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Autor: Mumford, David

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$\mathscr{I}_A^i$ are independent of i; we denote this ideal by $\mathscr{I}_A$. The hypothesis says that for large i

$$e(\mathscr{I}_A) \leqq \frac{\deg (C \times X)}{(n+1)(h^0(L^i) - r - 1)} \operatorname{codim} A$$
$$= \frac{(r+1) \deg X \deg L^{\otimes i}}{(n+1)(\deg L^{\otimes i} - g + 1) - r - 1} \cdot \sum_{l=0}^{n} \rho_l$$

and letting $i \to \infty$,

$$e(\mathscr{I}_A) \leqq \frac{(r+1) \deg X}{n+1} \sum_{l=0}^{n} \rho_l$$

But $C \times X$ along $p \times X$ is formally isomorphic to $\mathbf{A}^1 \times X$ along $0 \times X$ with corresponding $\mathscr{I}_A'$ s, so by Theorem 2.9., X is Chow-semi-stable.

§ 3. Effect of Singular Points on Stability

We begin with an application of Theorem 2.9.

Proposition 3.1. *Let $X^1 \subset \mathbf{P}^n$ be a curve with no embedded components such that* $\deg X/n+1 < 8/7$. *If X is Chow-semi-stable, then X has at most ordinary double points.*

Remarks. i) When $n = 2$, $\deg X/n+1 < 8/7 \Leftrightarrow \deg X < 4$ and the proposition confirms what we have seen in 1.10 and 1.11

ii) Suppose L is ample on X^1 and $X_m \subset \mathbf{P}^{N(m)}$ is the embedding of X defined by $\Gamma(X, L^{\otimes m})$. By Riemann-Roch, $\deg X_m/N(m) \to 1$ as $m \to \infty$, hence:

Corollary 3.2. *An asymptotically stable curve X has at most ordinary double points.*

In particular, if $X \subset \mathbf{P}^2$ has degree $\geqq 4$ and has one ordinary cusp, then, in $\mathbf{P}^2$, X is stable but when re-embedded in high enough space, X is unstable! The fact that this surprising flip happens was discovered by D. Gieseker and came as an amazing revelation to me, as I had previously assumed without proof the opposite.

iii) We will see in Proposition 3.14 that the constant 8/7 is best possible.

Proof of 3.1. We note first that a semi-stable X of any dimension cannot be contained in a hyperplane: if $X \subset V(X_0)$, then X has only positive weights with respect to the 1-PS

$$\lambda(t) = \begin{bmatrix} t^{-n} & & & & 0 \\ & t & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & t \end{bmatrix}$$

The plan is clear: by Theorem 2.9, it suffices to show that if x is a bad singularity of X, then there is a 1-PS.

$$\lambda(t) = \begin{bmatrix} t^{\rho_0} & & & 0 \\ & \cdot\,\cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & t^{\rho_n} \end{bmatrix}$$

such that

$$e(\mathscr{I}) \geqq \frac{16}{7} \sum_{i=0}^{n} \rho_i > \frac{\deg X\,.\,(r+1)}{(n+1)} \sum_{i=0}^{n} \rho_i\,.$$

First, if $x \in X$ has multiplicity at least three, then take coordinates $(X_0, \ldots, X_n)$ so that $x = (1, 0, \ldots, 0)$ and let $\lambda(t) = \begin{bmatrix} t & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & 1 \end{bmatrix}$ Then $\mathscr{I}\,.\,\mathcal{O}_{\mathbf{A}1 \times X}(1)$ is generated by $\{tX_0, X_1, \ldots, X_n\}$. Since $\{X_1, \ldots, X_n\}$ generate $\mathscr{M}_{x,X}$ and X_0 is a unit at x, $\mathscr{I} = (t, \mathscr{M}_x)\,\mathcal{O}_{\mathbf{A}1 \times X}$, i.e. $\mathscr{I}$ is the maximal ideal of $(0, x)$ on $\mathbf{A}^1 \times X$. Therefore, $e(\mathscr{I}) = \mathrm{mult}_{(0,x)}(\mathbf{A}^1 \times X) = \mathrm{mult}_x X \geqq 3$, which does what we want since $16/7 \sum_{i=0}^{n} \rho_i = 16/7 < 3$.

Now if $x \in V$ is a non-ordinary double point—i.e. a double point whose tangent cone is reduced to a single line—then $\dim(\mathscr{M}_{x,X}/\mathscr{M}^2_{x,X}) = 2$ and $\mathscr{M}_{x,X} \underset{\neq}{\supset} I \underset{\neq}{\supset} \mathscr{M}^2_{x,X}$ where I is the ideal of the tangent cone at x. Choose coordinates $(X_0, \ldots, X_n)$ such that

i) $X_0(x) \neq 0$

ii) $v = X_1/X_0$ and $u = X_2/X_0$ span $\mathscr{M}_{x,X}/\mathscr{M}^2_{x,X}$

iii) $u \in I$ so that $u^2 \in \mathscr{M}^3_{x,X}$.

iv) $X_3/X_0, \ldots, X_n/X_0 \in \mathscr{M}^2_{x,X}$

Then if $\lambda(t) = \begin{bmatrix} t^4 & & & & & & & 0 \\ & t^2 & & & & & & \\ & & t & & & & & \\ & & & 1 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ 0 & & & & & & & 1 \end{bmatrix}$ the associated ideal is

$\mathscr{I} = (t^4 X_0, t^2 X_1, tX_2, X_3, \ldots, X_n)$. But $\mathcal{O}_{\mathbf{A}^1 \times X}/\mathscr{I}$ is supported only at the point $(0, x)$ hence $e(\mathscr{I})$ is again Hilbert-Samuel multiplicity and is at least equal to the multiplicity of the possibly larger ideal $\mathscr{I}' = (t^4, t^2 v, tu, \mathscr{M}^2_{x,X})$. If I is the ideal $(t^4, \mathscr{M}^2_{x,X})$, then since

$$(t^2 v)^2 = t^4 v^2 \in I^2$$
$$(tu)^4 = t^4 (u^2)^2 \in t^4 (\mathscr{M}^3_{x,X})^2 \subset I^4 \quad \text{by iii)}$$

$\mathscr{I}'$ is integral over I. Hence

$$e(\mathscr{I}) \geqq e(\mathscr{I}') = e(I) = (4) \,.\, (2) \,.\, e(\mathscr{M}_{x,X}) = 16 = \frac{16}{7} \sum_{i=0}^{n} \rho_i$$

as required.

The attempt to systematize this theorem leads to a numerical measure of the degree of singularity of a point. The results that follow are part of a joint investigation of this concept by D. Eisenbud and myself. Full proofs will appear later. Many of these results have also been discovered independently by Jayant Shah.

DEFINITION 3.3. *If $\mathcal{O}$ is an equi-characteristic* [1]) *local ring of dimension r, and $k \geqq 0$ is an integer, then we define $e_k(\mathcal{O})$, the k^{th} flat multiplicity of $\mathcal{O}$, by*

$$e_0(\mathcal{O}) = \sup \left\{ \frac{e(I)}{r!\, \operatorname{col}(I)} \,\middle|\, I \text{ of finite colength in } \mathcal{O} \right\}$$
$$e_k(\mathcal{O}) = e_0(\mathcal{O}[[t_1, \ldots, t_k]])$$

It is obvious that if $\hat{\mathcal{O}}$ is the completion of $\mathcal{O}$, then $e_k(\hat{\mathcal{O}}) = e_k(\mathcal{O})$.

PROPOSITION 3.4. $e_k(\mathcal{O}) \geqq \max\left(1, e(\mathcal{O})/(r+k)\,!\right)$.

---

[1]) The hypothesis on $\mathcal{O}$ can be avoided, and the proof simplified, by a use of the associated graded ring instead of the Borel fixed point theorem (D. Eisenbud).

Proof. The second bound is obvious. To get the first note that if J is any ideal of finite colength then $e(J^n) = n^r e(J)$ and $\mathrm{col}(J^n) = \dfrac{e(J)\, n^r}{r!} + O(n^{r-1})$, hence

$$\frac{e(J^n)}{r!\,\mathrm{col}(J^n)} \to 1 \quad \text{as} \quad n \to \infty .$$

To get an upper bound on e_k we first obtain another lower bound!

PROPOSITION 3.5. $e_0(\mathcal{O}) \geqq e_0(\mathcal{O}[[t]])$; *moreover if* $r = \dim \mathcal{O} > 0$ *and there is equality, then the sup defining* $e_0(\mathcal{O}[[t]])$ *is not attained. Hence*

$$e_0(\mathcal{O}) \geqq e_1(\mathcal{O}) \geqq e_2(\mathcal{O}) \geqq \dots\dots\dots \geqq 1 .$$

Proof. We begin by giving a lemma which is useful in the applications of e_0 as well.

LEMMA 3.6. *Let* $\mathcal{J}$ *be the set of ideals of* $\mathcal{O}[[t]]$ *of the form* $I = \bigoplus_{i=0}^{\infty} I_i t^i$, *where* I_i *is an increasing sequence of ideals of finite colength in* $\mathcal{O}$ *such that* $I_N = \mathcal{O}$ *for some* N. *Then*

$$e_0(\mathcal{O}[[t]]) = \sup_{I \in \mathcal{J}} \frac{e(I)}{r!\,\mathrm{col}(I)}$$

Proof. For any equi-characteristic local ring R, let Hilb_R^n be the subscheme of the Grassmanian of codimension n subspaces of $R/\mathcal{M}_R^n$ parametrizing those subspaces which are ideals: since any ideal in R of colength n contains $\mathcal{M}_R^n$, Hilb_R^n parameterizes these ideals. Let $e \colon \mathrm{Hilb}_R^n \to \mathbf{Z}$ be the map assigning to an ideal its multiplicity. By results of Teissier and Lejeune [23], e is upper-semi-continuous.

The natural $\mathbf{G}_m$-action on $\mathcal{O}[[t]]$ by $t \to \lambda t$ induces a $\mathbf{G}_m$-action on $\mathrm{Hilb}^n_{\mathcal{O}[[t]]}$. By the Borel fixed point theorem, there is, for every I, an ideal fixed by this action in $O^{\mathbf{G}_m}(I)$. Such an ideal must, by the upper-semi-continuity of multiplicity have multiplicity at least as large as $e(I)$. Thus, to compute $e_0(\mathcal{O}[[t]])$ it suffices to look at $\mathbf{G}_m$-fixed ideals of finite colength and $\mathcal{J}$ is just the set of such ideals.

Fix $I = \bigoplus_{i=0}^{\infty} I_i t^i$, where $I_0 \subset I_1 \subset \dots \subset I_N = \mathcal{O}$ is an increasing sequence of ideals in $\mathcal{O}$. Clearly $\mathrm{col}(I) = \sum_{i=0}^{N-1} \mathrm{col}(I_i)$. To bound $e(I)$ we note that

$$
\begin{aligned}
I^n \supset & (I_0^n) \oplus (I_0^{n-1} I_1 t) \oplus (I_0^{n-2} I_1^2 t^2) \oplus \dots \oplus (I_0 I_1^{n-1} t^{n-1}) \oplus \\
& \oplus (I_1^n t^n) \oplus (I_1^{n-1} I_2 t^{n+1}) \oplus \dots\dots\dots \oplus (I_{N-2} I_{N-1}^{n-1} t^{(N-1)n-1}) \oplus \\
(3.7) \qquad & \oplus (I_{N-1}^n t^{(N-1)n}) \oplus (I_{N-1}^{n-1} t^{(N-1)n+1}) \oplus \dots\dots \oplus (I_{N-1} t^{Nn-1}) \\
& \oplus (\mathcal{O} t^{Nn}) \oplus \dots \\
\Rightarrow I^n \supset & (I_0^n \oplus I_0^n t \oplus \dots \oplus I_0^n t^{n-1}) \oplus (I_1^n t^n \oplus \dots \oplus I_1^n t^{2n-1}) \oplus \dots\dots \\
& \oplus (I_{N-1}^n t^{(N-1)n} \oplus I_{N-1}^{n-1} t^{(N-1)n+1} \oplus \dots . \oplus I_{N-1} t^{Nn-1}) \\
& \oplus \mathcal{O} t^{Nn} \oplus \dots
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\operatorname{col}(I^n) & \leq \sum_{i=0}^{N-2} n \operatorname{col}(I_i^n) + \sum_{j=1}^{n} \operatorname{col}(I_{n-1}^j) \\
& = \frac{n^{r+1}}{r!} \sum_{i=0}^{N-2} e(I_i) + \frac{n^{r+1}}{(r+1)!} e(I_{N-1}) + O(n^r)
\end{aligned}
$$

(We have evaluated the second sum by "integration"!)
Finally

$$
\frac{e(I)}{(r+1)! \operatorname{col}(I)} \leq \frac{(r+1) \sum_{i=0}^{N-2} e(I_i) + e(I_{N-1})}{(r+1)! \sum_{i=0}^{N-1} \operatorname{col}(I_i)} \leq \frac{\sum_{i=0}^{N-1} e(I_i)}{r! \sum_{i=0}^{N-1} \operatorname{col}(I_i)},
$$

with strict inequality if $r > 0$

$$
\leq \max_i \frac{e(I_i)}{r! \operatorname{col}(I_i)} \leq e_0(\mathcal{O}) .
$$

Corollary 3.8. *If $\mathcal{O}$ is regular, $e_0(\mathcal{O}) = 1$ and if $r > 1$, the defining sup is not attained.*

Corollary 3.9. (Lech [1]). *For all $\mathcal{O}$ and all $I \subset \mathcal{O}$, $e(I) \leq$* r ! *$e(\mathcal{O})$* col (I), *hence $e_0(\mathcal{O}) \leq e(\mathcal{O})$.*

Proof. None of the quantities involved change if we complete $\mathcal{O}$. But after doing this, we can write $\mathcal{O}$ as a finite module over $\mathcal{O}_0 = k[[t_1, \dots, t_r]]$ so that:

(*) There is a sub $\mathcal{O}_0$-module $\mathcal{O}_0^{e(\mathcal{O})} \subset \mathcal{O}$ such that the quotient $\mathcal{O}/\mathcal{O}_0$ is an $\mathcal{O}_0$-torsion module M.

[1]) Cf. [13], Theorem 3.

Let $I_0 = I \cap \mathcal{O}_0$. Then col $(I) \geqq$ col (I_0) and

$$\begin{aligned}\dim(\mathcal{O}/I^n) &\leqq \dim \mathcal{O}/I_0^n\,\mathcal{O}\\ &\leqq \dim(M/I_0^n\,M) + \dim(\mathcal{O}_0^{e(\mathcal{O})}/I_0^n\,\mathcal{O}^{e(\mathcal{O})})\end{aligned}$$

Condition (*) implies that $\dim(M/I_0^n\,M)$ is represented by a polynomial of degree less than r, hence

$$\begin{aligned}e(I) &\leqq e(\mathcal{O})\,e(I_0)\\ &\leqq r\,!\,e(\mathcal{O})\,\mathrm{col}(I_0) \text{ by Corollary 3.8}\\ &\leqq r\,!\,e(\mathcal{O})\,\mathrm{col}(I)\end{aligned}$$

We state two other useful properties of e_k:

PROPOSITION 3.10. i) *If* $\mathcal{O}$ *and* $\mathcal{O}'$ *are local domains with the same fraction field and* $\mathcal{O}'$ *is integral over* $\mathcal{O}$, *then* $e_k(\mathcal{O}') \leqq e_k(\mathcal{O})$.

ii) *If* $\mathcal{O} = (k[[t]] + \mathcal{P})$ *is an augmented* $k[[t]]$*-algebra, let* $\mathcal{O}_\eta = \mathcal{O}_{\mathcal{P}}$, *a local ring with residue field* $k((t))$ *and let* $\mathcal{O}_s = \mathcal{O}/t\mathcal{O}$ *be its specialization over* k; *then* $e_k(\mathcal{O}_\eta) \leqq e_k(\mathcal{O}_s)$.

We come now to the main definitions.

DEFINITION 3.11. $\mathcal{O}$ *is semi-stable if* $e_1(\mathcal{O}) = 1$; $\mathcal{O}$ *is stable if, in addition, the defining sup is not attained.*

This terminology is justified by the following proposition which shows that the semi-stability of the local rings on a variety X is just the local impact of the global condition of asymptotic semi-stability for X.

PROPOSITION 3.12. *Fix a variety* X^r, *an ample line bundle* $L = \mathcal{O}_X(D)$ *on* X, *and* $p \in X$. *Then if* $\mathcal{O}_{p,X}$ *is unstable,* (X, L) *is asymptotically unstable.*

Proof. Choose an ideal $I \subset \mathcal{O}_{p,X}[[t]]$ such that

i) $e(I) = (1+\varepsilon)(r+1)\,!\,\mathrm{col}(I)$, $\varepsilon > 0$

ii) $I = \bigoplus_{i=0}^{\infty} I_i t^i$, $I_0 \subset I_1 \subset \ldots \subset I_N = \mathcal{O}_{p,X}$ a sequence of ideals of finite colength. (This is possible because of Lemma 3.6).

Let Φ_m denote the projective embedding of X by $\Gamma(X, L^{\otimes m})$. Choose m large enough that

a) for all $Q \in X$, $\Gamma(X^r, L^m) \xrightarrow{\psi} \Gamma(X, L^m/I_0\mathscr{M}_{Q,X} \cdot L^m)$ is surjective

b) L^m is very ample

c) $h^0(X, L^m) > \dfrac{1}{1+\varepsilon} \dfrac{m^r(D^r)}{r!} = \dfrac{1}{1+\varepsilon} \dfrac{\deg \Phi_m(X)}{r!}$

(That the last condition can always be realized is a consequence of Riemann-Roch for X.)

Next choose a basis $X_{i,j}$, $0 \leqq i \leqq N$, of $\Gamma(X, L^m)$ such that

$$
\begin{aligned}
&X_{0,j} \text{ is a basis of } \psi^{-1}(I_0)\,,\\
&X_{1,j} \text{ is a basis of } \psi^{-1}(I_1)/\psi^{-1}(I_0)\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&X_{N,j} \text{ is a basis of } \Gamma(X, L^m)/\psi^{-1}(I_{N-1})\,,
\end{aligned}
$$

Finally, let λ be the 1-PS which multiplies $X_{i,j}$ by t^i : i.e. in the form of (2.8) $\rho^{(i,j)} = i$; then by assumption (a) the ideal $\mathscr{I}$ corresponding to λ in (2.8) is just I and is supported at the single point $(0, p) \in \mathbf{A}^1 \times X$. Moreover, by condition a)

$$
\begin{aligned}
\sum_{i,j} \rho^{(i,j)} &= N \dim(\mathcal{O}/I_{N-1}) + (N-1)\dim(I_{N-1}/I_{N-2})\\
&\quad + \dots + 2\dim(I_2/I_1) + \dim I_1/I_0 = \operatorname{col}(I)
\end{aligned}
$$

(This is Lemma 2.14 again). Hence,

$$
\begin{aligned}
e(\mathscr{I}) &= e(I)\\
&= (1+\varepsilon)\,.\,(r+1)\,!\operatorname{col}(I)\\
&> (1+\varepsilon)\,.\,(r+1)\,.\,\frac{\deg \Phi_m(X)}{(1+\varepsilon)\,h^0(L^m)} \cdot \sum_{i,j} \rho^{(i,j)}\\
&= \frac{(r+1)\deg \Phi_m(X)}{h^0(L^m)} \cdot \sum_{i,j} \rho^{(i,j)}
\end{aligned}
$$

By Theorem 2.9, $\Phi_m(X)$ is unstable.

Restating Corollary 3.7 gives us a trivial class of stable points:

PROPOSITION 3.13. *If $\mathcal{O}$ is regular and of positive dimension it is stable.*

The next step is to pindown the meaning of semi-stability for small dimensional local rings. For dimension 1, we can be quite explicit:

PROPOSITION 3.14. *If* $\dim \mathcal{O} = 1$ *and* $\mathcal{O}$ *is Cohen-Macauley (i.e.* Spec $\mathcal{O}$ *has no embedded components), then :*

i) $\mathcal{O}$ *stable* $\Leftrightarrow \mathcal{O}$ *regular* $\Leftrightarrow e(\mathcal{O}) = e_0(\mathcal{O}) = e_1(\mathcal{O}) = \ldots = 1$.

ii) $\mathcal{O}$ *semi-stable but not stable* $\Leftrightarrow \mathcal{O}$ *an ordinary double point* $\Leftrightarrow e(\mathcal{O}) = e_0(\mathcal{O}) = 2,\ e_1(\mathcal{O}) = e_2(\mathcal{O}) = \ldots = 1$.

iii) $\mathcal{O}$ *a higher double point* $\Rightarrow e_1(\mathcal{O}) \geqq 8/7$.

iv) $\mathcal{O}$ *a triple point or higher multiplicity* $\Rightarrow e_1(\mathcal{O}) \geqq 3/2$.

Proof. If $\mathcal{O}$ is a triple or higher point, so is $\mathcal{O}[[t]]$, hence $e(\mathcal{O}[[t]]) \geqq 3$, and by Proposition 3.4, $e_1(\mathcal{O}) = e_0(\mathcal{O}[[t]]) \geqq 3/2$.

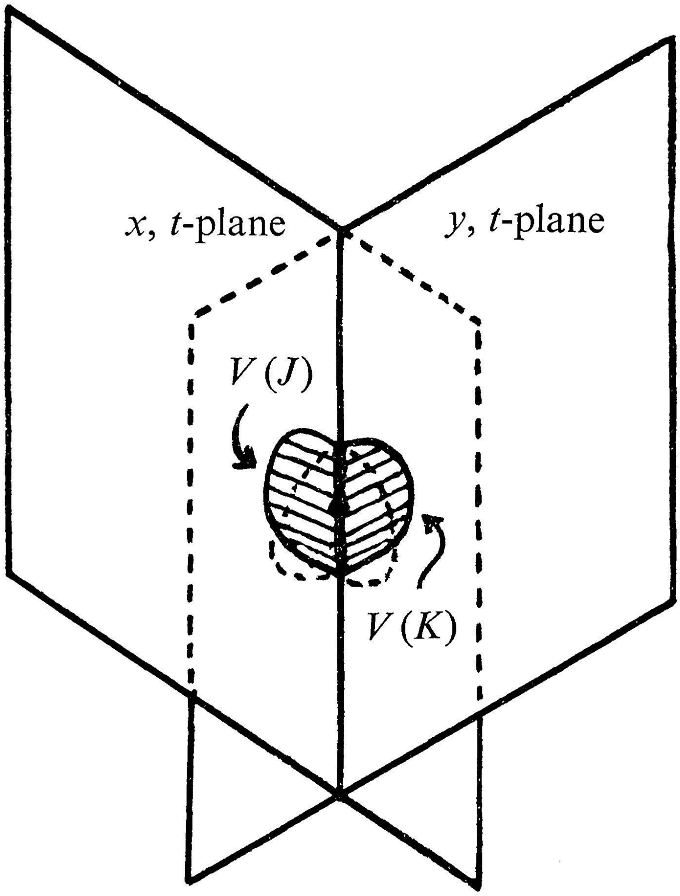
As for Cohen-Macaulay double points, when char. $\neq 2$ these are all of the form $\hat{\mathcal{O}} = k[[x, y]]/(x^2 - y^n)$, $2 \leqq n \leqq \infty$. (Think of $\hat{\mathcal{O}}$ as a quadratic free $k[[y]]$-algebra; the argument can be readily adapted to char. 2 also). If $n \geqq 3$, then in $k[[x, y, t]]/(x^2 - y^n)$, take $I = (x^2, xy, y^2, xt, yt^2, t^4)$. (This, of course, is the ideal of Proposition 3.1 again). I has complementary basis $(1, x, y, t, yt, t^2, t^3)$, hence $\mathrm{col}(I) = 7$. I claim $e(I) = 16$, which will imply iii). We first note that I is integral over (y^2, t^4). We compute the multiplicity of (y^2, t^4) as

$$\begin{aligned} &\textit{intersection-multiplicity at } \mathcal{M}\ \big((\mathrm{Spec}\ \mathcal{O}) \,.\, (y^2 = 0) \,.\, (t^4 = 0)\big) \\ &= \ 8 \,.\, \textit{intersection-multiplicity}\ \big((\mathrm{Spec}\ \mathcal{O}) \,.\, (y = 0) \,.\, (t = 0)\big) \\ &= \ 16 \end{aligned}$$

since $\mathcal{O}$ is a double point.

When $\mathcal{O}$ is an ordinary double point, I claim $e_0(\mathcal{O}[[t]]) = 1$. Since this value is attained by the maximal ideal $\mathcal{M}$: $\dfrac{e(\mathcal{M})}{2!\,\mathrm{col}(\mathcal{M})} = \dfrac{2}{2} = 1$, this will prove ii), hence i) in view of Proposition 3.13.

In general, if $\mathcal{O} = k[[x, y]]/(x \,.\, y)$, an ideal $I \subset \mathcal{O}[[t]]$ corresponds to a pair of ideals $J \subset k[[x, t]]$ and $K \subset k[[y, t]]$ such that $J + (x)/(x)$ and $K + (y)/(y)$ have the same image, say (t^n), in $k[[t]]$. A rough picture is given below: the condition on the two ideals ensures that they glue along the intersection of the two planes.



In this situation, $\mathrm{col}\,(I) = \mathrm{col}\,(J) + \mathrm{col}\,(K) - n$, and $e\,(I) = e\,(J) + e\,(K)$, so the inequality $e\,(I)/2 \,.\, \mathrm{col}\,(I) \leqq 1$ follows from:

LEMMA 3.15. *If* $I \subset k\,[[x, y]]$ *and* $I + (x) = (x, y^a)$, *then* $e\,(I) \leqq 2\,\mathrm{col}\,(I) - a$.

Proof. By applying Lemma 3.6, we can reduce to the case where I is generated by monomials:

$$I = \bigoplus_{l=0}^{\infty} (y^{r_l} . x^l) . k\,[[y]]\,, \text{ with } a = r_0 \geqq r_1 \geqq \ldots \geqq r_N = 0.$$

Then as in (3.7):

$$I^n \supset (y^{nr_0})\,k \oplus (y^{(n-1)r_0+r_1}x)\,k \oplus (y^{(n-2)r_0+2r_1}x^2)\,k \oplus \ldots$$
$$\oplus\,(y^{nr_1}x^n)\,k \oplus (y^{(n-1)r_1+r_2}x^{n+1})\,k \oplus \ldots\ldots \oplus (y^{nr_2}x^{2n})\,k \oplus \ldots\ldots\ldots\ldots$$

$$\Rightarrow \mathrm{col}\,(I^n) \leqq \frac{n\,(n+1)}{2}\,r_0 + n^2 r_1 + n^2 r_2 + \ldots\ldots + n^2 r_{N-1}$$

$$\Rightarrow \frac{e\,(I)}{2} \leqq \frac{r_0}{2} + r_1 + \ldots\ldots + r_{N-1} = \mathrm{col}\,(I) - \frac{a}{2}\,.$$

REMARK. If $I \subset \mathcal{O}\,[[t]]$ is of the form of Lemma 3.6, the expansion (3.7) for I^n, which we have used again here, can be used to give even better bounds for $e(I)$. To get these however, requires the more involved theory of mixed multiplicities which will be discussed in § 4.

The meaning of semi-stability for two dimensional singularities is not yet completely worked out, but what follows gives a good overview of the situation.

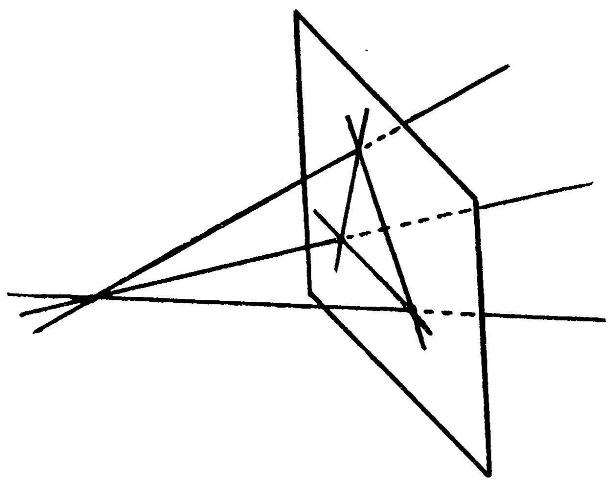
DEFINITION 3.16. *If* $\mathcal{O}$ *is a normal 2-dimensional local ring,* x *is the closed point of* Spec $\mathcal{O}$, *and* $X^* \xrightarrow{\pi}$ Spec $\mathcal{O}$ *is a resolution of* $\mathcal{O}$ *(i.e.* π *is proper and birational), then we define*

i) *big genus of* $\mathcal{O}$ = $\dim R^1 \pi_* (\mathcal{O}_{X^*})$
 ($R^1\pi_*$ *is a torsion* $\mathcal{O}$*-module supported at* x)

ii) *little genus of* $\mathcal{O}$ = $\sup_Z \left(p_a(\mathcal{O}_Z)\right)$, *where* Z *runs over the effective cycles on* $\pi^{-1}(x)$.

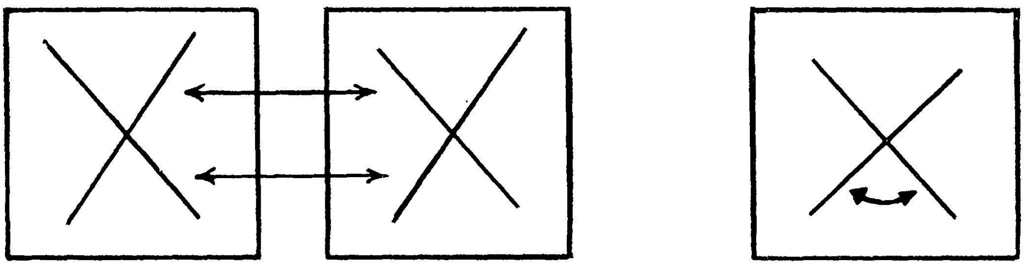
Wagreich [24] has shown that big genus $\geqq$ little genus—hence the names—and Artin [3] has shown that if the little genus is zero then so is the big genus. (But when little genus = 1, big genus may be > 1). We call $\mathcal{O}$: rational (resp. strongly elliptic) if its big genus is 0 (resp. 1), and weakly elliptic if its little genus is 1.

If there is to be any hope of constructing compact moduli spaces for semi-stable surfaces, the non-normal singularity $xyz = 0$ must be semi-stable—in fact, it is. But $xyz = 0$ is the cone over a plane triangle so the triple point on it is really a degenerate "elliptic" singularity. In fact, $xyz = 0$ is a limit of the family of non-singular cubics $xyz + t(x^3+y^3+z^3) = 0$. Similarly, the standard singularities A_{n-1}: $xy = z^n$ and D_n: $x^2=y^2z+z^n$ have non-normal limits $xy = 0$ and $x^2 = y^2z$ respectively as $n \to \infty$. We can summarize these considerations in the heuristic conjecture: the semi-stable singularities of surfaces will be a limited class of rational and strongly elliptic normal singularities and their non-normal limits.



We now list without proof some classes of semi-stable singularities.

3.17. ELLIPTIC POLYGONAL CONES. In $\mathbf{P}^{n-1}$ take a generic n-gon $\bigcup_{i=0}^{n} \overline{p_i p_{i+1}}$ ($p_0 = p_{n+1}$) and take the cone in $\mathbf{C}^n$ over it. This is a union of n-planes crossing normally in pairs and meeting at an n-fold point at the origin. We also allow the degenerate cases $n = 2$ (local equation $x^2 = y^2 z^2$) and $n = 1$ (local equation $x^2 = y^2 (y + z^2)$) which correspond respectively, to glueing two planes to each other along a pair of transversal lines, and to glueing a pair of transversal lines in a plane together as shown below.



$n = 2$ $\qquad\qquad\qquad\qquad$ $n = 1$

PROPOSITION 3.18. *Elliptic polygonal n-cones are semi-stable if and only if* $1 \leqq n \leqq 6$. *Moreover, all small deformations of these singularities are semi-stable.*

Examples of such singularities are:

i) Cone over a smooth elliptic curve with generic j in $\mathbf{P}^n$, $3 \leqq n \leqq 5$. (In fact, I expect this holds for arbitrary j). These are also called the simple elliptic (Saito) or parabolic (Arnold) singularities, and may be described as $\bigoplus_{m=0}^{\infty} \Gamma(E, L^m)$ where E is an elliptic curve and L is a line bundle of positive degree n: with this description, they are also defined for $n = 1, 2$. For small n, these have the form

$$x^2 + y^3 + z^6 + a(y^2 z^2) = 0 \qquad (n=1),$$
$$x^2 + y^4 + z^4 + a(y^2 z^2) = 0 \qquad (n=2),$$
$$x^3 + y^3 + z^3 + a(xyz) = 0 \qquad (n=3).$$

ii) The hyperbolic singularities of Arnold:

$$xyz + x^n + y^m + z^p = 0 \qquad \frac{1}{n} + \frac{1}{m} + \frac{1}{p} < 1.$$

iii) Rational double points.

iv) Pinch points: these have local equation $x^2 = y^2 z$.

3.18. RATIONAL POLYGONAL CONES. In $\mathbf{P}^{n-1}$ take $(n-1)$ generic line segments $\overline{P_0 P_1} \cup \overline{P_1 P_2} \ldots \cup \overline{P_{n-1} P_n}$ and in $\mathbf{C}^n$ take the cone over them: one obtains $(n-2)$ planes crossing normally in $(n-1)$ lines.

PROPOSITION 3.19. *Rational polygonal n-cones are semi-stable if and only if* $2 \leqq n \leqq 6$. *Hence, all small deformations of these singularities are semi-stable.*

A typical singularity which arises in this way is the cone over a rational normal curve in $\mathbf{P}^{n-1}$, $2 \leqq n \leqq 6$.

By applying the semi-stability condition to the ideal $I = \bigoplus_{j=0}^{i} t^{i-j} \cdot (\widetilde{I^j}) \subset \mathcal{O}\,[[t]]$, where I is an ideal in $\mathcal{O}$ and $\sim$ denotes integral closure in $\mathcal{O}$, one can prove the following necessary condition for semi-stability:

PROPOSITION 3.19. *If $\mathcal{O}^r$ is semi-stable, $I \subset \mathcal{O}$ and $P(i) = \dim(\mathcal{O}/(\widetilde{I^i}))$, then*

$$P(1) + \ldots\ldots + P(i) \geqq \frac{e(I)\, i^{r+1}}{(r+1)\,!}\,.$$

When $r = 2$, and $\mathcal{O}$ is Cohen-Macaulay this reduces us to *ten* basic types of singularities. In the first few cases we have listed the singularities of this type which are actually semi-stable.

1) Regular points: always stable.

2) Double coverings of $\mathbf{C}^2$ with branch curve of multiplicity $\leqq 4$: semi-stable here are,

   a) rational double points and their non-normal limits $xy = 0$, $x = y^2 z$,

   b) hyperbolic double points,

   c) parabolic double points.

3) Triple points in $\mathbf{C}^3$: Semi-stable are,

   a) cones over non-singular elliptic curves,

   b) hyperbolic triple points.

4-5) Triple and quadruple points in $\mathbf{C}^4$.

6-7) Quadruple and quintuple points in $\mathbf{C}^5$.

8-9) Quintuple and sextuple points in $\mathbf{C}^6$.

10) Sextuple points in $\mathbf{C}^7$.

REMARK. With Eisenbud, we made some computations by computor to eliminate cases; the computer came up with some amusing examples. For instance it found an ideal I in $k[[x, y, z, t]]/(x^2+y^3+z^7)$ with col $(I) = 63{,}398$, mult $(I) = 381{,}024$, showing that $e_0 \geqq 1.000167$, hence that the singularity $x^2 + y^3 + z^7 = 0$ is unstable.

Further restrictions, confirming the heuristic conjecture, on what singularities are semi-stable are provided by:

PROPOSITION 3.20. *If $\mathcal{O}$ is normal and semi-stable then $\mathcal{O}$ is rational or weakly elliptic. Moreover, there are no cuspidal curves, i.e. generically all singular curves are ordinary.*

We omit the proof except to note that the last statement comes from the observation that for large n the choices $I_n = (T^9, u^{9n}, v^{9n})^{\sim}$ show that $e_2(k[[T^2, T^3]]) \geqq 1 + 22/221$!

Now suppose $\mathcal{O}$ is not Cohen-Macaulay. We can create a slew of stable $\mathcal{O}$'s using i) of Proposition 3.10. For example if $k[[x, y]] \supset \mathcal{O} \supset k[[x, xy, y^2]]$, then $\mathcal{O}$ is semi-stable since the ring on the right which is the pinch point is semi-stable; a typical example is $\mathcal{O} = k[[x, xy, y^2, y^3]]$, a very partial pinch in which only the y-tangent has been removed. Fortunately most of these points cannot appear as singularities of varieties on boundary of moduli spaces as they have no smooth deformations. More precisely, (cf. [27]):

THEOREM 3.21. *If $\mathcal{O}$ is a 2-dimensional local ring which is not Cohen-Macauley such that $\mathcal{O} = \mathcal{O}'/t\mathcal{O}'$ where $\mathcal{O}'$ is a normal 3-dimensional local ring; let $\mathcal{O}_{\text{norm}}$ be its normalization and $\tilde{\mathcal{O}} = \{ a \in \mathcal{O}_{\text{norm}} \mid$ for some $n, \mathcal{M}_{\mathcal{O}}^n a \subset \mathcal{O} \}$.*

Then i) *$\tilde{\mathcal{O}}$ is a local ring*

ii) *If in addition $\mathcal{O}$ has characteristic* 0, *then*

$$\dim(\tilde{\mathcal{O}}/\mathcal{O}) \leqq \textit{big genus of } \tilde{\mathcal{O}}.$$

REMARK. If, as seems likely, in view of Proposition 3.20 the big genus of the Cohen-Macaulay ring $\tilde{\mathcal{O}}$ is 0 or 1, this means that $\mathcal{O}$ must be nearly Cohen-Macauley.

We conclude this section by outlining an as yet completely uninvestigated approach to deciding which singularities should be allowed on the objects of a moduli space.

Definition 3.22. *$\mathcal{O}^r$ is an insignificant limit singularity if, whenever $\mathcal{O}'$ is an $(r+1)$ dimensional local ring such that $\mathcal{O} = \mathcal{O}'/t\mathcal{O}'$ for some $t \in \mathcal{O}'$, $\pi: X \to \operatorname{Spec} \mathcal{O}'$ is a resolution of* Spec $\mathcal{O}'$ and *$E \subset X$ is an exceptional divisor (i.e.* $\dim \pi(E) < \dim E$), *then E is birationally ruled, that is, the function field of E is a purely transcendental extension of a proper subfield. Equivalently, setting $\mathcal{O}/\mathcal{M}_{\mathcal{O}} = k$, this says that whenever R is a discrete rank 1 valuation ring containing $\mathcal{O}'$ with tr.* $\operatorname{deg.}_k R/\mathcal{M}_R = r$, *then $R/\mathcal{M}_R = K(t)$, for some K such that* $\operatorname{tr.deg.}_k K = r - 1$.

Examples. 1) $xy = 0$ is insignificant because on deforming this only A_n singularities arise.

2) $x^2 + y^3 = 0$ is significant because the deformation $t^6 = x^2 + y^3$ blows up to a non-singular elliptic curve with $(E^2) = -1$. Similarly I can show that all higher plane curve singularities are significant.

3) $x^3 + y^3 + y^4 = 0$ is significant because $t^{12} = x^3 + y^3 + y^4$ blows up to a 3-fold containing a K3 surface.

4) Jayant Shah [26] has proven that rational double points and Arnold's parabolic and hyperbolic singularities are insignificant. As a limiting case, normal crossings $xyz = 0$ is insignificant.

Remarks. 1) Why should birational ruling of exceptional divisors be the right criterion for insignificance? The reason is that all exceptional divisors which arise from blow-ups of non-singular points are birationally ruled and all birationally ruled varieties arise in this way. So on the one hand, such exceptional divisors must be permitted, and on the other, the examples suggest that sufficiently tame singularities cannot "swallow" anything else.

2) The examples suggest that $\mathcal{O}$ semi-stable and $\mathcal{O}$ insignificant are closely related. For instance, perhaps these are the same when embedding-dim $\mathcal{O} = 1$. In dim 2 for example, after hyperbolic and parabolic singularities in the Dolgacev-Arnold list [2, 7] of 2-dimensional singularities come 31 special singularities. These are all unstable and in a recent letter to me Dolgacev remarks that all of these have deformations which blow up to K3 surfaces as in Example 3. If semi-stability and insignificance turn out to be roughly the same in arbitrary dimension, we would have a very powerful tool to apply to moduli problems.