

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 23 (1977)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** STABILITY OF PROJECTIVE VARIETIES  
**Autor:** Mumford, David  
**Kapitel:** §2. A CRITERION FOR  $X^r \subset P^n$  TO BE STABLE  
**DOI:** <https://doi.org/10.5169/seals-48919>

### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 22.02.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

§ 2. A CRITERION FOR  $X^r \subset \mathbf{P}^n$  TO BE STABLE

If  $f(a)$  is an integer-valued function which is represented by a rational polynomial of degree at most  $r$  in  $n$  for large  $n$ , we will denote by n.l.c. ( $f$ ) (the normalized leading coefficient of  $f$ ) the integer  $e$  for which  $f(n) = e \frac{n^r}{r!} + \text{lower order terms}$ . (What  $r$  is to be taken, will always be clear from the context.)

PROPOSITION 2.1<sup>1)</sup>. (*The "Hilbert-Hilbert-Samuel" Polynomial*). Suppose  $X$  is a  $k$ -variety (not necessarily complete),  $L$  is an invertible sheaf on  $X$  and  $\mathcal{I} \subset \mathcal{O}_X$  is an ideal sheaf such that  $Z = \text{Supp } \mathcal{O}_X/\mathcal{I}$  is proper over  $k$ . Then there is a polynomial  $P(n, m)$  of total degree  $\leq r$ , such that, for large  $m$

$$\chi(L^n/\mathcal{I}^m L^n) = P(n, m).$$

*Proof.* We can compactify  $X$  and extend  $L$  to a line bundle on this compactification, without altering the validity of the theorem so we may as well assume  $X$  proper over  $k$ . Let  $\pi: B \rightarrow X$  be the blow-up of  $X$  along  $\mathcal{I}$  (i.e.  $B = B_{\mathcal{I}}(X) = \text{Proj } (\mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \dots)$ ) and let  $E$  be the exceptional divisor on  $B$  so that  $\mathcal{I} \cdot \mathcal{O}_B = \mathcal{O}(-E)$ . The well-known theorems of F.A.C. (Serre [18]) for the vanishing of higher cohomology in the relative case imply that when  $m \gg 0$ :

- i)  $\pi_*(\mathcal{O}(-mE)) = \mathcal{I}^m$
- ii)  $R^i \pi_*(\mathcal{O}(-mE)) = (0), i > 0$

Now examine the exact sequence:

$$0 \longrightarrow \mathcal{I}^m L^n \longrightarrow L^n \longrightarrow L^n/\mathcal{I}^m L^n \longrightarrow 0$$

The Hilbert polynomial for  $\chi(L^n)$  certainly satisfies the conditions on  $P$ . Moreover, in view of i) and ii); we have for  $m \gg 0$ :

$$\chi(X, \mathcal{I}^m L^n) = \chi(B, \pi^* L^n(-mE)) = \chi(B, (\pi^* L)^{\otimes n} \otimes \mathcal{O}(-E)^{\otimes m})$$

so, a theorem of Snapper [5, 21] guarantees that this last Euler characteristic is also a polynomial of the required type for large  $m$  and  $n$ . By the additivity of  $\chi$  we are done.

<sup>1)</sup> This result and its geometric interpretation are essentially due to C. P. Ramanujam [16].

DEFINITION 2.2. In the situation of Proposition 2.1, we denote by  $e_L(\mathcal{I})$  (the multiplicity of  $\mathcal{I}$  measured via  $L$ ) the integer n.l.c.  $(\chi(L^n/\mathcal{I}^n L^n))$ .

EXAMPLES. i) If  $\mathcal{I} = 0$  and  $X$  is complete,  $P$  is the Hilbert polynomial of  $L$ . ii) If  $Z$  is set-theoretically a point  $x$  then  $P$  is the Hilbert-Samuel polynomial of  $\mathcal{I}$  as an ideal of  $\mathcal{O}_{x,X}$  and  $e(\mathcal{I})$  is its multiplicity there: in particular, it is independent of  $L$ . Note that, in general,  $e_L(\mathcal{I})$  depends on the formal completion of  $X$  along  $Z$  and the pull-backs of  $\mathcal{I}, L$  to this formal completion.

2.3. CLASSICAL GEOMETRIC INTERPRETATION. Let  $X^r \subset \mathbf{P}^n$  be a projective variety,  $L = \mathcal{O}_X(1)$ , and  $\Lambda$  be a subspace of  $\Gamma(\mathbf{P}^n, \mathcal{O}(1))$ . Define  $L_\Lambda$  to be the linear subspace of  $\mathbf{P}^n$  given by  $s = 0, s \in \Lambda$ . Define  $\mathcal{I}_\Lambda$  to be the ideal sheaf generated by the sections  $s \in \Lambda$ , i.e.  $\mathcal{I}_\Lambda \cdot L$  is the subsheaf of  $L$  generated by those sections and  $Z = \text{Supp}(\mathcal{O}_X/\mathcal{I}_\Lambda) = X \cap L_\Lambda$  is the set of their base points.

If  $p_\Lambda: \mathbf{P}^n - L_\Lambda \rightarrow \mathbf{P}(\Lambda) = \mathbf{P}^m$  is the canonical projection, and  $\pi$  is the blow-up of  $X$  along  $\mathcal{I}_\Lambda$  then there is a unique map  $q$  making the following diagram commute:

$$\begin{array}{ccc}
 X - Z & \xrightarrow{\text{res } p_\Lambda} & \mathbf{P}^m \\
 \cap & & \nearrow q \\
 X & \xleftarrow{\pi} & B = B_{\mathcal{I}_\Lambda}(X)
 \end{array}$$

Moreover, because sections of  $\mathcal{O}_{\mathbf{P}^m}(1)$  pull back to sections of  $\mathcal{I}_\Lambda \cdot L$  on  $X$  and are blown-up to sections of  $L$  twisted by minus the exceptional divisor  $E$ ,

$$(2.4) \quad q^*(\mathcal{O}_{\mathbf{P}^m}(1)) = (\pi^* L)(-E).$$

Define  $p_\Lambda(X)$ , the image of  $X$  by the projection  $p_\Lambda$ , to be  $[\text{cycle}(q(B))]$ : that is,  $q(B)$  with multiplicity equal to the degree of  $B$  over  $q(B)$  if these have the same dimension and 0 otherwise. I claim

PROPOSITION 2.5.  $e_L(\mathcal{I}_\Lambda) = \deg X - \deg p_\Lambda(X)$ .

*Proof.* If  $H$  is the divisor class of a hyperplane section on  $X$ , then

$$\deg X = (H^r) = \text{n.l.c.}(\chi(\mathcal{O}_X(n))).$$

By 2.4,  $q$  is defined by the linear system of divisors of the form  $\pi^{-1}(H) - E$ , hence

$$\deg p_A(x) = ((\pi^{-1}(H) - E)^r) = \text{n.l.c. } \chi(\pi^*(\mathcal{O}(n)(-nE))).$$

Finally, from its definition

$$\begin{aligned} e_L(\mathcal{I}_A) &= \text{n.l.c. } \chi(\mathcal{O}_X(n)/\mathcal{I}^n \mathcal{O}_X(n)) \\ &= \text{n.l.c. } \chi(\mathcal{O}_X(n)) - \text{n.l.c. } \chi(\mathcal{I}^n \mathcal{O}_X(n)) \\ &= \deg X - \deg p_A(X) \end{aligned}$$

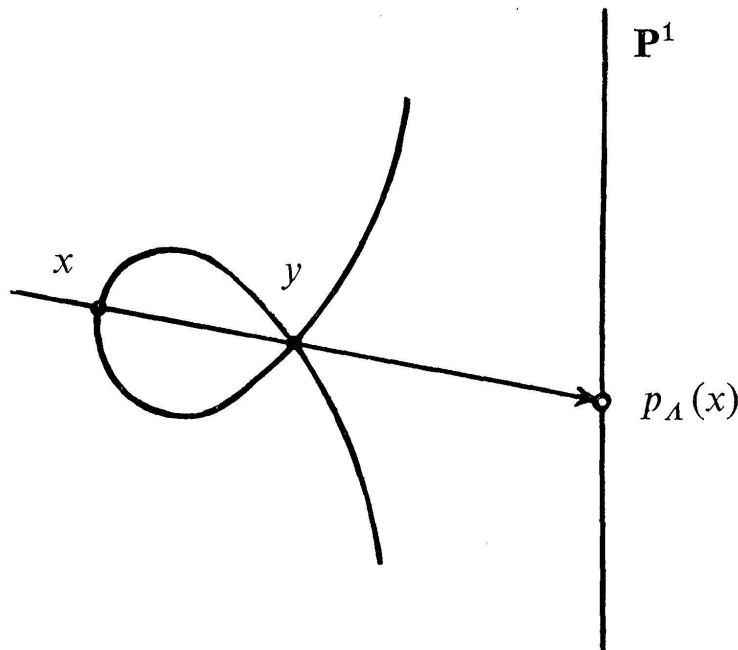
This proof brings out the geometry even more clearly. If  $H_1, \dots, H_r$  are generic hyperplanes in  $\mathbf{P}^r$  then

$$\deg(X) = \#(X \cap H_1 \cap \dots \cap H_r), \text{ (}\# \text{ denoting cardinality)}.$$

As the  $H_i$  specialize to hyperplanes  $H_i'$  of the form  $s = 0$ ,  $s \in \Lambda$  (remaining otherwise generic) the points in this intersection specialize to either:

- i) points outside  $Z$ : these points correspond to points in the intersection of  $\text{Im}(q)$  with  $r$  generic hyperplanes on  $\mathbf{P}^n$ , and each of these is the specialization of  $\deg q$  of the original points i.e.  $\deg p_A(X)$  points specialize in this way
- ii) points in  $Z$ :  $e_L(\mathcal{I}_A)$  measures the number of points which specialize in this way.

For example, if  $X^1 \subset \mathbf{P}^2$  is a curve of degree  $d$ ,  $y = (0, 0, 1)$  is on  $X$  and  $\Lambda = kX_0 + kX_1$ , then  $|Z| = \{y\}$ ,  $p_A(x_0, x_1, x_2) = (x_0, x_1)$  and the picture is:





Thus  $p_A(X) = (a\mathbf{P}^1)$ , where  $a$  is the degree of the covering  $p$ ; a generic line meets  $X$  in  $d$  points and as this line specializes to a non-tangent line through  $y$  it meets  $X$  at  $y$  on mult  $_y(X) = e_L(\mathcal{J}_A)$  points and meets  $X$  away from  $y$  in  $d - e_L(\mathcal{J}_A) = a$  points.

The following technical facts will be useful in calculating the the invariants  $e_L(\mathcal{J})$ .

PROPOSITION 2.6. a) If (in the situation of Proposition 2.1)  $L$  and  $\mathcal{J} \cdot L$  are generated by their sections then  $\left| h^0(L^n/\mathcal{J}^n L^n) - e_L(\mathcal{J}) \frac{n^r}{r!} \right| = O(n^{r-1})$ . (Thus we can calculate  $e_L(\mathcal{J})$  from the dimensions of spaces of sections.)

b) Suppose, in addition, we are given a diagram

$$\begin{array}{ccc} X & \supsetneq & X_0 = f^{-1}(0) \\ f \downarrow & & \downarrow \\ \text{Spec}(A) \ni & & 0 \end{array}$$

where  $f$  is proper, and a finite dimensional vector space  $W \subset \Gamma(X, \mathcal{J}L)$  which

- i) generates  $\mathcal{J} \cdot L$
- ii) defines a closed immersion  $X - X_0 \hookrightarrow \mathbf{P}(\hat{W})$

Then the dimensions of the kernel and cokernel of the map

$(\Gamma(X, L^n)/A\text{-submodule generated by the image of } W^{\otimes n} \rightarrow \Gamma(L^n/\mathcal{J}^n L^n))$  are both  $O(n^{r-1})$ .

*Proof.* The idea in a) is to show that  $h^i(L^n/\mathcal{J}^n \cdot L^n) = O(n^{r-1})$ ,  $i \geq 1$ . We first remark that is a compactification  $\bar{X}$  of  $X$  over which  $L$  extends to a line bundle  $\bar{L}$  such that

- i)  $\bar{L}$  is generated by its sections
- ii) some  $W \subset \Gamma(X, L)$  which generates  $\mathcal{J} \cdot L$  extends to a  $\bar{W} \subset \Gamma(\bar{X}, \bar{L})$ .

Indeed, on any compactification  $\bar{X}$ , there exists a coherent sheaf  $\bar{\mathcal{F}}$  such that  $\bar{\mathcal{F}}|_X \cong L$  and  $\bar{\mathcal{F}}$  has properties i) and ii), and the pullback of  $\bar{\mathcal{F}}$  to the blow-up  $B_{\bar{\mathcal{F}}_1}(\bar{X})$  is a line bundle with these properties: so we might as well replace  $\bar{X}$  by  $B_{\bar{\mathcal{F}}}(\bar{X})$ . Then if we take an ideal sheaf  $\bar{\mathcal{J}}$  such that  $\bar{W}$  generates  $\bar{\mathcal{J}} \cdot \bar{L}$ ,  $\bar{\mathcal{J}} = \mathcal{J} \cdot \mathcal{J}'$  where  $\mathcal{J}'$  is supported on  $\bar{X} - X$  only, and it suffices

to show  $h^i(\bar{L}^n/\mathcal{J}^n\bar{L}^n) = O(n^{r-1})$   $i \geq 1$  since  $\bar{L}^n/\mathcal{J}^n\bar{L}^n \cong \bar{L}^n/\mathcal{J}^n\bar{L}^n \oplus \bar{L}^n/\mathcal{J}'^n\bar{L}^n$  so this bounds  $h^i(L^n/\mathcal{J}^nL^n)$ . To do this, it suffices, in turn, to bound  $h^i(\bar{X}, \bar{L}^n)$  and  $h^i(\bar{X}, \mathcal{J}^n \cdot \bar{L}^n) = h^i(B_{\bar{\mathcal{J}}}(\bar{X}), \bar{L}(-\bar{E})^{\otimes n})$  (where  $E$  is the exceptional divisor on  $B_{\bar{\mathcal{J}}}(\bar{X})$ ). These bounds follow from:

LEMMA 2.7. *If  $X^r$  is proper over  $k$  and  $L$  is a line bundle on  $X$  generated by its sections, then  $h^i(L^{\otimes n}) = O(n^{r-1})$ ,  $i \geq 1$ .*

*Proof.* Let  $X_0$  be the image of  $X$  in  $\mathbf{P}^n$  under the map given by the sections of  $L$ . Then  $L = \pi^*(\mathcal{O}_{X_0}(1))$  and

$$\begin{aligned} H^i(X, L^{\otimes n}) &= H^i(X, \pi^*(\mathcal{O}_{X_0}(n))) \\ &\cong H^0(X_0, (R^i\pi_*\mathcal{O}_{X_0}) \otimes \mathcal{O}_{X_0}(n)) \\ &\text{for } n \text{ large.} \end{aligned}$$

The last isomorphism follows from first applying the Leray spectral sequence, and then noting that all the terms involving higher cohomology groups vanish for large  $n$ , by the ampleness of  $\mathcal{O}_{X_0}(1)$ . But if  $p \in \text{Supp } R^i\pi_*\mathcal{O}_{X_0}$  for  $i \geq 1$ , the fibre  $\pi^{-1}(p)$  has positive dimension, hence  $\dim \text{Supp } R^i\pi_*\mathcal{O}_{X_0} \leq r-1$  which gives the desired  $O(n^{r-1})$  bound on the dimension of the last space.

A suitable compactification and an argument like that in the proof of a), reduce the part of the statement of b) about the cokernel to bounding an  $h^1(\mathcal{J}^n \cdot L^n)$  and this is accompanied as in a) by a blow-up and the lemma. The procedure for dealing with the kernel is somewhat different: What we want to control is the dimension

$$(H^0(\mathcal{J}^n L^n)/A\text{-submodule generated by the image of } W^{\otimes n})$$

That is to say, for  $n \geq 0$ , the dimension of:

$$(H^0(B(X), \pi^*L^n(-nE))/A\text{-submodule generated by image of } W^{\otimes n})$$

Let  $B = B_{\mathcal{J}}(X)$  and  $q$  be the proper, birational map  $B \xrightarrow{q} B' \subset \mathbf{P}^n \times \text{Spec } A$  induced by  $W$ . Then  $q^*(\mathcal{O}_{B'}(1)) = \pi^*L(-E)$  and for large  $n$ , we have

$$H^0(B, L^n(-nE)) \cong H^0(B', q_*(\mathcal{O}_B) \otimes \mathcal{O}_{B'}(n))$$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ \left[ \begin{array}{l} A\text{-submodule} \\ \text{generated by} \\ \text{the image of } W^{\otimes n} \end{array} \right] & \cong & H^0(B', \mathcal{O}_{B'}(n)) \end{array}$$

The cokernel of the inclusion on the right is just  $H^0(B', q_*(\mathcal{O}_B)/\mathcal{O}_{B'}(n))$ . But the support of this last sheaf is proper over  $0 \in \text{Spec } A$ , hence of dimension less than  $r$ , so a final application of the lemma completes the proof.

2.8. Fix :  $X^r \subset \mathbf{P}^n$  a projective variety,

$X_0, \dots, X_n$  coordinates on  $\mathbf{P}^n$ ,

$\Phi_X$  the Chow form of  $X$ ,

$$\lambda(t) = \begin{bmatrix} t^{\rho_0} & & 0 \\ & \ddots & \\ 0 & & t^{\rho_n} \end{bmatrix} \dots t^{-k}, \quad \rho_0 \geq \rho_1 \geq \dots \geq \rho_n \geq 0,$$

$k$  chosen so that this is a 1-PS of  $SL(n+1)$ , i.e.  $k = -\sum \rho_i / n + 1$ .

We define an ideal sheaf  $\mathcal{I} \subset \mathcal{O}_{X \times \mathbf{A}^1}$  by

$$\mathcal{I} \cdot [\mathcal{O}_X(1) \otimes \mathcal{O}_{\mathbf{A}^1}] = \text{subsheaf generated by } \{t^{\rho_i} X_i\}, \quad i = 0, \dots, n.$$

REMARKS. i) From an examination of the generators of  $\mathcal{I}$ , one sees that the support of the subscheme  $Z = \mathcal{O}_{X \times \mathbf{A}^1} / \mathcal{I}$  is concentrated over  $0 \in \mathbf{A}^1$ ; if we normalize the  $\rho_i$  so that  $\rho_n = 0$  then the support of  $\mathcal{I}$  also lies over the section  $X_n = 0$  in  $X$ .

ii) Consider the weighted flag:

$$(X_1 = \dots = X_n = 0) \subset (X_2 = \dots = X_n = 0) \subset \dots \subset (X_n = 0)$$

||

||

||

$L_0$

$L_1$

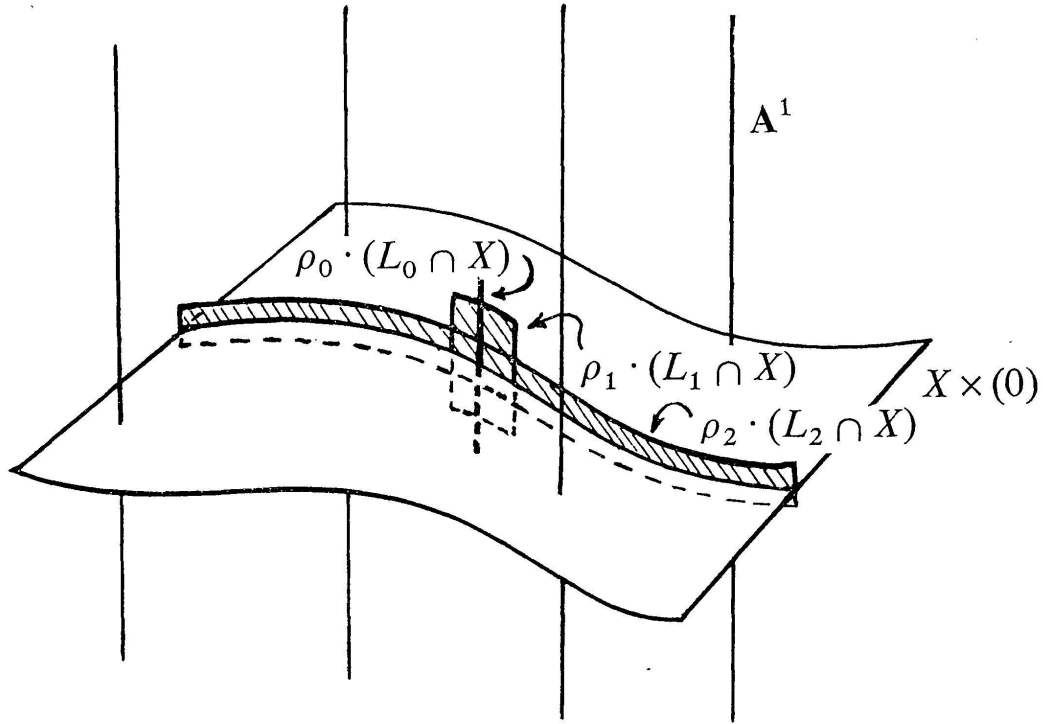
$L_{n-1}$

weight  $\rho_0$

weight  $\rho_1$

weight  $\rho_{n-1}$

The subscheme  $Z$  looks roughly like a union of  $\rho_i^{\text{th}}$ -order normal neighborhoods of  $L_i \cap X$ . It is easily seen to depend only on the weighted flag and not on the splitting defined by  $\lambda$ .



iii) Roughly speaking,  $e_{\mathcal{O}_{A^1} \otimes \mathcal{O}_X(1)}(\mathcal{F})$ , which we will denote  $e(\mathcal{F})$  measures the degree of contact of this weighted flag with  $X^1$ . The multiplicity of  $\mathcal{F}$  can be expected to get bigger, for example, if  $L_0$  becomes a more singular point of  $X$  or if  $L_{n-1}$  oscillates to  $X$  to higher degree. The main theorem of this chapter makes this more precise:

**THEOREM 2.9.** *In the situation of 2.8,  $\Phi_X$  is stable (resp.: semi-stable) with respect to  $\lambda$  if and only if:*

$$e(\mathcal{F}) < \frac{(r+1) \deg X}{n+1} \cdot \sum_{i=0}^n \rho_i$$

$$\left( \text{resp.: } e(\mathcal{F}) \leq \frac{(r+1) \deg X}{n+1} \cdot \sum_{i=0}^n \rho_i \right)$$

*Proof.* We begin with a definition.

**DEFINITION 2.10.** *If  $\mu: \mathbf{G}_m \rightarrow GL(W)$  is a representation of  $\mathbf{G}_m$  and  $W_i$  is the eigenspace where  $\mathbf{G}_m$  acts by the character  $t^i$ , then the  $\mu$ -weight of  $W$  is  $\sum_{i=-\infty}^{\infty} i \cdot \dim W_i$ . If  $w \in W_i$  then we say  $i$  is the  $\mu$ -weight of  $w$ .*

<sup>1)</sup> It seems to be a general fact of life that one must go up to some  $(r+1)$  dimensional variety—here  $X \times A^1$ —to measure such a contact on an  $r$ -dimensional variety.

1) THE LIMIT CYCLE. If  $X^{\lambda(t)}$  is the image of  $X$  by  $\lambda(t)$ , then taking  $\lim_{t \rightarrow 0} X^{\lambda(t)}$  gives a scheme  $X^{\lambda(0)}$  and an underlying cycle  $\tilde{X}$ , both of which are fixed by  $\lambda$ . Moreover,  $\Phi_{X^{\lambda(t)}} = (\Phi_X)^{\lambda(t)}$  so if  $\Phi_X = \sum_{i=a}^b \Phi_{X,i}$  where  $\Phi_{X,i}$  is the component of  $\Phi_X$  in the  $i^{th}$  weight space; then

$$\begin{aligned}\Phi_{X^{\lambda(t)}} &= \sum_{i=a}^b t^i \Phi_{X,i} \\ &= t^a [\Phi_{X,a} + t (\text{other terms})]\end{aligned}$$

Hence,  $\Phi_{\tilde{X}} = \Phi_{X,a}$  and  $a$  is the  $\lambda$ -weight of  $\Phi_{\tilde{X}}$ . By definition,  $\Phi_X$  is stable (resp: semi-stable) with respect to  $\lambda$  if and only if  $a < 0$  (resp:  $a \leq 0$ ) or equivalently if and only if the  $\lambda$ -weight of  $\Phi_{\tilde{X}}$  is  $< 0$  (resp:  $\leq 0$ ).

2) The next step is to connect this weight with a Hilbert polynomial; this is done by:

PROPOSITION 2.11. Let  $V^r \subset \mathbf{P}$  be fixed by a 1-PS  $\lambda$  of  $SL(n+1)$ , let  $I$  be the homogeneous ideal of  $V$  and let  $R_n = (k[x_0, \dots, X_n]/I)_n$  (i.e.  $V = \text{Proj}(\bigoplus_{n=0}^{\infty} R_n)$ ). Let  $a_V$  be the  $\lambda$ -weight of  $\Phi_V$  and  $r_n^V$  be the  $\lambda$ -weight of  $R_n$ . Then for large  $n$ ,  $r_n^V$  is represented by a polynomial in  $n$  of degree at most  $(r+1)$  with n.l.c.  $a_V$ .

*Proof.* a) Assume  $V$  is linear. In suitable coordinates, we can write

$$V = V(X_{r+1}, \dots, X_n) \text{ and } \lambda(t) = \begin{bmatrix} t^{a_0} & & 0 \\ & \ddots & \\ 0 & & t^{a_n} \end{bmatrix}. \text{ Then in the notation}$$

of 1.16, the Chow form of  $V$  is the monomial

$$\Phi_V = \det(U_i^{(j)}), \quad i, j = 0, \dots, n.$$

Hence  $\Phi_{\tilde{V}} = \Phi_V$  and has weight  $\sum_{i=0}^r a_i$ . On the other hand the  $\lambda$ -weight of  $R_n$  depends only on  $a_0 \dots a_r$ , is symmetric in these weights, and is linear in the vector  $(a_0, \dots, a_r)$ , hence depends only on  $\sum_{i=0}^r a_i$ . By considering the case  $a_0 = \dots = a_r$  we see that

$$r_n^V = \frac{n}{r+1} \left( \sum_{i=0}^r a_i \right) \dim R_n = a_V \cdot \frac{n}{r+1} \cdot \binom{n}{r}$$

which is certainly of the form claimed.

b)  $V$  is a positive cycle of linear spaces. Here it is more convenient to consider the ideal  $I$  instead of  $V$ . By noetherian induction, we can suppose the claim proven for all  $\lambda$ -fixed ideals  $I' \supsetneq I$ . Then if  $V = \sum a_i L_i$ , let  $J_1$  be the ideal of  $L_1$ , and choose an  $a \in k[X] - I$  which is a  $\lambda$ -eigenvector of weight, say,  $w$  and such that  $J_1 a \subset I$ . Now look at the exact sequence:

$$0 \rightarrow a + I/I \rightarrow k[x]/I \rightarrow k[x]/I + a \rightarrow 0$$

The claim is true for  $I + a$  by the noetherian induction. If  $I' = \{f \mid af \in I\} \supset J_1 \supsetneq I$ , then via the shift of weights by  $w$ ,  $a + I/I \cong k[x]/I'$ ; but this shift changes the  $\lambda$ -weight by an amount  $w$ .  $\dim[(k[x]/I')_n] = O(n^r)$ , hence does not affect the leading coefficient of the  $\lambda$ -weight. The claim for  $I'$ , which also follows from the noetherian induction, thus proves the claim for  $I$ .

c) Reduction to case b). Recall the Borel fixed point theorem: if  $G$  is a connected solvable algebraic group acting on a projective variety  $W$ , then there is a fixed point on  $\overline{O^G(y)}$  for every  $y \in W$ . Let  $[V]$  be the associated point of  $V$  in  $\text{Hilb}_{\mathbf{P}^n}$  and consider the orbit of  $[V]$  under the action of a maximal torus  $T \subset SL(n+1)$  containing  $\lambda(t)$ . Let  $[V_0]$  be a  $T$ -invariant point in  $\overline{O^T([V])}$ . Then  $V_0$  is a sum of linear spaces, since these are the only  $T$ -invariant subvarieties of  $\mathbf{P}^n$ . If we decompose  $\Phi_V$  by  $\Phi_V = \sum_{\alpha} \Phi_V^{\alpha}$ ,

where  $\alpha$  runs over the characters of  $T$  and  $\Phi_V^{\alpha}$  is the part of  $\Phi_V$  on which  $T$  acts with weight  $\alpha$ , then for any  $\tau \in T$ ,  $\Phi_V^{\tau} = \sum_{\alpha} c_{\alpha}^{\tau} \Phi_V^{\alpha}$  for suitable constants  $c_{\alpha}^{\tau}$ . Since  $\Phi_{V_0}$  is both  $T$ -invariant and a limit of forms  $\Phi_V^{\tau}$ ,  $\tau \in T$ ,  $\Phi_{V_0} = \Phi_V^{\alpha}$  for some  $\alpha$ . Moreover since  $V$  is a  $\lambda$ -invariant point, all the characters  $\alpha$  appearing in the decomposition of  $\Phi_V$  must have the same value on  $\lambda$ , hence the  $\lambda$ -weight of  $\Phi_{V_0}$  is the  $\lambda$ -weight of  $\Phi_V$ .

It remains only to compare the homogeneous coordinate rings. Now  $V$  and  $V_0$  are members of a flat family  $V_t$ ,  $t \in S$  for some connected parameter space  $S$ , so that if  $n \gg 0$ ,  $H^0(V_t, \mathcal{O}_{V_t}(n))$  are the fibres of a vector bundle over  $S$ . This means that the  $\lambda$ -action on these fibres varies continuously, hence that the  $\lambda$ -weights of all the fibres are equal. Now the claim for  $V$  follows from b).

REMARK. The relation between Chow forms and Hilbert points in c) is really much more general: in fact, Knudsen [12] has shown that there is a canonical isomorphism of 1-dimensional vector spaces  $k \cdot \Phi_V \cong [(r+1)^{\text{st}} \text{ "differences" } \text{---formed via } \otimes \text{---of successive spaces in the sequence } A^{\dim R_n} R_n]$ , and it is possible to base the whole proof of 2.11 on this.

3) Next we will see how to obtain  $X^{\lambda(0)}$  by blowing up  $\mathcal{J}$ . Consider the map

$$\begin{aligned} \Lambda_1 : \mathbf{G}_m \times X &\rightarrow \mathbf{P} \\ (t, X) &\mapsto \lambda(t)(x). \end{aligned}$$

If the embedding of  $X$  is defined by  $s_0, \dots, s_n \in \Gamma[X, \mathcal{O}_X(1)]$  and the action of  $\lambda(t)$  is by  $(a_0, \dots, a_n) \mapsto (t^{r_0}a_0, \dots, t^{r_n}a_n)$  with  $r_0 \geq r_1 \geq \dots \geq r_n$  and  $\sum_{i=0}^n r_i = 0$  (i.e.  $(0, \dots, 0, 1)$  is an attractive fixed point and  $(1, 0, \dots, 0)$  is a repulsive fixed point), then  $\Lambda_1^*(X_1) = t^{r_i}s_i$ . Now  $t^{-\gamma}$  is a unit on  $\mathbf{G}_m \times X$ , so changing the identification  $\Lambda_1^*(\mathcal{O}_{\mathbf{P}^n}(1)) \cong \mathcal{O}_{\mathbf{G}_m} \otimes \mathcal{O}_X(1)$  by this unit we can assume  $\Lambda_1^*(X_1) = t^{\rho_i}s_i$  where  $\rho_i = r_i - \gamma$  is normalized as in 2.8 so that  $\rho_n \geq 0$ . Then  $\Lambda_1$  "extends" to a rational map  $\mathbf{A}^1 \times X \rightarrow \mathbf{P}^n$  which is defined by the section  $\{t^{\rho_i}s_i\} \in \Gamma(\mathbf{A}^1 \times X, p_2^*\mathcal{O}_X(1))$ .  $\mathcal{J}$  is just the ideal sheaf these generate in  $\mathcal{O}_{\mathbf{A}^1 \times X}$  and  $Z$  is just the set of base points of the rational map. Blowing up along  $\mathcal{J}$  gives the picture

$$\begin{array}{ccccc} & E & & B = B_{\mathcal{J}}(\mathbf{A}^1 \times X) & \\ & \text{exceptional} & \searrow & \swarrow \pi & \searrow \Lambda \\ & \text{divisor} & & \mathbf{A}^1 \times X & \mathbf{A}^1 \times \mathbf{P}^n \\ & & \swarrow p_2 & \searrow p_1 & \swarrow p_1 \\ X & & & \mathbf{A}^1 & \end{array}$$

where the morphism  $\Lambda$  is defined by the sections  $\{t^{\rho_i}s_i\}$  in  $\Gamma[B, (p_2\pi)^*(\mathcal{O}(1))(-E)]$ . Now  $\text{Im}(\Lambda)$  is the closed subscheme of  $\mathbf{A}^1 \times \mathbf{P}^n$  given by  $\text{Proj}(\bigoplus_{m=0}^m R_m)$  where

$$(2.12) \quad R_m = \left[ \begin{array}{l} k[t]\text{-submodule of } \Gamma(X, \mathcal{O}(m)) \otimes_k k[t] \\ \text{generated by } m^{\text{th}} \text{ degree monomials in } \{t^{\rho_i} s_i\} \end{array} \right]$$

In fact,  $\text{Im } \Lambda$  is flat over  $\mathbf{A}^1$ , because of:

LEMMA 2.13. *Let  $S$  be a non-singular curve,  $X$  flat over  $S$  and  $f: X \rightarrow Y$  be a proper map over  $S$ . Then the scheme  $(f(X), \mathcal{O}_Y/\ker f^*)$  is flat over  $S$ .*

*Proof.* We may as well suppose  $S = \text{Spec } R$ ; and then this amounts to showing the  $\mathcal{O}_Y/\ker f^*$  has no  $R$ -torsion: if  $a \in \mathcal{O}_Y/\ker f^*$ ,  $r \in R$ , then  $r \cdot a = 0 \Rightarrow r \cdot f^* a = 0 \Rightarrow f^* a = 0 \Rightarrow a = 0$ .

In particular, we see that  $X^{\lambda(0)}$  is the fibre of  $\text{Im } \Lambda$  over  $t = 0$ , i.e.  $X^{\lambda(0)} = \text{Proj} \left( \bigoplus_{m=0}^{\infty} R_m/tR_m \right)$ .

4) The proof is completed by making precise the relation between  $\mathcal{J}$  and the  $\lambda$ -weight of  $\Phi_{\tilde{X}}$ . One must be careful however because there are two  $\mathbf{G}_m$ -actions on  $R_m/tR_m$ , that given by the identification  $R_1/tR_1 = \bigoplus (t^r s_i) k$ , which is just  $\lambda$ , and that given by the identification  $R_1/tR_1 = \bigoplus (t^{\rho_i} s_i) k$ ; call this action  $\mu$ . The weights of  $\mu$  on  $R_m/tR_m$  are just those of  $\lambda$  translated by  $m\gamma$ . By Proposition 2.11

$$\begin{aligned} \lambda\text{-weight of } \Phi_{\tilde{X}} &= \text{n.l.c. } (\lambda\text{-weight of } R_m/tR_m) \\ &= \text{n.l.c. } (\mu\text{-weight of } R_m/tR_m + \gamma m \dim(R_m/tR_m)) \\ &= \text{n.l.c. } (\mu\text{-weight of } R_m/tR_m) - \left( \frac{r+1}{n+1} \deg X \sum_{i=0}^n \rho_i \right) \end{aligned}$$

using  $\gamma = -\frac{1}{n+1} \sum \rho_i$  and

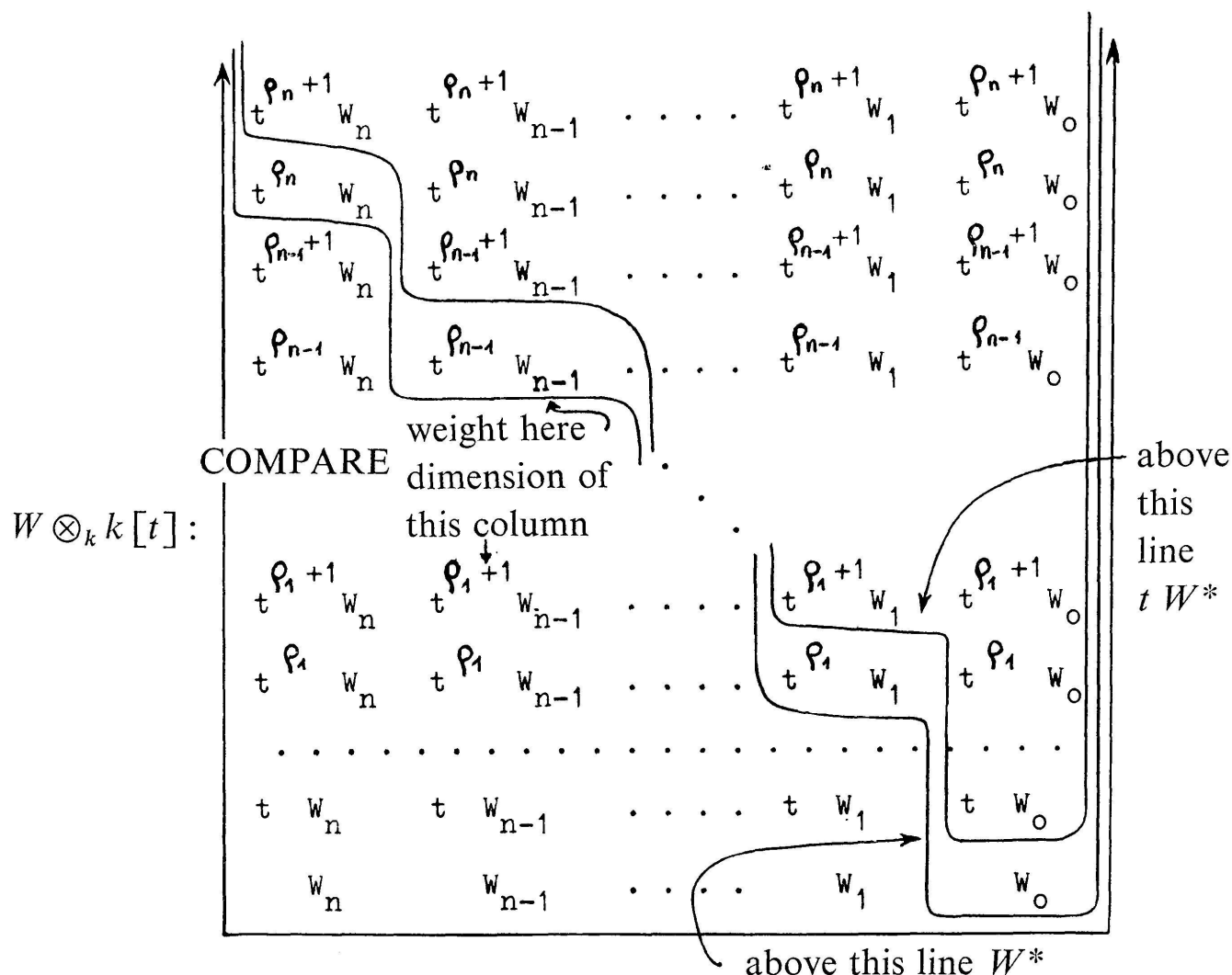
$$\begin{aligned} \dim(R_m/tR_m) &= (\deg X_{\lambda(0)}) \frac{m^r}{r!} + \text{lower terms} \\ &= \frac{(\deg X) m^r}{r!} + \text{lower terms.} \end{aligned}$$

A droll lemma allows us to re-express the  $\mu$ -weight of  $R_m/tR_m$ .

LEMMA 2.14. *Let  $W$  be a  $k$ -vector space and let  $\mathbf{G}_m$  act by  $\mu$  on  $W$  with weights  $\rho_n \geq \rho_{n-1} \dots \geq \rho_0 = 0$ . Let  $W_i$  be the eigenspace of weight  $\rho_i$  and let  $W^*$  be the  $k[t]$ -submodule of  $W \otimes k[t]$  generated by  $\bigoplus t^{\rho_i} W_i$ . Then  $\dim(k[t] \otimes W/W^*) = \mu\text{-weight of } W^*/tW^*$ .*



*Proof by Diagram :*



Recalling the definition of  $R_m$  (2.12), and applying this to the  $\mu$ -action on  $R_m/tR_m$ , we see that the  $\mu$ -weight of  $R_m/tR_m$  is just:  $\dim(\Gamma(X, \mathcal{O}(m)) \otimes_k k[t]/R_m)$ . But the sections  $\{t^{\rho_i} s_i\}$  whose  $m^{\text{th}}$  tensor powers generate  $R_m$ , also generate  $\mathcal{S} \cdot p_2^*(\mathcal{O}_{X(1)})$  so by a) and b) of Proposition 2.6, this last dimension can be used to calculate  $e(\mathcal{S})$ . Putting all this together, we see that:

$\Phi_X$  is stable with respect to  $\lambda$

$$\Leftrightarrow \lambda\text{-weight of } \Phi_X < 0$$

$$\Leftrightarrow e_L(\mathcal{S}) - \frac{(r+1)}{(n+1)} \deg X \sum_{i=0}^n \rho_i < 0$$

which, with the analogous statement for semi-stability, is our theorem.

2.15. INTERPRETATION VIA REDUCED DEGREE. If  $X^r \subset \mathbf{P}^n$  is a variety, its reduced degree is defined to be:

$$\text{red. deg } (X) = \frac{\deg X}{n + 1 - r}$$

A very old theorem says that if  $X$  is not contained in any hyperplane then  $\text{red. deg } (X) \geq 1$ . Reduced degree measures, in some sense, how complicatedly  $X$  sits in  $\mathbf{P}^n$ , and there are classical classifications of varieties with small reduced degree. For example if  $X$  has reduced degree 1 and is not contained in any hyperplane then  $X$  is either

- a) a quadric hypersurface
- b) the Veronese surface in  $\mathbf{P}^5$  or a cone over it
- c) a rational scroll:  $X = \mathbf{P} \left( \bigoplus_{i=0}^r \mathcal{O}_{\mathbf{P}^1}(n_i) \right) \subset \mathbf{P}^N$ ,  $n_i > 0$

where  $N = \sum_{i=0}^r (n_i + 1) - 1$ , or a cone over it. (This is called a scroll because the fibres  $\mathbf{P}^{r-1}$  of  $X$  over  $\mathbf{P}_1$  are linearly embedded.)

Some other facts about reduced degree are:

- i) canonical curves, K3-surfaces and Fano 3-folds have  $\text{red. deg} = 2$ ;
- ii) all non-ruled surfaces and all special curves have  $\text{red. deg} \geq 2$ . (For special curves, this is just a restatement of Clifford's theorem.)
- iii) for ample  $L$  on  $X^r$ , the embedding by  $L^{\otimes r}$  has reduced degree asymptotic to  $r!$  as  $n \rightarrow \infty$ ;
- iv) red-deg is preserved under taking of proper hyperplane sections.

It would be very interesting to know whether almost all 3-folds (in a sense similar to that of ii) for surfaces) have  $\text{red. deg} \geq 2 + \varepsilon$ . The following definition is introduced only tentatively as a means of linking the present ideas to older ideas (e.g. Albanese's method to simplify singularities of varieties):

2.16. DEFINITION. *A variety  $X^r \subset \mathbf{P}^n$  is linearly stable (resp. linearly semi-stable) if, whenever  $L^{n-m-1} \subset \mathbf{P}^n$  is a linear space such that the image cycle  $p_L(X)$  of  $X$  under the projection  $p_L : \mathbf{P}^n - L \rightarrow \mathbf{P}^m$  has dimension  $r$ , then  $\text{red deg } p_L(X) > \text{red deg } X$  (resp.  $\text{red-deg } p_L(X) \geq \text{red deg } X$ ).*

Attention:  $p_L$  is allowed to be finite to 1, and which case  $p_L(X)$  must be taken to be the image cycle. Linear stability is a property of the linear system embedding  $X$ ; if  $X^r \subset \mathbf{P}^n$  is embedded by  $\Gamma(X, L)$ , then  $X$  linearly stable means that for all subspaces  $\Lambda \subset \Gamma(X, L)$

$$\frac{\deg p_L(X)}{\dim \Lambda - r} > \frac{\deg X}{n + 1 - r}$$

or equivalently, by applying Proposition 2.5,

$$e(\mathcal{I}_\Lambda) < \frac{\deg X}{n + 1 - r} (\text{codim } \Lambda)$$

EXAMPLES. i) when  $X$  is a curve of genus 0, it is linearly semi-stable but not stable. When  $g \geq 1$ , Clifford's theorem shows that  $X$  is linearly stable whenever it is embedded by a complete non-special linear system (see § 4 below).

ii)  $\mathbf{P}^2$  is linearly unstable when embedded by  $\mathcal{O}(n)$ ,  $n \geq 3$  because it projects to the Veronese surface. In view of the next proposition, a very interesting problem is that of finding large classes of linearly (semi)-stable surfaces.

(It may, however, turn out that linear stability is really too strong, or unpredictable, a property for surfaces in which case this Proposition is not very interesting !)

PROPOSITION 2.17. Fix  $X^r \subset \mathbf{P}^n$ , let  $C$  be any smooth curve and let  $L$  be an ample line bundle on  $C$ . Let  $\Phi_i : C \times X \rightarrow \mathbf{P}^{N(i)}$  be the embedding defined by  $\{S_j \otimes X_l\}$  where  $\{S_j\}$  is a basis of  $\Gamma(L^{\otimes i})$  and  $X_l \in \Gamma(X, \mathcal{O}_X(1))$  are the homogeneous coordinates. If  $\Phi_i(C \times X)$  is linearly semi-stable for all large  $i$ , then  $X^r$  is Chow-semi-stable.

*Proof.* Choose a 1-PS:  $\lambda(t) = \begin{bmatrix} t^{\rho_0} & & 0 \\ & \ddots & \\ 0 & & t^{\rho_n} \end{bmatrix} t^{-\frac{\sum \rho_i}{n+1}}$

as in (2.8).

Choose a point  $p \in C$  an isomorphism  $L_p \cong \mathcal{O}_p$  and an  $i$  large enough that  $L^{\otimes i}$  is very ample and  $L^{\otimes i}(-\rho_0 p)$  is non-special. Then the map

$$\bigoplus_{l=1}^n \Gamma(C, L^{\otimes i}) \cdot X_l \xrightarrow{\Phi_i} \bigoplus_{l=0}^n [\mathcal{O}_{p,C} / \mathcal{M}_{p,C}^{\rho_0}] \cdot X_l$$

is surjective. Let  $\Lambda^i$  be the inverse image of  $\bigoplus_{l=0}^n [(\mathcal{M}_{p,C}^{\rho_l} / \mathcal{M}_{p,C}^{\rho_0}) \cdot X_l]$  under this map and let  $\mathcal{I}_\Lambda^i \subset \mathcal{O}_{C \times X}$  be the induced ideal. Since all the  $L^{\otimes i}$  are trivial near  $p$  and  $\mathcal{I}_\Lambda^i$  has support on the fibre of  $X \times C$  over  $P$ , the ideals

$\mathcal{I}_A^i$  are independent of  $i$ ; we denote this ideal by  $\mathcal{I}_A$ . The hypothesis says that for large  $i$

$$\begin{aligned} e(\mathcal{I}_A) &\leq \frac{\deg(C \times X)}{(n+1)(h^0(L^i) - r - 1)} \operatorname{codim} A \\ &= \frac{(r+1) \deg X \deg L^{\otimes i}}{(n+1)(\deg L^{\otimes i} - g + 1) - r - 1} \cdot \sum_{l=0}^n \rho_l \end{aligned}$$

and letting  $i \rightarrow \infty$ ,

$$e(\mathcal{I}_A) \leq \frac{(r+1) \deg X}{n+1} \sum_{l=0}^n \rho_l$$

But  $C \times X$  along  $p \times X$  is formally isomorphic to  $\mathbf{A}^1 \times X$  along  $0 \times X$  with corresponding  $\mathcal{I}_A'$ s, so by Theorem 2.9.,  $X$  is Chow-semi-stable.

### § 3. EFFECT OF SINGULAR POINTS ON STABILITY

We begin with an application of Theorem 2.9.

**PROPOSITION 3.1.** *Let  $X^1 \subset \mathbf{P}^n$  be a curve with no embedded components such that  $\deg X/n+1 < 8/7$ . If  $X$  is Chow-semi-stable, then  $X$  has at most ordinary double points.*

**REMARKS.** i) When  $n = 2$ ,  $\deg X/n+1 < 8/7 \Leftrightarrow \deg X < 4$  and the proposition confirms what we have seen in 1.10 and 1.11

ii) Suppose  $L$  is ample on  $X^1$  and  $X_m \subset \mathbf{P}^{N(m)}$  is the embedding of  $X$  defined by  $\Gamma(X, L^{\otimes m})$ . By Riemann-Roch,  $\deg X_m/N(m) \rightarrow 1$  as  $m \rightarrow \infty$ , hence:

**COROLLARY 3.2.** *An asymptotically stable curve  $X$  has at most ordinary double points.*

In particular, if  $X \subset \mathbf{P}^2$  has degree  $\geq 4$  and has one ordinary cusp, then, in  $\mathbf{P}^2$ ,  $X$  is stable but when re-embedded in high enough space,  $X$  is unstable! The fact that this surprising flip happens was discovered by D. Gieseker and came as an amazing revelation to me, as I had previously assumed without proof the opposite.

iii) We will see in Proposition 3.14 that the constant  $8/7$  is best possible.

*Proof of 3.1.* We note first that a semi-stable  $X$  of any dimension cannot be contained in a hyperplane: if  $X \subset V(X_0)$ , then  $X$  has only positive weights with respect to the 1-PS