Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 23 (1977)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: STABILITY OF PROJECTIVE VARIETIES
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Kapitel: §2. A CRITERION FOR $X^r \subset P^n$ TO BE STABLE
DOI: https://doi.org/10.5169/seals-48919

§ 2. A criterion for $X^r \subset \mathbf{P}^n$ to be stable

If $f(a)$ is an integer-valued function which is represented by a rational polynomial of degree at most r in n for large n, we will denote by n.l.c. (f) (the normalized leading coefficient of f) the integer e for which $f(n) = e \frac{n^r}{r!}$ + lower order terms. (What r is to be taken, will always be clear from the context.)

Proposition 2.1 [1]). *(The "Hilbert-Hilbert-Samuel" Polynomial). Suppose X is a k-variety (not necessarily complete), L is an invertible sheaf on X and $\mathcal{I} \subset \mathcal{O}_X$ is an ideal sheaf such that $Z =$ Supp $\mathcal{O}_X/\mathcal{I}$ is proper over k. Then there is a polynomial $P(n, m)$ of total degree $\leq r$, such that, for large m*

$$\chi(L^n/\mathcal{I}^m L^n) = P(n, m).$$

Proof. We can compactify X and extend L to a line bundle on this compactification, without altering the validity of the theorem so we may as well assume X proper over k. Let $\pi: B \to X$ be the blow-up of X along $\mathcal{I}$ (i.e. $B = B_{\mathcal{I}}(X) = \operatorname{Proj}(\mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \ldots)$) and let E be the exceptional divisor on B so that $\mathcal{I} . \mathcal{O}_B = \mathcal{O}(-E)$. The well-known theorems of F.A.C. (Serre [18]) for the vanishing of higher cohomology in the relative case imply that when $m \gg 0$:

i) $\pi_*(\mathcal{O}(-mE)) = \mathcal{I}^m$

ii) $R^i\pi_*(\mathcal{O}(-mE)) = (0), \ i > 0$

Now examine the exact sequence:

$$0 \longrightarrow \mathcal{I}^m L^n \longrightarrow L^n \longrightarrow L^n/\mathcal{I}^m L^n \longrightarrow 0$$

The Hilbert polynomial for $\chi(L^n)$ certainly satisfies the conditions on P. Moreover, in view of i) and ii); we have for $m \gg 0$:

$$\chi(X, \mathcal{I}^m L^n) = \chi(B, \pi^* L^n(-mE)) = \chi(B, (\pi^* L)^{\otimes n} \otimes \mathcal{O}(-E)^{\otimes m})$$

so, a theorem of Snapper [5, 21] guarantees that this last Euler characteristic is also a polynomial of the required type for large m and n. By the additivity of χ we are done.

1) This result and its geometric interpretation are essentially due to C. P. Ramanujam [16].

DEFINITION 2.2. *In the situation of Proposition* 2.1, *we denote by* $e_L(\mathscr{I})$ *(the multiplicity of* $\mathscr{I}$ *measured via* L*) the integer* n.l.c. $(\chi(L^n/\mathscr{I}^n L^n))$.

EXAMPLES. i) If $\mathscr{I} = 0$ and X is complete, P is the Hilbert polynomial of L. ii) If Z is set-theoretically a point x then P is the Hilbert-Samuel polynomial of $\mathscr{I}$ as an ideal of $\mathcal{O}_{x,X}$ and $e(\mathscr{I})$ is its multiplicity there: in particular, it is independent of L. Note that, in general, $e_L(\mathscr{I})$ depends on the formal completion of X along Z and the pull-backs of $\mathscr{I}, L$ to this formal completion.

2.3. CLASSICAL GEOMETRIC INTERPRETATION. Let $X^r \subset \mathbf{P}^n$ be a projective variety, $L = \mathcal{O}_X(1)$, and Λ be a subspace of $\Gamma(\mathbf{P}^n, \mathcal{O}(1))$. Define L_Λ to be the linear subspace of $\mathbf{P}^n$ given by $s = 0$, $s \in \Lambda$. Define $\mathscr{I}_\Lambda$ to be the ideal sheaf generated by the sections $s \in \Lambda$, i.e. $\mathscr{I}_\Lambda . L$ is the subsheaf of L generated by those sections and $Z = \operatorname{Supp}(\mathcal{O}_X/\mathscr{I}_\Lambda) = X \cap L_\Lambda$ is the set of their base points.

If $p_\Lambda : \mathbf{P}^n - L_\Lambda \to \mathbf{P}(\Lambda) = \mathbf{P}^m$ is the canonical projection, and π is the blow-up of X along $\mathscr{I}_\Lambda$ then there is a unique map q making the following diagram commute:

$$\begin{array}{ccc} X - Z & \xrightarrow{\text{res } p_\Lambda} & \mathbf{P}^m \\ \cap & \cap & \nearrow q \\ X & \xleftarrow{\pi} & B = B_{\mathscr{I}_\Lambda}(X) \end{array}$$

Moreover, because sections of $\mathcal{O}_{\mathbf{P}^m}(1)$ pull back to sections of $\mathscr{I}_\Lambda . L$ on X and are blown-up to sections of L twisted by minus the exceptional divisor E,

$$q^*(\mathcal{O}_{\mathbf{P}^m}(1)) = (\pi^* L)(-E) . \qquad (2.4)$$

Define $p_\Lambda(X)$, the image of X by the projection p_Λ, to be $[\text{cycle}(q(B))]$: that is, $q(B)$ with multiplicity equal to the degree of B over $q(B)$ if these have the same dimension and 0 otherwise. I claim

PROPOSITION 2.5. $e_L(\mathscr{I}_\Lambda) = \deg X - \deg p_\Lambda(X)$.

Proof. If H is the divisor class of a hyperplane section on X, then

$$\deg X = (H^r) = \text{n.l.c.}(\chi(\mathcal{O}_X(n))) .$$

By 2.4, q is defined by the linear system of divisors of the form $\pi^{-1}(H) - E$, hence

$$\deg p_A(x) = ((\pi^{-1}(H) - E)^r) = \text{n.l.c. } \chi(\pi^*(\mathcal{O}(n)(-nE))) .$$

Finally, from its definition

$$\begin{aligned} e_L(\mathscr{I}_A) &= \text{n.l.c. } \chi(\mathcal{O}_X(n)/\mathscr{I}^n\mathcal{O}_X(n)) \\ &= \text{n.l.c. } \chi(\mathcal{O}_X(n)) - \text{n.l.c. } \chi(\mathscr{I}^n\mathcal{O}_X(n)) \\ &= \deg X - \deg p_A(X) \end{aligned}$$

This proof brings out the geometry even more clearly. If $H_1, \ldots, H_r$ are generic hyperplanes in $\mathbf{P}^r$ then

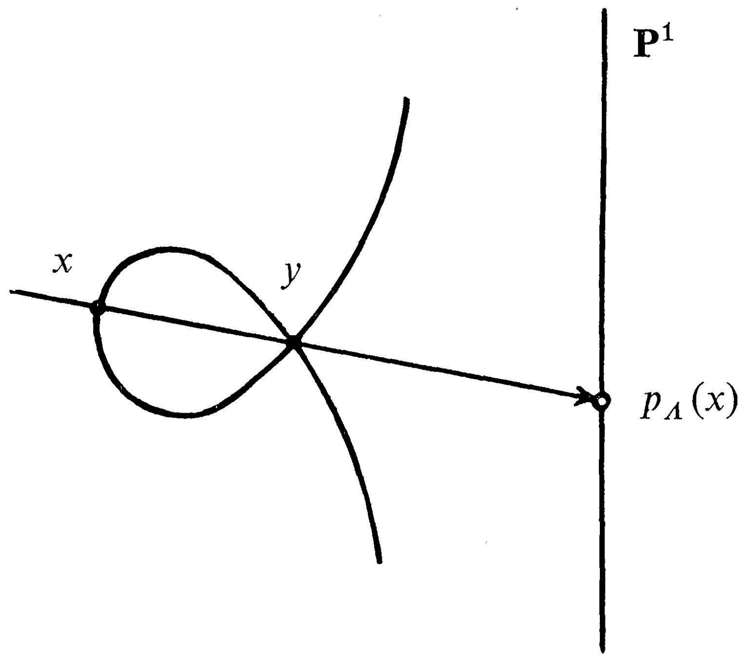
$$\deg(X) = \#(X \cap H_1 \cap \ldots \cap H_r), \ (\# \text{ denoting cardinality}) .$$

As the H_i specialize to hyperplanes H_i' of the form $s = 0$, $s \in A$ (remaining otherwise generic) the points in this intersection specialize to either:

i) points outside Z: these points correspond to points in the intersection of $\operatorname{Im}(q)$ with r generic hyperplanes on $\mathbf{P}^n$, and each of these is the specialization of $\deg q$ of the original points i.e. $\deg p_A(X)$ points specialize in this way

ii) points in Z: $e_L(\mathscr{I}_A)$ measures the number of points which specialize in this way.

For example, if $X^1 \subset \mathbf{P}^2$ is a curve of degree d, $y = (0, 0, 1)$ is on X and $A = kX_0 + kX_1$, then $|Z| = \{y\}$, $p_A(x_0, x_1, x_2) = (x_0, x_1)$ and the picture is:



Thus $p_A(X) = (a\mathbf{P}^1)$, where a is the degree of the covering p; a generic line meets X in d points and as this line specializes to a non-tangent line through y it meets X at y on $\mathrm{mult}_y(X) = e_L(\mathcal{I}_A)$ points and meets X away from y in $d - e_L(\mathcal{I}_A) = a$ points.

The following technical facts will be useful in calculating the the invariants $e_L(\mathcal{I})$.

PROPOSITION 2.6. a) *If (in the situation of Proposition 2.1) L and $\mathcal{I} . L$ are generated by their sections then* $\left| h^0(L^n/\mathcal{I}^n L^n) - e_L(\mathcal{I}) \dfrac{n^r}{r!} \right| = O(n^{r-1})$. *(Thus we can calculate $e_L(\mathcal{I})$ from the dimensions of spaces of sections.)*

b) *Suppose, in addition, we are given a diagram*

$$\begin{array}{ccc} X & \supsetneq & X_0 = f^{-1}(0) \\ f\big\downarrow & & \big\downarrow \\ \mathrm{Spec}(A) & \ni & 0 \end{array}$$

where f is proper, and a finite dimensional vector space $W \subset \Gamma(X, \mathcal{I} L)$ which

i) *generates $\mathcal{I} . L$*

ii) *defines a closed immersion $X - X_0 \hookrightarrow \mathbf{P}(\hat{W})$*

Then the dimensions of the kernel and cokernel of the map

$(\Gamma(X, L^n)/A$-submodule generated by the image of $W^{\otimes n} \to \Gamma(L^n/\mathcal{I}^n L^n)$

are both $O(n^{r-1})$.

Proof. The idea in a) is to show that $h^i(L^n/\mathcal{I}^n . L^n) = O(n^{r-1})$, $i \geq 1$. We first remark that is a compactification $\bar{X}$ of X over which L extends to a line bundle $\bar{L}$ such that

i) $\bar{L}$ is generated by its sections

ii) some $W \subset \Gamma(X, L)$ which generates $\mathcal{I} . L$ extends to a $\bar{W} \subset \Gamma(\bar{X}, \bar{L})$.

Indeed, on any compactification $\bar{X}$, there exists a coherent sheaf $\bar{\mathcal{F}}$ such that $\bar{\mathcal{F}}|_X \cong L$ and $\bar{\mathcal{F}}$ has properties i) and ii), and the pullback of $\bar{\mathcal{F}}$ to the blow-up $B_{\bar{\mathcal{F}}_1}(\bar{X})$ is a line bundle with these properties: so we might as well replace $\bar{X}$ by $B_{\bar{\mathcal{F}}}(\bar{X})$. Then if we take an ideal sheaf $\bar{\mathcal{I}}$ such that $\bar{W}$ generates $\bar{\mathcal{I}} . \bar{L}$, $\bar{\mathcal{I}} = \mathcal{I} . \mathcal{I}'$ where $\mathcal{I}'$ is supported on $\bar{X} - X$ only, and it suffices to show $h^i(\bar{L}^n/\mathcal{J}^n\bar{L}^n) = O(n^{r-1})$ $i \geqq 1$ since $\bar{L}^n/\mathcal{J}^n\bar{L}^n \cong \bar{L}^n/\mathcal{J}^n\bar{L}^n \oplus \bar{L}^n/\mathcal{J}'^n.\bar{L}^n$ so this bounds $h^i(L^n/\mathcal{J}^nL^n)$. To do this, it suffices, in turn, to bound $h^i(\bar{X}, \bar{L}^n)$ and $h^i(\bar{X}, \bar{\mathcal{J}}^n . \bar{L}^n) = h^i(B_{\bar{\mathcal{J}}}(\bar{X}), \bar{L}(-\bar{E})^{\otimes n})$ (where E is the exceptional divisor on $B_{\bar{\mathcal{J}}}(\bar{X})$). These bounds follow from:

LEMMA 2.7. *If X^r is proper over k and L is a line bundle on X generated by its sections, then* $h^i(L^{\otimes n}) = O(n^{r-1})$, $i \geqq 1$.

Proof. Let X_0 be the image of X in $\mathbf{P}^n$ under the map given by the sections of L. Then $L = \pi^*(\mathcal{O}_{X_0}(1))$ and

$$\begin{aligned} H^i(X, L^{\otimes n}) &= H^i(X, \pi^*(\mathcal{O}_{X_0}(n))) \\ &\cong H^0(X_0, (R^i\pi_*\mathcal{O}_{x_0}) \otimes \mathcal{O}_{X_0}(n)) \\ &\quad \text{for } n \text{ large.} \end{aligned}$$

The last isomorphism follows from first applying the Leray spectral sequence, and then noting that all the terms involving higher cohomology groups vanish for large n, by the ampleness of $\mathcal{O}_{X_0}(1)$. But if $p \in \operatorname{Supp} R^i\pi_*\mathcal{O}_{X_0}$ for $i \geqq 1$, the fibre $\pi^{-1}(p)$ has positive dimension, hence $\dim \operatorname{Supp} R^i\pi_*\mathcal{O}_{X_0} \leqq r - 1$ which gives the desired $O(n^{r-1})$ bound on the dimension of the last space.

A suitable compactification and an argument like that in the proof of a), reduce the part of the statement of b) about the cokernel to bounding an $h^1(\mathcal{J}^n . L^n)$ and this is accompanied as in a) by a blow-up and the lemma. The procedure for dealing with the kernel is somewhat different: What we want to control is the dimension

$$(H^0(\mathcal{J}^nL^n)/A\text{-submodule generated by the image of } W^{\otimes n})$$

That is to say, for $n \gg 0$, the dimension of:

$$(H^0(B(X), \pi^*L^n(-nE))/A\text{-submodule generated by image of } W^{\otimes n})$$

Let $B = B_{\mathcal{J}}(X)$ and q be the proper, birational map $B \xrightarrow{q} B' \subset \mathbf{P}^n \times \operatorname{Spec} A$ induced by W. Then $q^*(\mathcal{O}_{B'}(1)) = \pi^*L(-E)$ and for large n, we have

$$\begin{array}{ccc} H^0(B, L^n(-nE)) & \cong & H^0(B', q_*(\mathcal{O}_B) \otimes \mathcal{O}_{B'}(n)) \\ \bigcup\!\!\uparrow & & \uparrow\!\!\bigcup \\ \left[\begin{array}{l} A\text{-submodule} \\ \text{generated by} \\ \text{the image of } W^{\otimes n} \end{array}\right] & \cong & H^0(B', \mathcal{O}_{B'}(n)) \end{array}$$

The cokernel of the inclusion on the right is just $H^0(B', q_*(\mathcal{O}_B)/\mathcal{O}_{B'}(n))$. But the support of this last sheaf is proper over $0 \in \operatorname{Spec} A$, hence of dimension less than r, so a final application of the lemma completes the proof.

2.8. Fix : $X^r \subset \mathbf{P}^n$ a projective variety,

$X_0, \ldots, X_n$ coordinates on $\mathbf{P}^n$,

Φ_X the Chow form of X,

$$\lambda(t) = \begin{bmatrix} t^{\rho_0} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & t^{\rho_n} \end{bmatrix} \cdot t^{-k}, \quad \rho_0 \geqq \rho_1 \geqq \ldots \geqq \rho_n \geqq 0,$$

k chosen so that this is a 1-PS of $SL(n+1)$, i.e. $k = -\sum \rho_i/n+1$.

We define an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{X \times A^1}$ by

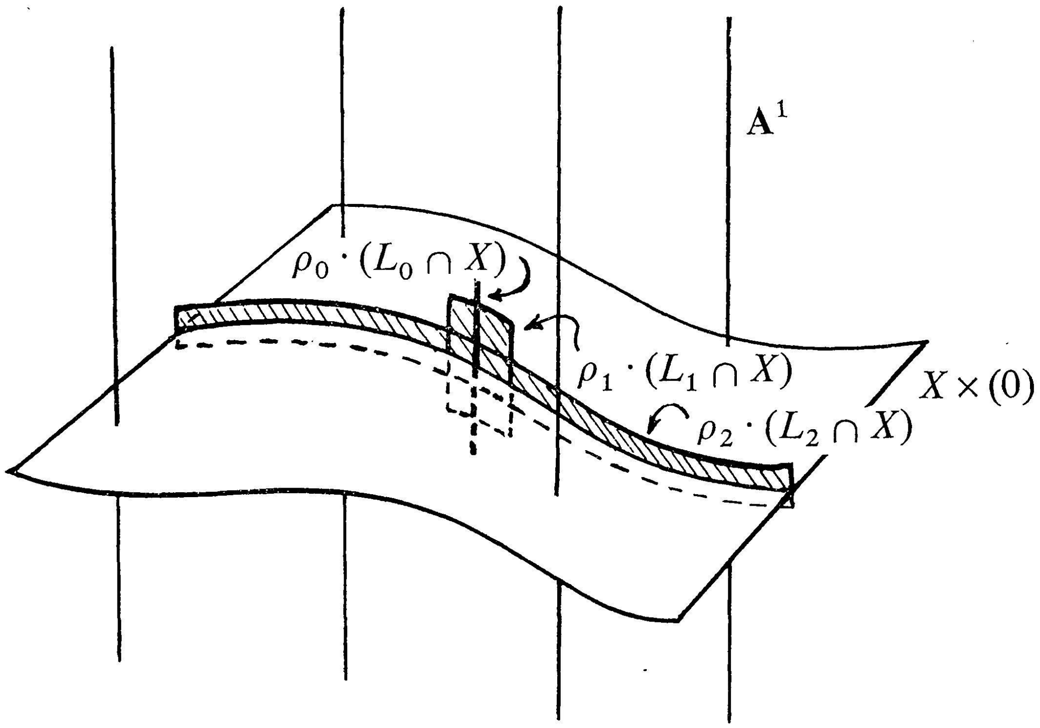
$$\mathcal{I} \cdot [\mathcal{O}_X(1) \otimes \mathcal{O}_{\mathbf{A}^1}] = \text{subsheaf generated by } \{t^{\rho_i} X_i\},\ i = 0, \ldots, n.$$

Remarks. i) From an examination of the generators of $\mathcal{I}$, one sees that the support of the subscheme $Z = \mathcal{O}_{X \times \mathbf{A}^1}/\mathcal{I}$ is concentrated over $0 \in \mathbf{A}^1$; if we normalize the ρ_i so that $\rho_n = 0$ then the support of $\mathcal{I}$ also lies over the section $X_n = 0$ in X.

ii) Consider the weighted flag:

$$\begin{array}{ccccc} (X_1 = \ldots = X_n = 0) & \subset & (X_2 = \ldots = X_n = 0) & \subset \ldots \subset & (X_n = 0) \\ \| & & \| & & \| \\ L_0 & & L_1 & & L_{n-1} \\ \text{weight } \rho_0 & & \text{weight } \rho_1 & & \text{weight } \rho_{n-1} \end{array}$$

The subscheme Z looks roughly like a union of ρ_i^{th}-order normal neighborhoods of $L_i \cap X$. It is easily seen to depend only on the weighted flag and not on the splitting defined by λ.



iii) Roughly speaking, $e_{\mathcal{O}_{\mathbf{A}^1} \otimes \mathcal{O}_X(1)}(\mathcal{I})$, which we will denote $e(\mathcal{I})$ measures the degree of contact of this weighted flag with X[1]. The multiplicity of $\mathcal{I}$ can be expected to get bigger, for example, if L_0 becomes a more singular point of X or if L_{n-1} oscillates to X to higher degree. The main theorem of this chapter makes this more precise:

THEOREM 2.9. *In the situation of* 2.8, Φ_X *is stable (resp.: semi-stable) with respect to* λ *if and only if:*

$$e(\mathcal{I}) < \frac{(r+1)\deg X}{n+1} \cdot \sum_{i=0}^{n} \rho_i$$

$$\left(\text{resp.: } e(\mathcal{I}) \leqq \frac{(r+1)\deg X}{n+1} \cdot \sum_{i=0}^{n} \rho_i\right)$$

Proof. We begin with a definition.

DEFINITION 2.10. *If* $\mu\colon \mathbf{G}_m \to GL(W)$ *is a representation of* $\mathbf{G}_m$ *and* W_i *is the eigenspace where* $\mathbf{G}_m$ *acts by the character* t^i, *then the* μ*-weight of* W *is* $\sum_{i=-\infty}^{\infty} i \,.\, \dim W_i$. *If* $w \in W_i$ *then we say* i *is the* μ*-weight of* w.

[1]) It seems to be a general fact of life that one must go up to some $(r+1)$ dimensional variety—here $X \times \mathbf{A}^1$—to measure such a contact on an r-dimensional variety.

1) THE LIMIT CYCLE. If $X^{\lambda(t)}$ is the image of X by $\lambda(t)$, then taking $\lim_{t\to 0} X^{\lambda(t)}$ gives a scheme $X^{\lambda(0)}$ and an underlying cycle $\tilde{X}$, both of which are fixed by λ. Moreover, $\Phi_{X^{\lambda(t)}} = (\Phi_X)^{\lambda(t)}$ so if $\Phi_X = \sum_{i=a}^{b} \Phi_{X,i}$ where $\Phi_{X,i}$ is the component of Φ_X in the i^{th} weight space; then

$$\begin{aligned}\Phi_{X^{\lambda(t)}} &= \sum_{i=a}^{b} t^i \, \Phi_{X,i} \\ &= t^a [\Phi_{X,a} + t \text{ (other terms)}]\end{aligned}$$

Hence, $\Phi_{\tilde{X}} = \Phi_{X,a}$ and a is the λ-weight of $\Phi_{\tilde{X}}$. By definition, Φ_X is stable (resp: semi-stable) with respect to λ if and only if $a < 0$ (resp: $a \leqq 0$) or equivalently if and only if the λ-weight of $\Phi_{\tilde{X}}$ is < 0 (resp: $\leqq 0$).

2) The next step is to connect this weight with a Hilbert polynomial; this is done by:

PROPOSITION 2.11. *Let* $V^r \subset \mathbf{P}$ *be fixed by a* 1-PS λ *of* $SL(n+1)$, *let* I *be the homogeneous ideal of* V *and let* $R_n = (k[x_0, \ldots, X_n]/I)_n$ (i.e. $V = \operatorname{Proj}(\bigoplus_{n=0}^{\infty} R_n)$). *Let* a_V *be the* λ*-weight of* Φ_V *and* r_n^V *be the* λ*-weight of* R_n. *Then for large* n, r_n^V *is represented by a polynomial in* n *of degree at most* $(r+1)$ *with* n.l.c. a_V.

Proof. a) Assume V is linear. In suitable coordinates, we can write

$V = V(X_{r+1}, \ldots, X_n)$ and $\lambda(t) = \begin{bmatrix} t^{a_0} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & t^{a_n} \end{bmatrix}$. Then in the notation of 1.16, the Chow form of V is the monomial

$$\Phi_V = \det(U_i^{(j)}), \quad i, j = 0, \ldots, n .$$

Hence $\Phi_{\tilde{V}} = \Phi_V$ and has weight $\sum_{i=0}^{r} a_i$. On the other hand the λ-weight of R_n depends only on $a_0 \ldots a_r$, is symmetric in these weights, and is linear in the vector $(a_0, \ldots, a_r)$, hence depends only on $\sum_{i=0}^{r} a_i$. By considering the case $a_0 = \ldots = a_r$ we see that

$$r_n^V = \frac{n}{r+1} \left(\sum_{i=0}^{r} a_i \right) \dim R_n = a_V \cdot \frac{n}{r+1} \cdot \binom{n}{r}$$

which is certainly of the form claimed.

b) V is a positive cycle of linear spaces. Here it is more convenient to consider the ideal I instead of V. By noetherian induction, we can suppose the claim proven for all λ-fixed ideals $I' \supsetneqq I$. Then if $V = \sum a_i L_i$, let J_1 be the ideal of L_1, and choose an $a \in k[X] - I$ which is a λ-eigenvector of weight, say, w and such that $J_1 a \subset I$. Now look at the exact sequence:

$$0 \to a + I/I \to k[x]/I \to k[x]/I + a \to 0$$

The claim is true for $I + a$ by the noetherian induction. If $I' = \{f \mid af \in I\}$ $\supset J_1 \supsetneqq I$, then via the shift of weights by w, $a + I/I \cong k[x]/I'$; but this shift changes the λ-weight by an amount $w \cdot \dim [(k[x]/I')_n]) = O(n^r)$, hence does not affect the leading coefficient of the λ-weight. The claim for I', which also follows from the noetherian induction, thus proves the claim for I.

c) Reduction to case b). Recall the Borel fixed point theorem: if G is a connected solvable algebraic group acting on a projective variety W, then there is a fixed point on $\overline{O^G(y)}$ for every $y \in W$. Let $[V]$ be the associated point of V in $\mathrm{Hilb}_{\mathbf{P}^n}$ and consider the orbit of $[V]$ under the action of a maximal torus $T \subset SL(n+1)$ containing $\lambda(t)$. Let $[V_0]$ be a T-invariant point in $\overline{O^T([V])}$. Then V_0 is a sum of linear spaces, since these are the only T-invariant subvarieties of $\mathbf{P}^n$. If we decompose Φ_V by $\Phi_V = \sum_\alpha \Phi_V^\alpha$, where α runs over the characters of T and Φ_V^α is the part of Φ_V on which T acts with weight α, then for any $\tau \in T$, $\Phi_V^\tau = \sum_\alpha c_\alpha^\tau \Phi_V^\alpha$ for suitable constants c_α^τ. Since Φ_{V_0} is both T-invariant and a limit of forms Φ_V^τ, $\tau \in T$, $\Phi_{V_0} = \Phi^\alpha$ for some α. Moreover since V is a λ-invariant point, all the characters α appearing in the decomposition of Φ_V must have the same value on λ, hence the λ-weight of Φ_{V_0} is the λ-weight of Φ_V.

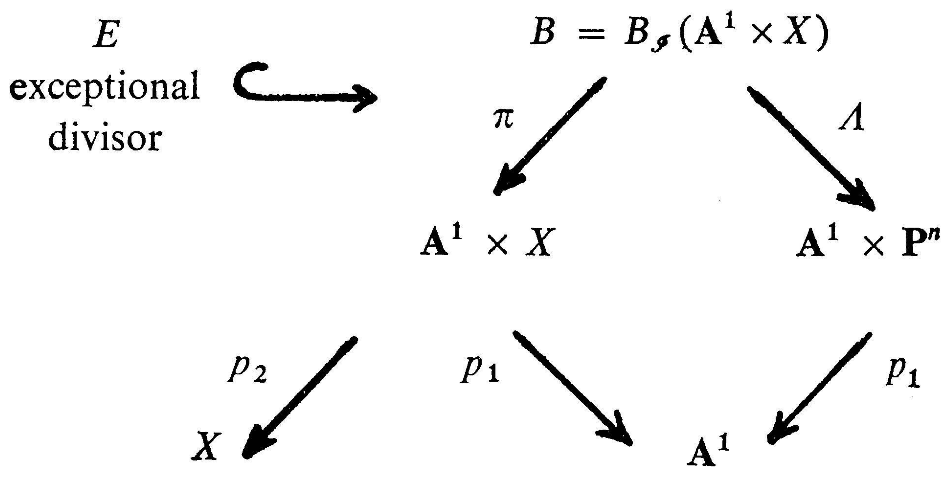
It remains only to compare the homogeneous coordinate rings. Now V and V_0 are members of a flat family V_t, $t \in S$ for some connected parameter space S, so that if $n \gg 0$, $H^0(V_t, \mathcal{O}_{V_t}(n))$ are the fibres of a vector bundle over S. This means that the λ-action on these fibres varies continuously, hence that the λ-weights of all the fibres are equal. Now the claim for V follows from b).

REMARK. The relation between Chow forms and Hilbert points in c) is really much more general: in fact, Knudsen [12] has shown that there is a canonical isomorphism of 1-dimensional vector spaces $k \cdot \Phi_V \cong [(r+1)^{\text{st}}$ "differences"—formed via $\otimes$—of successive spaces in the sequence $\Lambda^{\dim R_n} R_n]$, and it is possible to base the whole proof of 2.11 on this.

3) Next we will see how to obtain $X^{\lambda(0)}$ by blowing up $\mathscr{I}$. Consider the map

$$\begin{aligned} \Lambda_1 \ : \ \mathbf{G}_m \times X &\to \mathbf{P} \\ (t, X) &\mapsto \lambda(t)(x) . \end{aligned}$$

If the embedding of X is defined by $s_0, \ldots, s_n \in \Gamma\,[X, \mathcal{O}_X(1)]$ and the action of $\lambda(t)$ is by $(a_0, \ldots, a_n) \mapsto (t^{r_0} a_0, \ldots, t^{r_n} a_n)$ with $r_0 \geqq r_1 \geqq \ldots \geqq r_n$ and $\sum_{i=0}^{n} r_i = 0$ (i.e. (0, ..., 0, 1) is an attractive fixed point and (1, 0, ..., 0) is a repulsive fixed point), then $\Lambda^*{}_1(X_1) = t^{r_i} s_i$. Now $t^{-\gamma}$ is a unit on $\mathbf{G}_m \times X$, so changing the identification $\Lambda_1^*(\mathcal{O}_{\mathbf{P}^n}(1)) \cong \mathcal{O}_{\mathbf{G}_m} \otimes \mathcal{O}_X(1)$ by this unit we can assume $\Lambda_1^*(X_1) = t^{\rho_i} s_i$ where $\rho_i = r_i - \gamma$ is normalized as in 2.8 so that $\rho_n \geqq 0$. Then Λ_1 "extends" to a rational map $\mathbf{A}^1 \times X \to \mathbf{P}^n$ which is defined by the section $\{ t^{\rho_i} s_i \} \in \Gamma\,(\mathbf{A}^1 \times X, p_2^* \mathcal{O}_X(1)$. $\mathscr{I}$ is just the ideal sheaf these generate in $\mathcal{O}_{\mathbf{A}^1 \times X}$ and Z is just the set of base points of the rational map. Blowing up along $\mathscr{I}$ gives the picture



where the morphism Λ is defined by the sections $\{ t^{\rho_i} s_i \}$ in $\Gamma\,[B, (p_2 \pi)^* (\mathcal{O}(1))(-E)]$. Now Im (Λ) is the closed subscheme of $\mathbf{A}^1 \times \mathbf{P}^n$ given by Proj $(\bigoplus_{m=0}^{m} R_m)$ where

$$(2.12)\qquad R_m = \begin{bmatrix} k\,[t]\text{-submodule of } \Gamma\big(X, \mathcal{O}(m)\big) \otimes_k k\,[t] \\ \text{generated by } m^{\text{th}} \text{ degree monomials in } \{\, t^{\rho_i} s_i \,\} \end{bmatrix}$$

In fact, Im Λ is *flat* over $\mathbf{A}^1$, because of:

LEMMA 2.13. *Let S be a non-singular curve, X flat over S and $f: X \to Y$ be a proper map over S. Then the scheme $(f(X), \mathcal{O}_Y/\ker f^*)$ is flat over S.*

Proof. We may as well suppose S = Spec R; and then this amounts to showing the $\mathcal{O}_Y/\ker f^*$ has no R-torsion: if $a \in \mathcal{O}_Y/\ker f^*$, $r \in R$, then $r \,.\, a = 0 \Rightarrow r \,.\, f^* a = 0 \Rightarrow f^* a = 0 \Rightarrow a = 0$.

In particular, we see that $X^{\lambda(0)}$ is the fibre of Im Λ over $t = 0$, i.e. $X^{\lambda(0)} = \operatorname{Proj}\big(\bigoplus_{m=0}^{m} R_m/tR_m\big)$.

4) The proof is completed by making precise the relation between $\mathscr{I}$ and the λ-weight of $\Phi_{\tilde{X}}$. One must be careful however because there are two $\mathbf{G}_m$-actions on R_m/tR_m, that given by the identification $R_1/tR_1 = \oplus\,(t^{r_i} s_i)\,k$, which is just λ, and that given by the identification $R_1/tR_1 = \oplus\,(t^{\rho_i} s_i)\,k$; call this action μ. The weights of μ on R_m/tR_m are just those of λ translated by $m\gamma$. By Proposition 2.11

$$\begin{aligned} \lambda\text{-weight of } \Phi_{\tilde{X}} &= \text{n.l.c. } (\lambda\text{-weight of } R_m/tR_m) \\ &= \text{n.l.c. } (\mu\text{-weight of } R_m/tR_m + \gamma m \dim (R_m/tR_m)) \\ &= \text{n.l.c. } (\mu\text{-weight of } R_m/tR_m) - \left(\frac{r+1 \deg X}{n+1} \sum_{i=0}^{n} \rho_i\right) \end{aligned}$$

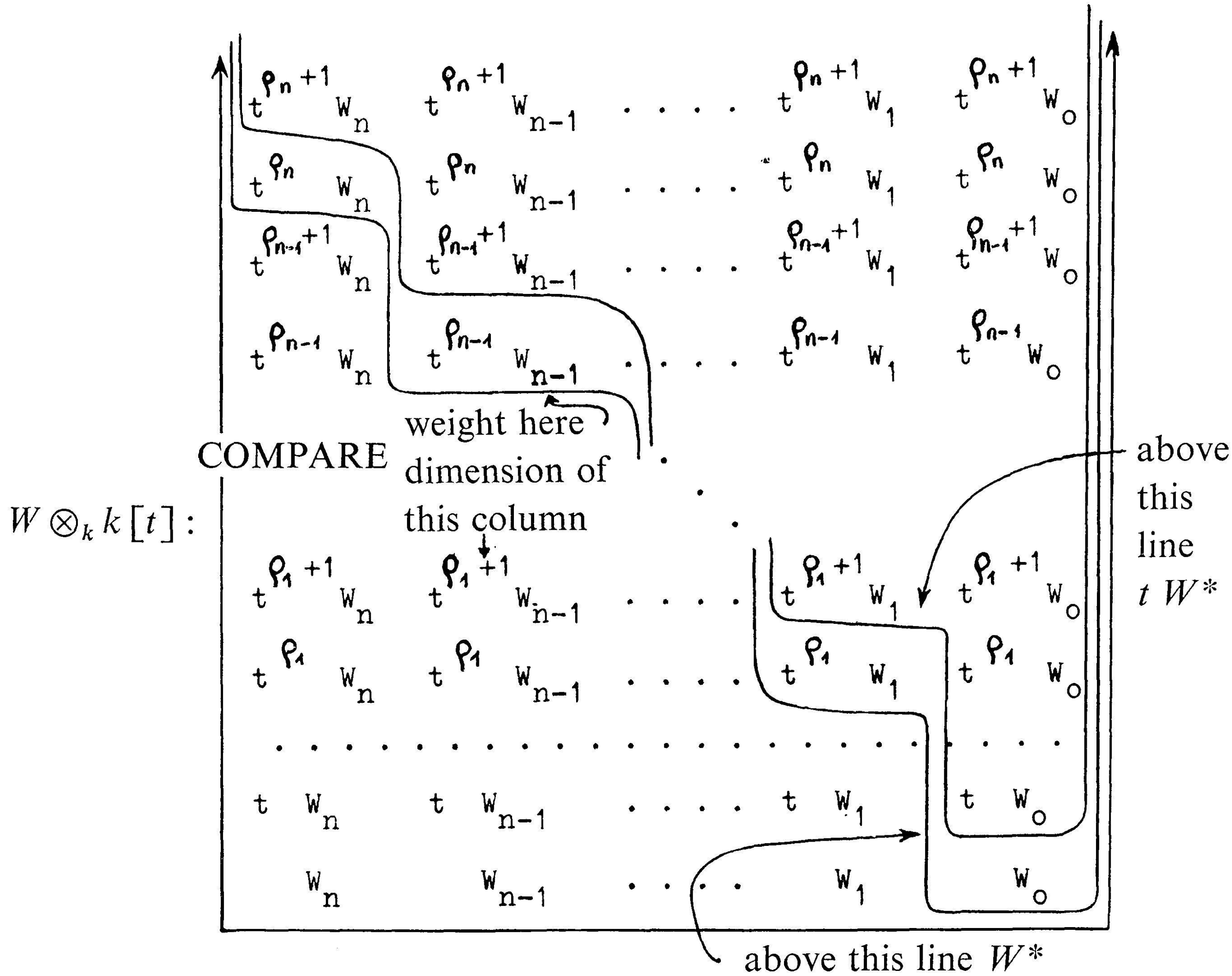
using $\gamma = -\dfrac{1}{n+1} \sum \rho_i$ and

$$\begin{aligned} \dim (R_m/tR_m) &= (\deg X_{\lambda(0)}) \frac{m^r}{r\,!} + \text{lower terms} \\ &= \frac{(\deg X)\, m^r}{r\,!} + \text{lower terms.} \end{aligned}$$

A droll lemma allows us to re-express the μ-weight of R_m/tR_m.

LEMMA 2.14. *Let W be a k-vector space and let $\mathbf{G}_m$ act by μ on W with weights $\rho_n \geqq \rho_{n-1} \cdots \geqq \rho_0 = 0$. Let W_i be the eigenspace of weight ρ_i and let W^* be the $k\,[t]$-submodule of $W \otimes k\,[t]$ generated by $\oplus\, t^{\rho_i} W_i$. Then $\dim (k\,[t] \otimes W/W^*) = \mu$-weight of W^*/tW^*.*

Proof by Diagram :



Recalling the definition of R_m (2.12), and applying this to the μ-action on R_m/tR_m, we see that the μ-weight of R_m/tR_m is just: $\dim\big(\Gamma\ (X,\ \mathcal{O}\ (m))\ \otimes_k k\ [t]/R_m\big)$. But the sections $\{ t^{\rho_i} s_i \}$ whose m^{th} tensor powers generate R_m, also generate $\mathscr{I} \cdot p_2^* (\mathcal{O}_{X(1)})$ so by a) and b) of Proposition 2.6, this last dimension can be used to calculate $e(\mathscr{I})$. Putting all this together, we see that:

$$\Phi_X \text{ is stable with respect to } \lambda$$
$$\Leftrightarrow \lambda\text{-weight of } \Phi_X < 0$$
$$\Leftrightarrow e_L(\mathscr{I}) - \frac{(r+1)}{(n+1)} \deg X \sum_{i=0}^{n} \rho_i < 0$$

which, with the analogous statement for semi-stability, is our theorem.

2.15. Interpretation via reduced degree. If $X^r \subset \mathbf{P}^n$ is a variety, its reduced degree is defined to be:

$$\text{red. deg }(X) = \frac{\deg X}{n+1-r}$$

A very old theorem says that if X is not contained in any hyperplane then red. deg $(X) \geqq 1$. Reduced degree measures, in some sense, how complicatedly X sits in $\mathbf{P}^n$, and there are classical classifications of varieties with small reduced degree. For example if X has reduced degree 1 and is not contained in any hyperplane then X is either

a) a quadric hypersurface

b) the Veronese surface in $\mathbf{P}^5$ or a cone over it

c) a rational scroll: $X = \mathbf{P}\left(\bigoplus_{i=0}^{r} \mathcal{O}_{\mathbf{P}^1}(n_i)\right) \subset \mathbf{P}^N,\ n_i > 0$

where $N = \sum_{i=0}^{r} (n_i+1) - 1$, or a cone over it. (This is called a scroll because the fibres $\mathbf{P}^{r-1}$ of X over $\mathbf{P}_1$ are linearly embedded.)

Some other facts about reduced degree are:

i) canonical curves, K3-surfaces and Fano 3-folds have red. deg $= 2$;

ii) all non-ruled surfaces and all special curves have red. deg $\geqq 2$. (For special curves, this is just a restatement of Clifford's theorem.)

iii) for ample L on X^r, the embedding by $L^{\otimes r}$ has reduced degree asymptotic to $r\,!$ as $n \to \infty$;

iv) red-deg is preserved under taking of proper hyperplane sections.

It would be very interesting to know whether almost all 3-folds (in a sense similar to that of ii) for surfaces) have red. deg $\geqq 2 + \varepsilon$. The following definition is introduced only tentatively as a means of linking the present ideas to older ideas (e.g. Albanese's method to simplify singularities of varieties):

2.16. Definition. *A variety $X^r \subset \mathbf{P}^n$ is linearly stable (resp. linearly semi-stable) if, whenever $L^{n-m-1} \subset \mathbf{P}^n$ is a linear space such that the image cycle $p_L(X)$ of X under the projection $p_L : \mathbf{P}^n - L \to \mathbf{P}^m$ has dimension r, then* red deg $p_L(X) >$ red deg X (*resp.* red-deg $p_L(X) \geqq$ red deg X).

Attention: p_L is allowed to be finite to 1, and which case $p_L(X)$ must be taken to be the image cycle. Linear stability is a property of the linear system embedding X; if $X^r \subset \mathbf{P}^n$ is embedded by $\Gamma(X, L)$, then X linearly stable means that for all subspaces $\Lambda \subset \Gamma(X, L)$

$$\frac{\deg p_L(X)}{\dim \Lambda - r} > \frac{\deg X}{n + 1 - r}$$

or equivalently, by applying Proposition 2.5,

$$e(\mathscr{I}_\Lambda) < \frac{\deg X}{n + 1 - r} (\text{codim } \Lambda)$$

EXAMPLES. i) when X is a curve of genus 0, it is linearly semi-stable but not stable. When $g \geqq 1$, Clifford's theorem shows that X is linearly stable whenever it is embedded by a complete non-special linear system (see § 4 below).

ii) $\mathbf{P}^2$ is linearly unstable when embedded by $\mathcal{O}(n)$, $n \geqq 3$ because it projects to the Veronese surface. In view of the next proposition, a very interesting problem is that of finding large classes of linearly (semi)-stable surfaces.
(It may, however, turn out that linear stability is really too strong, or unpredictable, a property for surfaces in which case this Proposition is not very interesting !)

PROPOSITION 2.17. *Fix* $X^r \subset \mathbf{P}^n$, *let* C *be any smooth curve and let* L *be an ample line bundle on* C. *Let* $\Phi_i : C \times X \to \mathbf{P}^{N(i)}$ *be the embedding defined by* $\{ S_j \otimes X_l \}$ *where* $\{ S_j \}$ *is a basis of* $\Gamma(L^{\otimes i})$ *and* $X_l \in \Gamma(X, \mathcal{O}_X(1))$ *are the homogeneous coordinates. If* $\Phi_i(C \times X)$ *is linearly semi-stable for all large* i, *then* X^r *is Chow-semi-stable.*

Proof. Choose a 1-PS: $\lambda(t) = \begin{bmatrix} t^{\rho_0} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & t^{\rho_n} \end{bmatrix} t^{-\frac{\Sigma \rho_i}{n+1}}$

as in (2.8).
Choose a point $p \in C$ an isomorphism $L_p \cong \mathcal{O}_p$ and an i large enough that $L^{\otimes i}$ is very ample and $L^{\otimes i}(-\rho_0 p)$ is non-special. Then the map

$$\bigoplus_{l=1}^{n} \Gamma(C, L^{\otimes i}) . X_l \xrightarrow{\Phi_i} \bigoplus_{l=0}^{n} [\mathcal{O}_{p,C}/\mathcal{M}_{p,C}^{\rho_0}] . X_i$$

is surjective. Let Λ^i be the inverse image of $\bigoplus_{l=0}^{n} [(\mathcal{M}_{p,C}^{\rho_l}/\mathcal{M}_{p,C}^{\rho_0}) \cdot X_l]$ under this map and let $\mathscr{I}_\Lambda^i \subset \mathcal{O}_{C \times X}$ be the induced ideal. Since all the $L^{\otimes i}$ are trivial near p and $\mathscr{I}_\Lambda^i$ has support on the fibre of $X \times C$ over P, the ideals

$\mathscr{I}_A^i$ are independent of i; we denote this ideal by $\mathscr{I}_A$. The hypothesis says that for large i

$$e(\mathscr{I}_A) \leqq \frac{\deg(C \times X)}{(n+1)(h^0(L^i)-r-1)} \operatorname{codim} A$$
$$= \frac{(r+1)\deg X \deg L^{\otimes i}}{(n+1)(\deg L^{\otimes i} - g + 1) - r - 1} \cdot \sum_{l=0}^{n} \rho_l$$

and letting $i \to \infty$,

$$e(\mathscr{I}_A) \leqq \frac{(r+1)\deg X}{n+1} \sum_{l=0}^{n} \rho_l$$

But $C \times X$ along $p \times X$ is formally isomorphic to $\mathbf{A}^1 \times X$ along $0 \times X$ with corresponding $\mathscr{I}_A'$ s, so by Theorem 2.9., X is Chow-semi-stable.

§ 3. Effect of Singular Points on Stability

We begin with an application of Theorem 2.9.

PROPOSITION 3.1. *Let $X^1 \subset \mathbf{P}^n$ be a curve with no embedded components such that* $\deg X/n+1 < 8/7$. *If X is Chow-semi-stable, then X has at most ordinary double points.*

REMARKS. i) When $n = 2$, $\deg X/n+1 < 8/7 \Leftrightarrow \deg X < 4$ and the proposition confirms what we have seen in 1.10 and 1.11

ii) Suppose L is ample on X^1 and $X_m \subset \mathbf{P}^{N(m)}$ is the embedding of X defined by $\Gamma(X, L^{\otimes m})$. By Riemann-Roch, $\deg X_m/N(m) \to 1$ as $m \to \infty$, hence:

COROLLARY 3.2. *An asymptotically stable curve X has at most ordinary double points.*

In particular, if $X \subset \mathbf{P}^2$ has degree $\geqq 4$ and has one ordinary cusp, then, in $\mathbf{P}^2$, X is stable but when re-embedded in high enough space, X is unstable! The fact that this surprising flip happens was discovered by D. Gieseker and came as an amazing revelation to me, as I had previously assumed without proof the opposite.

iii) We will see in Proposition 3.14 that the constant 8/7 is best possible.

Proof of 3.1. We note first that a semi-stable X of any dimension cannot be contained in a hyperplane: if $X \subset V(X_0)$, then X has only positive weights with respect to the 1-PS