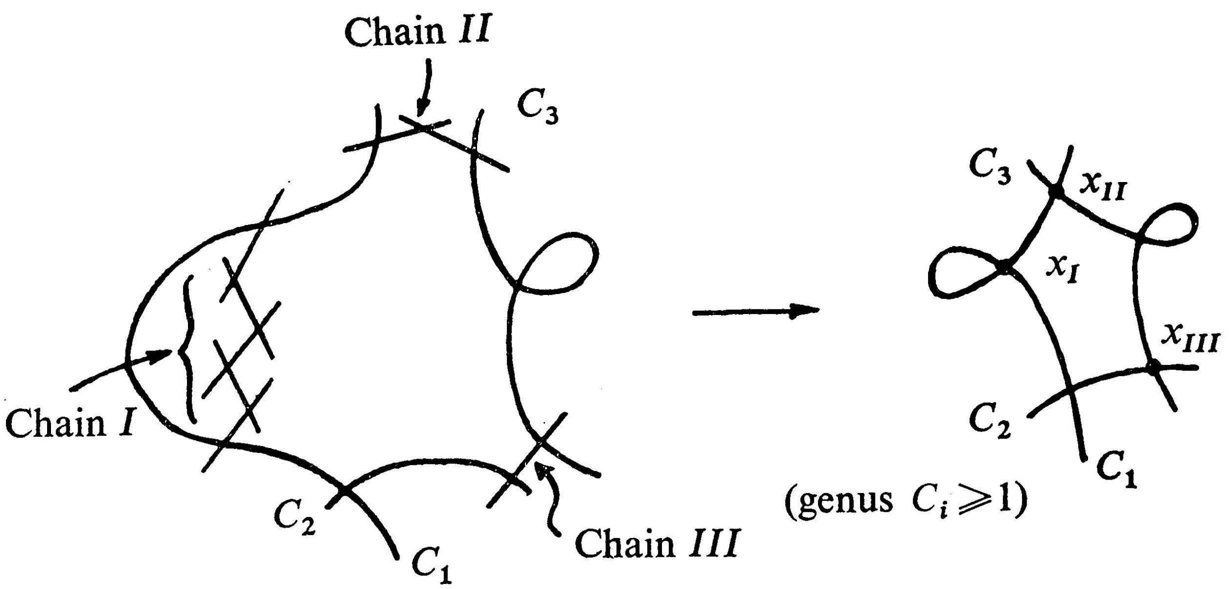
(1) C has at worst ordinary double points (by Proposition 3.12, this is necessary for the asymptotic semi-stability of C) and is connected,

(2) C has no smooth rational components meeting the rest of the curve in fewer than three points:
this condition is necessary to ensure that C has only finitely many automorphisms.

We will call C moduli semi-stable if it satisfies (1) and

(2′) C has no smooth rational components meeting the rest of the curve in only one point.

Note that if C is moduli semi-stable, then the set of its smooth rational components meeting the rest of the curve in exactly 2 points form a finite set of chains and if each of these is replaced by a point, we get a moduli stable curve:



We will case these the rational chains of C.

It would be more satisfactory to have a direct proof of Theorem 5.1 similar to the proof of the stability of smooth curves given in § 4. But curves with double points are not usually linearly stable (cf. the remark following Theorem 4.14) and, in fact, the estimates in Corollary 4.11 do not suffice to prove stability for such curves. We will therefore take an indirect approach.

Proof of 5.1. We begin by recalling the useful valuative criterion:

LEMMA 5.3. *Suppose a reductive group G acts on a k-vector space V. Let $K = k((t))$ and suppose $x \in V_K$ is G-stable. Then there is a finite extension $K' = k'((t')) \supset K$, and elements $g \in G_{K'}$, $\lambda \in (K')^*$ such that the point $\lambda g(x) \in V \otimes_k K'$ lies in $V \otimes_k k'[[t']]$ and specializes as $t \to 0$ to a point $\overline{\lambda g(x)}$ with closed orbit. Thus $\overline{\lambda g(x)}$ is either stable or semi-stable with a positive dimensional stabilizer.*

Proof. The diagram below is defined over k:

$$\begin{array}{l} \mathbf{P}(V) \supset \mathbf{P}(V)_{ss} \\ \qquad\qquad \big\downarrow \pi \\ \qquad\qquad X = \text{Proj (graded ring of invariants on } V) \end{array}$$

The point $\pi(x) \in X_K$ specializes to a point $\overline{\pi(x)} \in X_k$. Let $\bar{y}$ be a lifting of this point to V_{ss} with $O^G(\bar{y})$ closed. In the scheme $V \times \operatorname{Spec} k[[t]]$ form the closure Z of $\mathbf{G}_m \,.\, O^G(x)$. The lemma follows if we prove that $\bar{y} \in Z$. If $\bar{y} \notin Z$, then Z and $O^G(\bar{y})$ are closed disjoint G invariant subsets of $V \times \operatorname{Spec} k[[t]]$, hence there exists a homogeneous G-invariant f such that $f(x) = 0$ but $f(\bar{y}) \neq 0$. Then for some n, $f^{\otimes n}$ descends to a section of some line bundle on $X \times \operatorname{Spec} k[[t]]$. But then $f(\pi(x)) = 0$ and $\overline{f(\pi(x))} \neq 0$ are contradictory.

Now suppose that C is a moduli stable curve of genus g over k. Let $\mathscr{C}/k[[t]]$ be a family of curves with fibre C_0 over $t = 0$ equal to C and generic fibre C_η smooth. At the double points of C_0, $\mathscr{C}$ looks formally like $xy = t^n$, that is has only A_{n-1}-type singularities and hence is normal. Embed C_η in $\mathbf{P}^N$ $\big(N = (2n-1)(g-1)-1\big)$ by $\Gamma(C_\eta\, \omega_{C_\eta}{}^{\otimes n})$ and let $\Phi(C_\eta)$ denote its image there. Then Lemma 5.3 says that by replacing $k[[t]]$ with some finite extension and choosing a suitable basis of $\Gamma(C_\eta, \omega_{C_\eta}{}^{\otimes n})$—this corresponds to choosing g, λ—we may assume that the closure $\mathscr{D}$ in $\mathbf{P}^N \times \operatorname{Spec} k[[t]]$ of $\Phi(C_\eta)$ satisfies

i) $D_\eta = C_\eta$

ii) D_0 Chow-stable or Chow semi-stable with positive dimensional stabilizer.

I now claim:

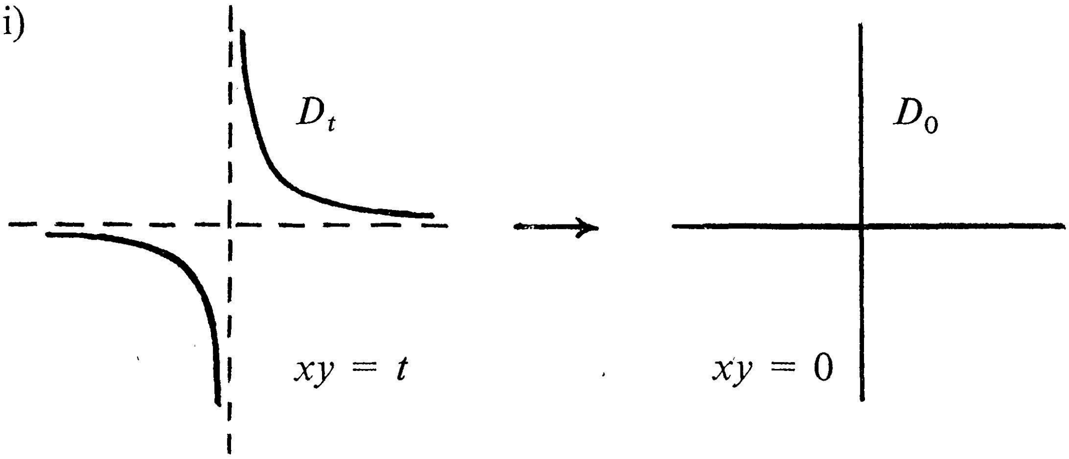
(5.4) $\mathscr{D} = \Phi(\mathscr{C})$, the image of $\mathscr{C}$ under a $k[[t]]$ basis of

$$\Gamma(\mathscr{C}, \omega_{\mathscr{C}}{}^n{}_{/k[[t]]})$$

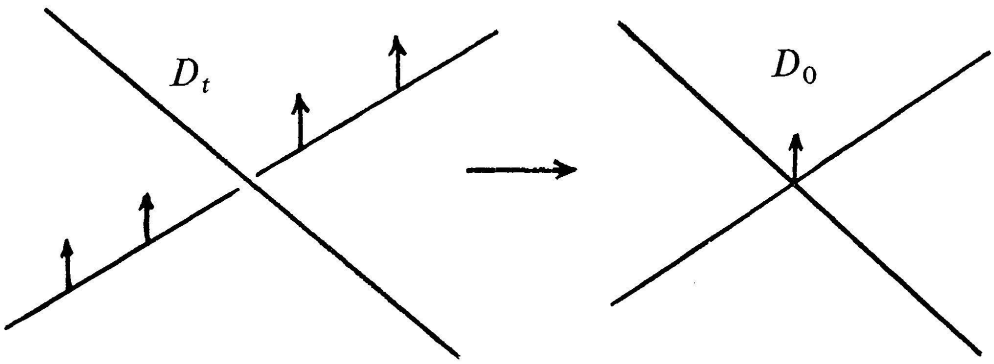
In particular this implies $D_0 = C_0 = C$ and since C has finite stabilizer this means D_0, hence C, is Chow stable.

The main step in the proof of (5.4) is to show that D_0 is moduli semi-stable as a scheme, and the key difficulty in doing this is to show that D_0 has only ordinary double points. At first glance, this seems rather obvious, since from Proposition 3.12 it follows easily that as a cycle D_0 has no multiplicities and has only ordinary double points. But ordinary double points on a limit cycle arise in two ways:

i)



ii)



ideal $= \langle x, z \rangle . \langle y, z-t \rangle$ ideal $= (xy, xz, yz, z^2)$

In the second case the scheme D_0 has an embedded component (the first order normal neighbourhood in the z-direction) at the double point so in the limit scheme the double point is not ordinary. If case (ii) occurred for D_0, then since D_0 is Chow semi-stable, it must span $\mathbf{P}^N$ set-theoretically. But $\Gamma(D_0, \mathcal{O}_{D_0}(1))$ has a torsion section supported at the double point: so D_0 would have to be embedded by a non-complete linear system $\sum \subset \Gamma(D_0, \mathcal{O}_{D_0}(1))$ of torsion-free sections, $\dim \sum = \dim H^0(D_\eta, \mathcal{O}_{D_\eta}(1))$. Consequently $H^1(D_0, \mathcal{O}_{D_0}(1)) \neq (0)$ too. That this cannot happen in the situation of (5.4) follows from:

PROPOSITION 5.5. *Let* $C \subset \mathbf{P}^n$ *be a* 1*-dimensional scheme such that*

a) $n + 1 = \deg C + \chi(\mathcal{O}_C), \chi(\mathcal{O}_C) < 0$,

b) C *is Chow semi-stable,*

c) $\dfrac{\deg C}{n+1} < \dfrac{8}{7}$.

Then i) C *is embedded by a complete non-special* [1]) *linear system,*

ii) C *is a moduli semi-stable curve with rational chains of length at most one consisting of straight lines.*

Moreover if $v = \dfrac{\deg C}{\deg \omega_C}$ *(where* ω_C *is the Grothendieck dualizing sheaf) and* $C = C_1 \cup C_2$ *is a decomposition of* C *into two sets of components such that* $\mathscr{W} = C_1 \cap C_2$ *and* $w = \# \mathscr{W}$ *then*

iii) $$|\deg C_1 - v \deg_{C_1}(\omega_C)| \leq \frac{w}{2}$$

REMARKS. 1) It is clear that D_0 satisfies the hypotheses of the lemma. Indeed a) is satisfied by D_η and is preserved under specialization. The key point of the Proposition to replace this by the stronger condition i)

2) Roughly, iii) says that the degrees of the components of C are roughly in proposition to their "natural" degrees. We will see later on that this is enough to force $\mathscr{D} = \mathscr{C}$.

Proof. From b), c) and Proposition 3.1 we know that the cycle of C has no multiplicity and only ordinary double points. Hence C_{red} is a scheme

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[1]) Non-special means $H^1(C, \mathcal{O}_C(1)) = (0)$.

having only ordinary double points and differing from C only by embedded components.

Suppose we are given a decomposition $C_{\text{red}} = C_1 \cup C_2$; let $\mathcal{W} = C_1 \cap C_2$, $w = \# \mathcal{W}$, L_i be the smallest linear subspace containing C_i and $n_i = \dim L_i$. We can assume $L_1 = V(X_{n_1+1} \dots X_n)$. For the 1-PS λ given by

$$t \mapsto \begin{array}{c|c} \left[\begin{array}{ccccc|ccccc} t & & & & & & & & & \\ & \cdot & & & & & & 0 & & \\ & & \cdot & & & & & & & \\ & & & \cdot & & & & & & \\ & & & & t & & & & & \\ \hline & & & & & 1 & & & & \\ & & & & & & \cdot & & & \\ & & 0 & & & & & \cdot & & \\ & & & & & & & & \cdot & \\ & & & & & & & & & 1 \end{array}\right] & \begin{array}{l} n_1 + 1 \\ \\ \\ \\ \\ \\ \\ n - n_1 \end{array} \end{array}, \qquad \sum \rho_i = n_1 + 1$$

the associated ideal $\mathcal{I}$ in $\mathcal{O}_{C_{\text{red}} \times \mathbf{A}^1}$ is given by $\mathcal{I} = (t, I(L_1))$. To evaluate $e(\mathcal{I})$ we use an easy lemma whose proof is left to the reader

LEMMA 5.6. *If $X' \xrightarrow{f} X$ is a proper morphism of r-dimensional, possibly reducible "varieties", birational on each component, L is a line bundle on X, and $\mathcal{I}$ is an ideal sheaf on X such that* $\operatorname{supp}(\mathcal{O}_X/\mathcal{I})$ *is proper, then* $e_{f^*(L)}(f^*(\mathcal{I})) = e_L(\mathcal{I})$.

Letting $\mathcal{I}_i$ be the pullback of $\mathcal{I}$ to C_i, the lemma says $e_L(\mathcal{I}) = e_{L_1}(\mathcal{I}_1) + e_{L_2}(\mathcal{I}_2)$. But $\mathcal{I}_1 = t \,.\, \mathcal{O}_{C_1 \times \mathbf{A}^1}$ and support $\mathcal{I}_2$ contains $(0) \times \mathcal{W}$ so this implies[1]) $e_L(\mathcal{I}) \geqq 2 \deg C_1 + w$. Using b) and Theorem 2.8 this gives

$$(5.7) \qquad w + 2 \deg C_1 \leqq \frac{\deg C}{n+1} \,.\, 2 \,.\, (n_1 + 1) \leqq \frac{16}{7} (n_1 + 1)$$

If C_1 as any component of C_{red}, then this implies:

a) $H^1(C_1, \mathcal{O}_{C_1}(1)) = 0$: if not, then by Clifford's theorem

$$h^0(C_1, \mathcal{O}_{C_1}(1)) \leqq \frac{\deg C_1}{2} + 1$$

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[1]) This argument has a gap: see Appendix, p. 108.

so by (5.7)

$$\deg C_1 \leqq \frac{8}{7} h^0(C_1, \mathcal{O}_{C_1}(1)) \leqq \frac{8}{14} \deg C_1 + \frac{8}{7},$$

which implies $\deg C_1 \leqq 2$, hence C_1 is rational and then $H^1(C_1, \mathcal{O}_{C_1}(1)) = (0)$ anyway.

b) $H^1(C_1, \mathcal{O}_{C_1}(1)(-\mathcal{W})) = (0)$: indeed from (5.7) and Riemann-Roch,

$$\deg C_1 + \frac{1}{2} w \leqq \frac{8}{7} (\deg C_1 - g_1 + 1), \text{ whence}$$

$$\deg \mathcal{O}_{C_1}(1)(-\mathcal{W}) = \deg C_1 - w \geqq 8(g_1 - 1) + \frac{5}{2} w.$$

The last expression is greater than $2g_1 - 2$ unless $w = 0$, when b) reduces to a), or $g_1 = 0$ and $w = 1$ or 2. But in this case $\mathcal{O}_{C_1}(1)(-\mathcal{W}) = \mathcal{O}_{\mathbf{P}^1}(e)$, with $e \geqq 1 - 2 = -1$.

Together a) and b) imply $H^1(C, \mathcal{O}_C(1)) = 0$. In fact, if C_{red} has components C_i, then there is an exact sequence

$$0 \to \oplus \mathcal{O}_{C_i}(1)(-\mathcal{W}_i) \to \mathcal{O}_{C_{\mathrm{red}}}(1) \to \mathcal{M} \to 0$$

where $\mathcal{M}$ has 0-dimensional support, hence $H^1(C_{\mathrm{red}}, \mathcal{O}_{C_{\mathrm{red}}}(1)) = 0$, and if $\mathcal{N}$ is the sheaf of nilpotents in $\mathcal{O}_C$, then $\mathcal{N}$ has 0-dimensional support and the conclusion follows from an examination of the exact sequence

$$0 \to \mathcal{N} \to \mathcal{O}_C \to \mathcal{O}_{C\mathrm{red}} \to 0.$$

Therefore hypothesis (a) can be rewritten $n + 1 = h^0(\mathcal{O}_C(1))$. Since C is not contained in a hyperplane, C is embedded by a complete linear system. But now if $\mathcal{N} \neq (0)$, then set-theoretically C will still be contained in a hyperplane, contradicting its Chow semi-stability; so $C = C_{\mathrm{red}}$ and all that we have said about C_{red} above is true of C.

Using the fact that

$$\chi(\mathcal{O}_C) = -\chi(\omega_C) = -(\deg \omega_C + \chi(\mathcal{O}_C))$$

it follows that $\deg C/n + 1 = 2v/2v - 1$ and we can rewrite (5.7) in terms of v as

$$\frac{w}{2} + \deg C_1 \leqq \left(\frac{2v}{2v-1}\right) (\deg C_1 - g_1 + 1)$$

or equivalently

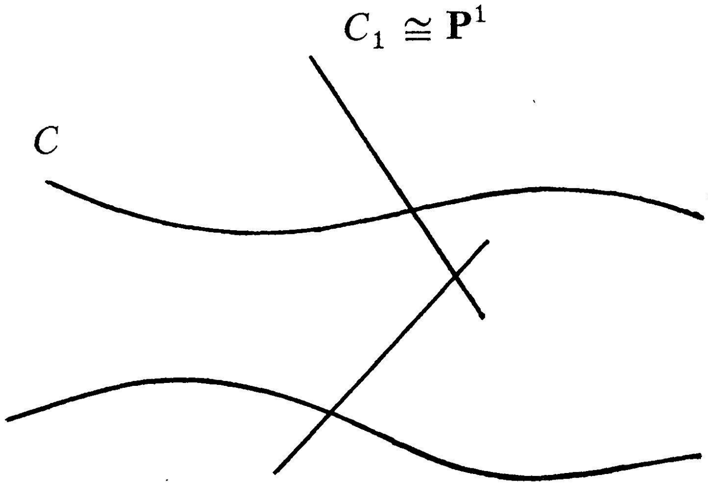
$$\frac{w}{2} \geqq \nu(2g_1 - 2 + w) - \deg C_1 = \nu \deg_{C_1}(\omega_C) - \deg C_1 .$$

Then since

$$\begin{aligned} 0 &= \nu\big(\deg(\omega_C)\big) - \deg C \\ &= \nu \deg_{C_1}(\omega_C) + \nu \deg_{C_2}(\omega_C) - \deg C_1 - \deg C_2 , \end{aligned}$$

we obtain iii): $\dfrac{w}{2} \geqq \left| \nu \deg_{C_1}(\omega_C) - \deg C_1 \right|$.

Now suppose C has a smooth rational component C_1 meeting the rest of the curve in w points $P_1, \ldots, P_w$. Then $\omega_C \mid C_1$ is just the sheaf of differentials on C_1 with poles at $P_1, \ldots, P_w$, so if $w \leqq 2$, $\deg_{C_1}(\omega_C) \leqq 0$. Using iii) this shows $\deg C_1 \leqq \frac{1}{2}$ if $w = 1$, absurd, and $\deg C_1 \leqq 1$ if $w = 2$. Moreover, if, in this last case one of the P_1 lies on a smooth rational curve C_2 meeting the rest of C in only 1 other point, as in the diagram below



then $\omega_C \mid {}_{C_1} \cong \mathcal{O}_{C_1}$ and $\omega_C \mid {}_{C_1} \cong \mathcal{O}_{C_2}$ so $\deg_{C_1 \cup C_2}(\omega_C) = 0$. Using iii) again, we find $\deg(C_1 \cup C_2) \leqq \frac{1}{2}\, 2 = 1$, and as this is absurd, we have proved all parts of the Proposition.

We are now ready to show that $\mathscr{D} = \mathscr{C}$. Since D_0 is moduli semi-stable, it follows that $\mathscr{D}$ is a normal two-dimensional scheme with only type A_n singularities. Moreover $\omega_{\mathscr{D}}^{\otimes n}{}_{/k[[t]]}$ is generated by its sections if $n \geqq 3$ and defines a morphism from $\mathscr{D}$ to a scheme $\mathscr{D}'/k\,[[t]]$, where $D'_\eta = D_\eta$, $D'_0 = D_0$ with rational chains blown down to points. Thus $\mathscr{D}'$ is a family of moduli-stable curves over $k\,[[t]]$ with generic fibre $\mathscr{C}_\eta$. Since there is only one such (cf. [6]), it follows that $\mathscr{D}' = \mathscr{C}$. Thus we have a diagram:

$$\begin{array}{ccccc} C_\eta & \xleftarrow{\approx} & D_\eta & \xrightarrow{\Phi_\eta} & \mathbf{P}^N \times \operatorname{Spec} k((t)) \\ \cap & & \cap & & \cap \\ \mathscr{C} & \longleftarrow & \mathscr{D} & \xrightarrow[\Phi]{} & \mathbf{P}^N \times \operatorname{Spec} k[[t]] \end{array}$$

$$\Phi_\eta^*\left(\mathcal{O}_{\mathbf{P}^N}(1)\right) = \omega_{D_\eta / k((t))}^{\otimes n} .$$

Let $L = \mathcal{O}_{\mathscr{D}}(1)$. It follows that $L \cong \omega_{\mathscr{D}/k[[t]]}^{\otimes n}(-\sum r_i D_i)$, where D_i are the components of D_0. Multiplying the isomorphism by $t^{\min(r_i)}$, we can assume $r_i \geqq 0$, $\min r_i = 0$. Let $D_1 = \bigcup_{r_i=0} D_i$, $D_2 = \bigcup_{r_i>0} D_i$. If f is a local equation of $\sum r_i D_i$, then $f \not\equiv 0$ in any component of D_1 since $r_i = 0$ on all these while $f(x) = 0$, all $x \in D_1 \cap D_2$, so

$$\#\,(D_1 \cap D_2) \leqq \deg_{D_1}\left(\mathcal{O}_{\mathscr{D}_0}\left(\textstyle\sum r_i D_i\right)\right) .$$

But this last degree equals $\left(\deg D_1 - n \deg_{D_1}(\omega_{D_0})\right)$ which contradicts iii) of Proposition 5.5 unless all r_i are zero. Hence $L = \omega_{\mathscr{C}}^{\otimes n}$ which shows $\mathscr{D} = \mathscr{C}$.

Line bundles on the moduli space

For the remainder of this section we examine $\operatorname{Pic}(\overline{\mathscr{M}}_g)$. We fix a genus $g \geqq 2$ and an $e \geqq 3$. Then for all stable C, $\omega_C^{\otimes e}$ is very ample and in this embedding C has degree $d = 2e(g-1)$, the ambient space has dimension $\nu - 1$ where $\nu = (2e-1)(g-1)$ and C has Hilbert polynomial $P(X) = dX - (g-1)$. Let $H \subset \mathrm{Hilb}^P_{\mathbf{P}^{\nu-1}}$ be the locally closed smooth subscheme of e-canonical stable curves C, let $C \subset H \times \mathbf{P}^{\nu-1}$ be the universal curve and let

$$\mathrm{ch} : H \to \mathrm{Div} = \mathrm{Div}^{d,d} = \left\{\begin{array}{l}\text{projective space of bihomogeneous forms} \\ \text{of bidegree } (d, d) \text{ in dual coordinates} \\ u, v \text{ (cf. § 1).}\end{array}\right\}$$

be the Chow map. These are related by the diagram

$$\begin{array}{ccccc} & & C & & \\ & & \big\downarrow{\scriptstyle \pi} & & \\ \mathrm{Div} & \xleftarrow{\mathrm{ch}} & H & \xrightarrow{\rho} & \overline{\mathscr{M}}_g = H/PGL(\nu) \end{array}$$

If $\operatorname{Pic}\left(H, PGL(\nu)\right)$ is the Picard group of invertible sheaves on H with $PGL(\nu)$-action, we have a diagram