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STABILITY OF PROJECTIVE VARIETIES [1])

by David Mumford

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Introduction

The most direct approach to the construction of moduli spaces of algebraic varieties is via the theory of invariants: one describes the varieties by some sort of numerical projective data, canonically up to the action of some algebraic group, and then seeks to make these numbers canonical by applying invariant polynomials to the data, or equivalently by forming a quotient of the data by the group action. The main difficulty in this approach is to prove that "enough invariants exist": their values on the projective data must distinguish non-isomorphic varieties.

Take as an example the moduli space $\mathscr{M}_g$ of curves of genus $g \geqq 2$ over some algebraically closed field k. Given C, such a curve, we obtain by choosing a basis B of $\Gamma(C, (\Omega_c^1)^{\otimes l})$, an embedding $\Phi\colon C \to \mathbf{P}^{(2l-1)(g-1)-1}$

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[1]) Lectures given at the "Institut des Hautes Etudes Scientifiques", Bures-sur-Yvette (France), March-April 1976, under the sponsorship of the International Mathematical Union. Notes by Ian Morrison.

$= \mathbf{P}^N$. Let F be the Chow form of $\Phi(C)$ (cf. 1.16). Changing the basis B subjects $\Phi(C)$ to a projective transformation and F to the corresponding contragradient transformation. So if we could find "enough" polynomials I_λ in the coefficients of F which are invariant under this action of $SL(N+1)$ then the image of the map given by $C \mapsto (..., I_\lambda(F), ...)$ would be $\mathcal{M}_g$.

As of two years ago, this process could be carried out only when char $k = 0$ and C was smooth; and moduli spaces in characteristic p had to be constructed via the much more explicit theory of moduli of abelian varieties (cf. [14] and [15]). Since then, however, two very nice things have been proven:

a) W. Haboush [10] by making a systematic use of Steinberg representations has shown that all reductive groups are geometrically reductive (cf. Remark 1.2. vi). This was independently shown for $SL(n)$, by Processi and Formanek [25], using the idea that the group ring of an infinite permutation group has "radical" zero: i.e. for each $x \in R$, $x \neq 0$, there exists $y \in R$ such that xy is not nilpotent. For a complete treatment of the new situation in characteristic p moduli problems see Seshadri [20].

b) D. Gieseker [9] using the concept of asymptotic stability (cf. 1.17) has established the numerical criterion for stability (c_s of 1.1) for surfaces of general type. Inspired by Gieseker's ideas, the author has extended this method to the "stable" curves of Deligne and Mumford [6]. (These are curves C with dim $H^1(C, \mathcal{O}_C) = g$, ordinary double points but no worse singularities and no smooth rational components meeting the remainder of the curve in fewer than three points; they are important because the most natural compactification $\overline{\mathcal{M}_g}$ of $\mathcal{M}_g$ is the moduli space for stable curves of genus g.) The power of the ideas of Gieseker is by no means exhausted. It looks like nice results may be possible for other surfaces, perhaps even for singular surfaces and the technique suggests several nice problems: in particular, it may lead to a proof of the surjectivity of the period map for K3 surfaces. The new ideas and results of these lectures are largely inspired by Gieseker's results (cf. especially corollary 3.2 below).

My goal is to outline this method and its applications, especially to the completed moduli spaces of curves $\overline{\mathcal{M}_g}$, indicating open problems. The field is moving ahead rapidly and may be greatly simplified in the near future.

We will work in general over an arbitrary ground field k.

§ 1. Stable points of representation, examples and Chow forms

For more details on the notations, definitions and properties which follow see Mumford [14], which we will call G.I.T. or Seshadri [20].

Fix k an algebraically closed field,

G a reductive algebraic group over k (i.e. $G =$ [semi-simple group $\times$ $\mathbf{G}_m^n$]/finite central subgroup),

V an n-dimensional representation of G,

$x \in V$.

There are three possibilities for x whose equivalent formulations are summarized in table 1.1 below.

1.1.

x unstable	x semi-stable	x stable
(a_u) $0 \in \overline{O^G(x)}$	(a_{ss}) $0 \notin \overline{O^G(x)}$	(a_s) i) $O^G(x)$ is closed in V ii) stab (x) is finite
(b_u) $\forall$ non-constant G-invariant homogeneous polynomials f $f(x) = 0$	(b_{ss}) $\exists$ a non-constant G-invariant homogeneous polynomial f s.t. $f(x) \neq 0$	(b_s) i) $\forall y \in V - O^G(x)$, $\exists$ a G-invariant polynomial f s.t. $f(x) \neq f(y)$ ii) tr $\deg_k k(V)^G$ $= \dim V\text{-}\dim G$
(c_u) $\exists$ a 1-PS λ of G s.t. the weights of x with respect to λ are all positive	(c_{ss}) $\forall$ 1-PS's λ of G the weights of x with respect to λ are not all positive	(c_s) for all non-trivial 1-PS's λ of G, x has both positive and negative weights with respect to λ

1.2. REMARKS. i) Recall that a 1-PS (one parameter subgroup) λ of G is just a homomorphism $\lambda\colon \mathbf{G}_m \to G$. Such λ can always be diagonalized in a suitable basis:

$$\lambda(t) = \begin{bmatrix} t^{r_1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & t^{r_n} \end{bmatrix}$$

If in this basis $x = (x_1, ..., x_n)$, the set of weights of x with respect to λ is the set of r_i for which $x_i \neq 0$.

ii) Unstable is *not* the opposite of stable, but of semi-stable. We will use non-stable as the opposite of stable.

iii) The important part of stability is the condition: $O^G(x)$ closed in V. In virtually all the cases that will interest us the finiteness of stab (x) will be automatic (but cf. the remark following 1.15).

iv) A point x is stable if it merely has negative weights with respect to every non-trivial 1-PS λ, for then it also has positive weights with respect to λ, namely, its negative weights with respect to λ^{-1}.

v) The proofs of $c_u \Rightarrow a_u \Rightarrow b_u$ and of $b_s \Rightarrow a_s \Rightarrow c_s$ are obvious: for example, if λ is a 1-PS for which all weights of x are positive, then $\lambda(t)\, x \to 0$ at $t \to 0$; i.e. $c_u \Rightarrow a_u$.

vi) The proofs of $a_s \Rightarrow b_s$ and $b_u \Rightarrow a_u$ are achieved by reduction to the special case called geometric reducivity of G. A group G is called geometrically reductive if

a) whenever V_0 is an invariant codimension-1 subspace of a vector space V in which G is represented, there exists an n for which the codimension-1 invariant subspace $V^0 \cdot \mathrm{Symm}^{n-1} V \subset \mathrm{Symm}^n V$ has an invariant 1-dimensional complement.
But notice that this is the same as saying that

b) whenever $x \neq 0$ is a G-invariant point, then there exists a G-invariant polynomial f such that $f(x) \neq 0$ and $f(0) = 0$. (Just consider x as a functional on the dual $\hat{V}$ and apply a) to its kernel there).

And b) is a special case of $a_s \Rightarrow b_s$. When char $k = 0$ we can take the polynomial f to be linear, for by complete reducibility the invariant subspace generated by x is invariantly complemented. A simple example shows this does not happen in char p. Take $p = 2$, $G = SL(2)$, $V =$ the space of symmetric bilinear functions on k^2, and x a non-degenerate skew-symmetric form ($x \in V$ because $p = 2$!). Then x is $SL(2)$-invariant and there are no G-invariant non-zero linear functionals on V. A quadratic f which does work is the determinant.

vii) The remaining implications $c_s \Rightarrow a_s$ and $a_u \Rightarrow c_u$ are essentially consequences of the surjectivity of the natural map

$$\left\{\begin{matrix} \text{1PS's } \lambda \text{ of } G \\ \lambda : \mathbf{G}_m \to G \end{matrix}\right\} \longrightarrow G(k[[t]]) \Big\backslash \; G(k((t))) \; \Big/ \; G(k[[t]])$$

where λ is considered as a $k((t))$-valued point of G by composition with the canonical map

$$\operatorname{Spec} k((t)) \to \operatorname{Spec} k[t, t^{-1}] = \mathbf{G}_m$$

1.3. Let V_{ss} (resp. V_s) denote the Zariski-open cones of semi-stable (resp. stable) points. $V - V_{ss}$ is the Zariski-closed cone of unstable points. The conditions b of 1.1 tell us that if we try to map $\mathbf{P}(V)$ to a projective space by invariant polynomials, we can only hope to achieve a well-defined map on $\mathbf{P}(V)_{ss}$ and an embedding on $\mathbf{P}(V)_s$. From the point of view of quotients this can be expressed by:

PROPOSITION 1.3. *Let* $X = \operatorname{Proj} k[V]^G$. *Then there is a diagram*

$$\begin{array}{ccccc} \mathbf{P}(V) & \supset & \mathbf{P}(V)_{ss} & \supset & \mathbf{P}(V)_s \\ & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_s} \\ & & X & \supset & X_s \end{array}$$

such that i) *if* $x, y \in \mathbf{P}(V)_s$, $\pi_s(x) = \pi_s(y) \Leftrightarrow \exists\, g \in G$ s.t. $x = g\,y$

ii) *if* $x, y \in \mathbf{P}(V)_{ss}$, $\pi(x) = \pi(y) \Leftrightarrow \overline{O^G(x)} \cap \overline{O^G(y)} \cap \mathbf{P}(V)_{ss} \neq \varnothing$.

We now want to look at some examples to illustrate the application of these ideas.

1.4. "BAD" ACTIONS. Using results of T. Kimura and M. Sato [11] [1]), we can give a list of all representations of simple algebraic groups in charac-

[1]) Plus help given by J. Tits.

teristic 0 in which all vectors are unstable. The point is that there are *very* few such representations.

G	V
$SL(W)$	$W^i, \hat{W}^i, 1 \leqslant k < \dim W$
$\left\{\begin{array}{l} SL(W) \\ \dim W \text{ odd} \end{array}\right.$	$\Lambda^2 W, \Lambda^2 \hat{W}$ $\Lambda^2 W \oplus \hat{W}, \Lambda^2 \hat{W} \oplus W$
$\mathrm{Sp}(W)$	W
$\mathrm{Spin}(10)$	W or $W \oplus W$ where W is a 16-dimensional half-spin representation

1.5. Discriminant. If G is semi-simple and char $k = 0$ then any irreducible representation V has the form $V = \Gamma(G/B, L)$ for a suitable line bundle L on G/B (B is a Borel subgroup of G). To a point x in V associate the divisor H_x on G/B which is the zero set of the corresponding section. Except in the extremely unusual case that the set of singular H_x is of codimension > 1, there is an irreducible invariant polynomial δ, the discriminant, such that

1) $\delta(x) = 0 \Leftrightarrow H_x$ is singular

2) $V - (\delta = 0)$ consists of semi-stable points.

An interesting case is

Lemma 1.6. *Let* $G = SL(n)$, $V = \Lambda^l(k^n)$. *If* $W \subset k^n$ *is a subspace of codimension* l *then let* Φ_W *denote the natural map* $\Lambda^2 W \otimes \Lambda^{l-2}(k^n) \to \Lambda^l(k^n)$. *If* $2 < l < n - 2$ *or* n *is even* $l = 2$ *or* $n - 2$, *then there is a* G-*invariant* δ *such that* $\delta(x) = 0 \Leftrightarrow x \in \mathrm{Im}(\Phi_W)$ *for some* W.

When $l = 2$ and $n = 2m + 1$ we have seen that there are no invariants; corresponding to these cases the Grassmanian of lines in $\mathbf{P}^{2m}$ in its Plücker embedding in projective space has the unusual property that the singular hyperplane sections are of codimension $\geqq 2$ in the set of all such sections.

Question: if not every point of V is unstable, then is the set of singular hyperplane sections H_x of codimension 1 ?

For $l = 2$ and n even or $l = 3$, $n \leqq 8$, one can check that x is unstable $\Leftrightarrow \delta(x) = 0$, hence δ generates the ring of invariants. It would be nice to have a necessary and sufficient condition for a 3-form to be unstable for higher n as well.

1.7. 0-Cycles. For $G = SL(W)$, $\dim W = 2$,

$$\begin{aligned} V_n &= \mathrm{Symm}^n(\hat{W}) \\ &= \text{vector space of homogeneous polynomials } f \text{ of degree } n \text{ on } W, \\ \mathbf{P}(V_n) &= \text{space of 0-cycles of } n \text{ unordered points on the projective line } \mathbf{P}(W)\text{, the roots of an } f \text{ determining the cycle.} \end{aligned}$$

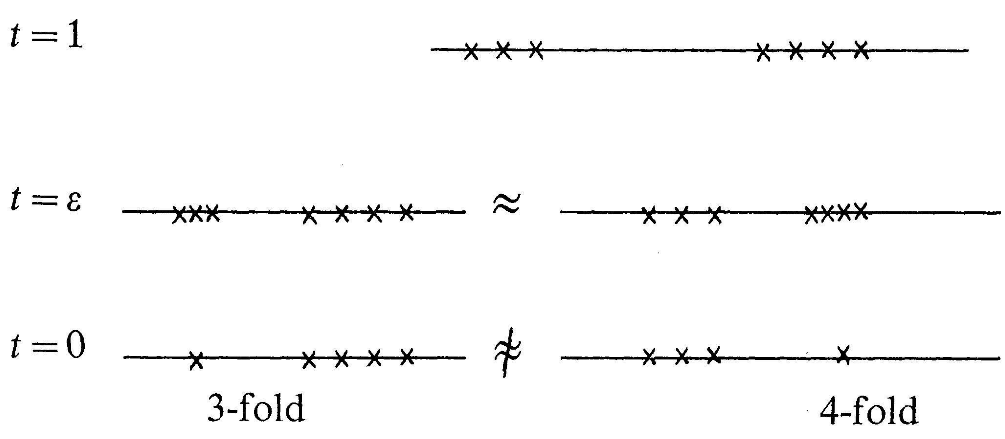
If $f = \sum_{i=0}^{n} a_i x^{n-i} y^i$ and λ is the one-parameter subgroup given by $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ in these coordinates, then $\lambda(t) f = \sum_{i=0}^{n} a_i t^{n-2i} x^{n-i} y^i$. For f to be stable, the weights $(n-2i)$ associated to the non-zero coefficients of f must lie on both sides of 0: i.e. if $j \geqq n/2$, neither x^j nor y^j divide f.

In fact, the stability of f is equivalent to the same condition with respect to all linear forms l: $l^j \nmid f$ if $j \geqq n/2$.

Thus $\mathbf{P}(V_n)_s = \{$0-cycles with no points of multiplicity $\geqq n/2\}$
$\mathbf{P}(V_n)_{ss} = \{$0-cycles with no points of multiplicity $> n/2\}$.

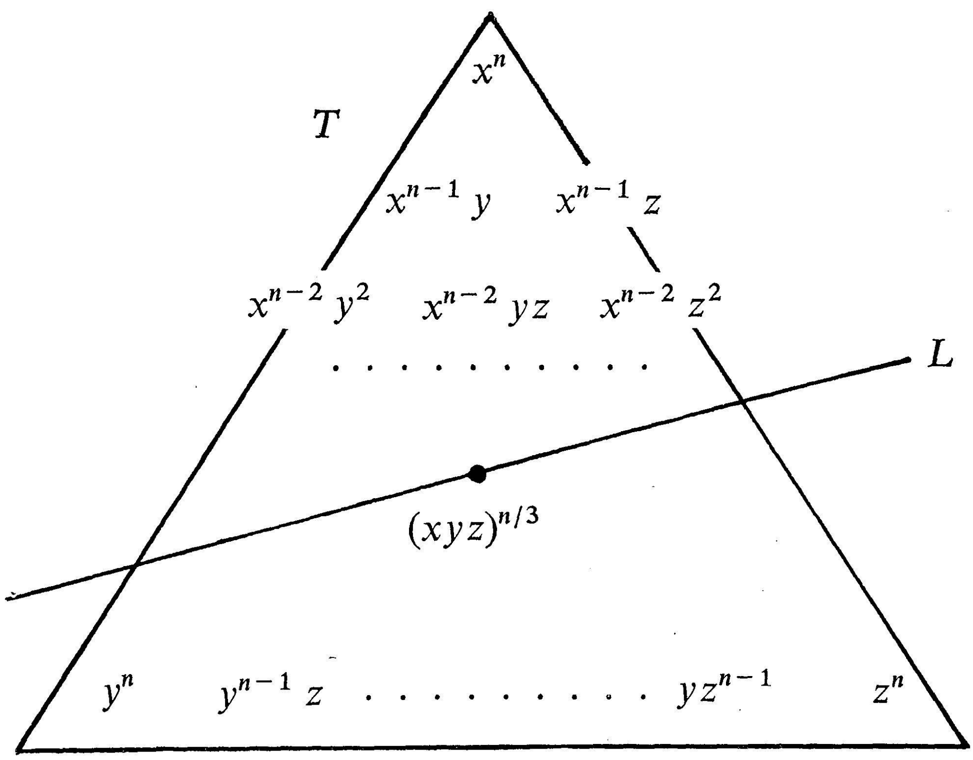
1.8. Remark. In the example above we can also prove that semi-stability is a purely topological character. I claim that if n is odd and f is unstable then the action of G near $\bar{f} \in \mathbf{P}(V_n)$ is bad: on all open neighbourhoods of the orbit of $\bar{f}$, G acts non-properly and the orbit space is non-Haussdorf. Let's see this for $n = 7$. Consider the following deformations of a 7-point cycle.

(Subscripts indicate multiplicities)



At each intermediate stage the two cycles are projectively equivalent, but the unstable limiting cycle in the right is clearly not equivalent to the limit on the left. In fact, any pair of cycles with the multiplicities indicated on the line $t = 0$ arise in this way as simultaneous limits of projectively equivalent 0-cycles. Moreover, there are cycles of the same type as the left hand limit in any neighbourhood of the orbit of the right limit—just bring a multiplicity one point in towards the triple point; so the orbit space cannot be Hausdorff near the right limit.

1.9. Curves. Here $G = SL(W)$, $\dim W = 3$, $V_n = \text{Symm}^n(\hat{W})$, as before, and a point $f \in V_n$ defines a plane curve of degree n. There is a very simple way to decide the stability of f. Represent f as below by a triangle of coefficients, T.



We can coordinatize this triangle by 3 coordinates i_x, i_y, i_z (the exponents of x, y and z respectively) related by $i_x + i_y + i_z = n$. The condition that a line L with equation $ai_x + bi_y + ci_z = 0$, $(a, b, c) \neq (0, 0, 0)$, should pass through the centre of this triangle is just $a + b + c = 0$; if L also passes through a point with integral coordinates then a, b and c can be chosen integral. It is now easy to check that the weights of the 1-PS

$$t \longmapsto \begin{pmatrix} t^a & & 0 \\ & t^b & \\ 0 & & t^c \end{pmatrix}$$

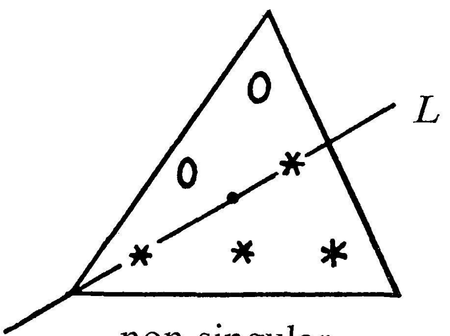
at f are just the values of the form defining L at the non-zero coefficients of f. In suitable coordinates every 1-PS is of this form so:

f is unstable $\Leftrightarrow$ in some coordinates, all non-zero coefficients of f lie to one side of some L

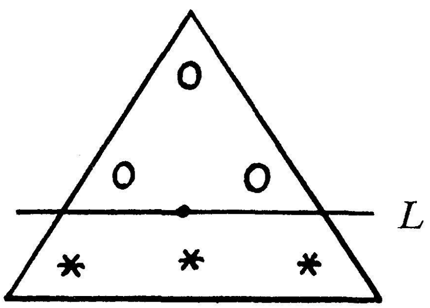
f is stable (resp. semi-stable) $\Leftrightarrow$ for all choices of coordinates and all L, f has non-zero coordinates on both sides of L (resp. f has non-zero coordinates on both sides of L or has non-zero coefficients on L).

Roughly speaking, a stable f can only have certain restricted singularities. We summarize what happens for small n, showing the "worst" triangle T for f with given singularities, and the associated L when f is not stable.

1.10. $n = 2$: We can achieve the diagram below for a non-singular quadric f by choosing coordinates so that $(1, 0, 0) \in f$ and $z = 0$ is the tangent line there, so f is never stable. We cannot make the xz coefficient of f zero without making f singular so f is always semi-stable; indeed, we know f always has non-zero discriminant. A singular quadric always has a diagram like that on the right: make $(1, 0, 0)$ the double point. Henceforth, we leave the checking of the diagrams to the reader.

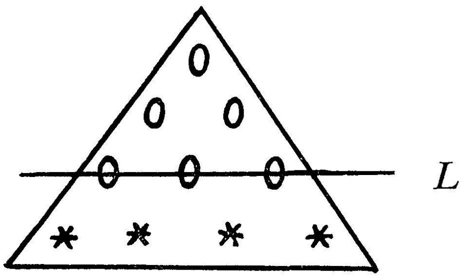
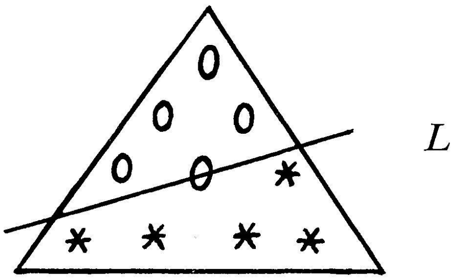
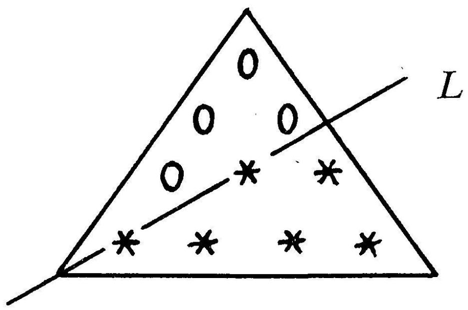
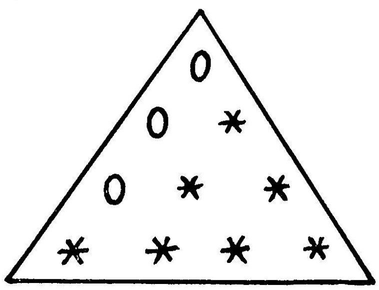


non-singular
quadric



singular
quadric

1.11. $n = 3$: It is well known that in this case the ring of invariants is generated by two invariants, A of degree 4 and B of degree 6. If we set $\varDelta = 27A^3 + 4B^2$, then up to a constant the classical j-invariant is just $A^3/\varDelta$. The possibilities are:

SINGULARITIES OF f	"WORST" TRIANGLE	STABILITY AND INVARIANTS
f has triple point		unstable $A = B = 0$ j undefined
f has a cusp or two components tangent at a point.		unstable $A = B = 0$ j undefined
f has ordinary double points (this includes the reducible cases: f is a conic and a transversal line, f is a triangle)		semi-stable and not stable $\varDelta = 0$ but $A, B \neq 0$ hence $j = \infty$
f smooth		stable $\varDelta \neq 0$ j finite

We remark that in this case, we have

$$\begin{array}{ccc} \mathcal{M}_1 & \cong & \mathbf{A}^1 \\ \cap & & \cap \\ \overline{\mathcal{M}_1} & = & \mathbf{P}^2 \end{array}$$

and that the j-invariant is a true modulus. Note that from a moduli point of view all three semi-stable types are equivalent.

1.12. $n = 4$: There are already quite a few diagram types here. Their enumeration can be summarized by saying that f is unstable if and only if f has a triple point or consists of a cubic and an inflectional tangent line; f is stable if and only if f has only ordinary double points or ordinary cusps (i.e. singularities with local equation $y^2 = x^3 +$ higher terms). The remaining f's with a tacnode (a double point with local equation $y^2 = x^4 +$ higher terms) are strictly semi-stable.

1.13. Remark. The fact that for $n \geqq 4$ curves with sufficiently tame cusps are semi-stable (or even stable!) is a definite problem because

i) such curves do not appear in the good compactification $\overline{\mathcal{M}}_g$ of the moduli space of non-singular curves of genus g. But

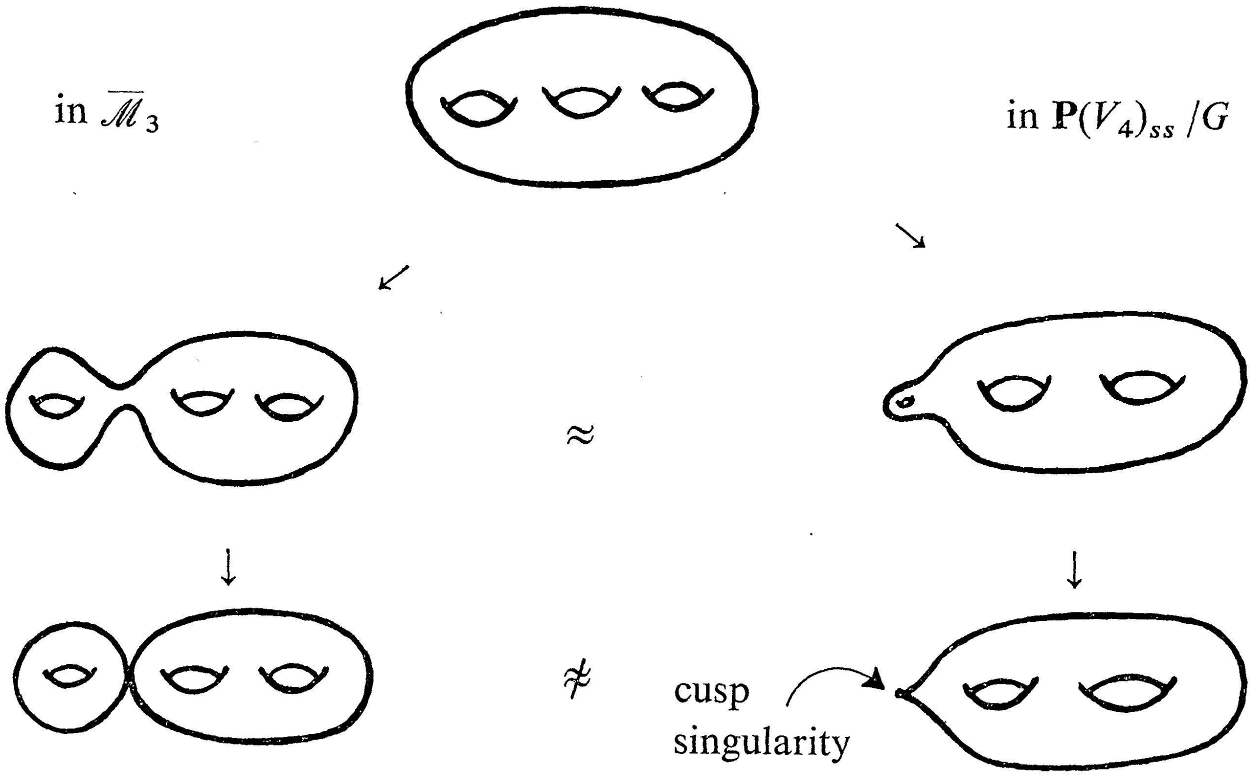
ii) if we wish to obtain a compactification of $\mathcal{M}_g$ as the quotient space of some subset of $\mathbf{P}(V_n)$ by G, the natural candidate is $\mathbf{P}(V_n)_{ss}$; so these curves must be let in.

For example, when $n = 4$, we have

$$\begin{array}{ccc} \mathcal{M}_{3,\ \text{non-hyperelliptic}} & \cong & [\mathbf{P}(V_4) - (\delta = 0)]/G \\ \cap & & \cap \\ \mathcal{M}_3 & & \mathbf{P}(V_4)_s/G \\ \cap & & \cap \\ \overline{\mathcal{M}}_3 & \overset{\alpha}{\underset{\beta}{\leftrightarrows}} & \mathbf{P}(V_4)_{ss}/G \end{array}$$

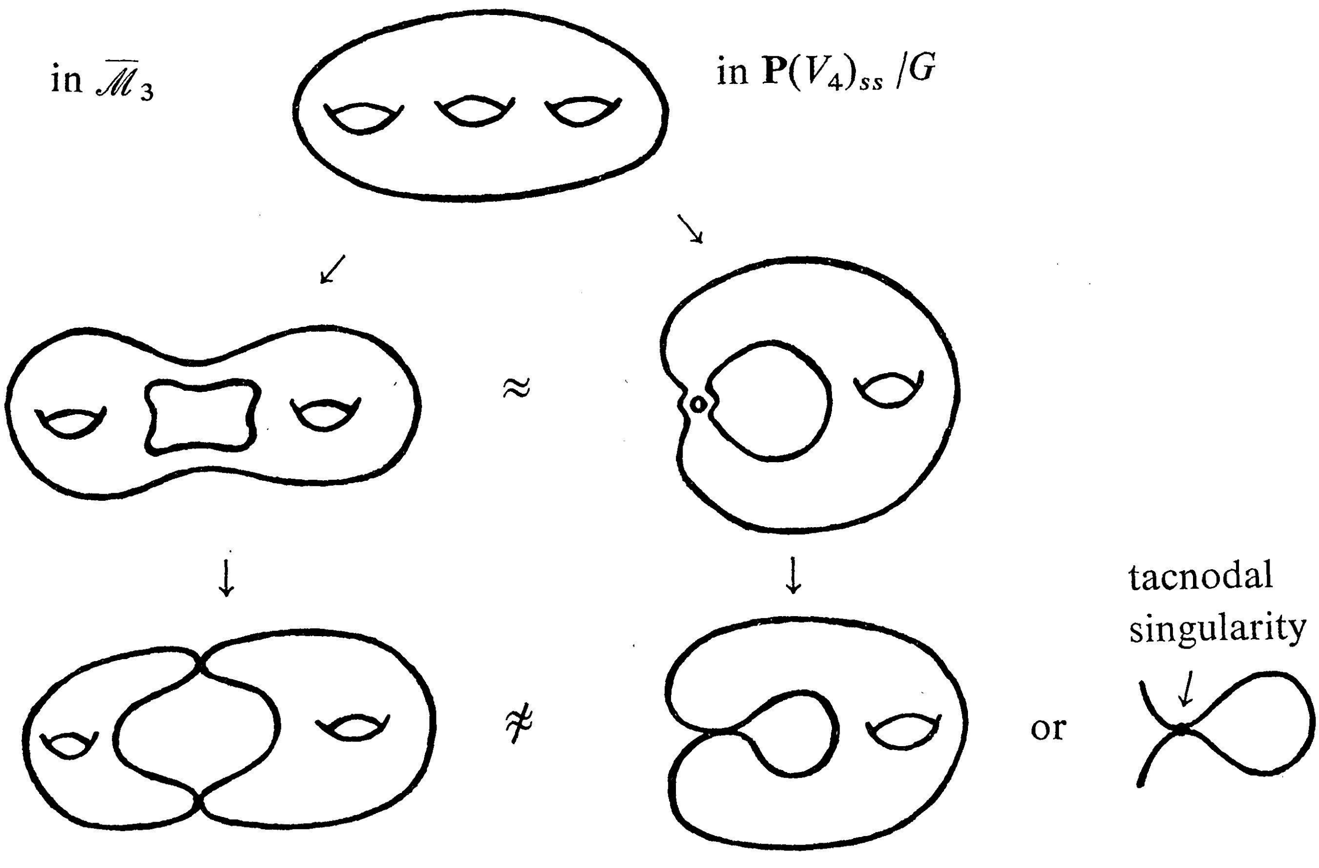
$\overline{\mathcal{M}}_3$ is the moduli space for "stable" curves of genus 3: (see introduction). Recall from Proposition 1.3 that $\mathbf{P}(V_4)_{ss}/G$ is just the projectivization of the full rings of invariants of $\mathbf{P}(V_4)$. The rational maps α and β induced by the top isomorphism enable us to make a topological comparison of these two compactifications. Let's see geometrically how cuspidal curves in $\mathbf{P}(V_4)_{ss}$ prevent α and β from being continuous.

First α: the diagram below shows on the left a deformation on $\mathcal{M}_3$ with limit in $\overline{\mathcal{M}}_3$, and on the right the same deformation followed to its limit in $\mathbf{P}(V_4)_{ss}/G$.



In the limit on the right, the value of the j-invariant of the shrinking elliptic curve has been lost! So α blows up a point representing a curve C with a cusp to the set of points representing joins of an arbitrary elliptic curve with the desingularization $\tilde{C}$ of C. α also blows up the point representing a double conic to the family of all hyperelliptic curves.

As for β, look at the double pinching below:



Here it is the manner in which the tangent spaces of the two branches have been glued at the tacnodal point which has been lost in the limiting curve on the left: this glueing corresponds on the left to the relative rate at which the two pinches are made. Thus β has blown up the point corresponding to the double join of two elliptic curves to a family of tacnodal quartics.

1.14. SURFACES. Here $G = SL(W)$, $\dim W = 4$ and $V_n = \mathrm{Symm}^n(\hat{W})$ as before. The technique for determining stability here is essentially that given for curves in 1.9 except that one has a tetrahedron T of coefficients and 1-PS's determine central planes, L: and, of course, the computations required to apply the technique are much more complicated (cf. the case $n = 4$ below). For small n, the situation is summarized below.

n	TYPE OF SINGULARITIES	STABILITY
$n = 2$	non-singular	semi-stable, not stable
	singular	unstable
$n = 3$	non-singular or with ordinary double points of type A1	stable
	ordinary double points of type A2	semi-stable, not stable
	triple points, double curve, higher double points	unstable
$n = 4$ (due to Jayant Shah [26])	singularities at most rational double points, or ordinary double curves possibly with pinch points, but no double *line*, and if reducible then no component a plane, no multiple components	stable
	A triple point whose tangent cone has only ordinary double points; or a double line not as below; or an irrational double point not as below; or a plane plus a cubic meeting in a plane cubic curve with only ordinary double points; or a non-singular quadric counted twice	semi-stable — but not stable
	a) quadruple point, or triple point whose tangent cone has cusp, b) $x = y = 0$ is double line and $f \in (x^2, xyz^2, xy^2, y^3)$ c) a higher double point of form: $f \in (x^2, xy^2, xyz^2, xz^3, y^3z, y^4)$	unstable

1.15. Adjoint stability.

Proposition 1.15. *Let G be any semi-simple group with Lie algebra* $\mathfrak{g}$. *Then $X \in \mathfrak{g}$ is unstable* $\Leftrightarrow$ ad X *is nilpotent.*

Proof: ($\Rightarrow$) From the formula $\operatorname{ad}(\operatorname{Ad} g\,(x)) = \operatorname{Ad} g \circ \operatorname{ad} x \circ \operatorname{Ad} g^{-1}$ it is immediate that the characteristic polynomial $\det(tI - \operatorname{ad} x)$ is G-invariant, hence that is coefficients are invariant functions. If x is unstable, these all vanish so $\operatorname{ad} x$ is nilpotent.
($\Leftarrow$) If $\operatorname{ad} x$ is nilpotent then the $\{\exp t\,(x) \mid t \in k\}$ is a unipotent subgroup of G which must be contained in the unipotent radical $R_u\,(B)$ of some Borel subgroup B of G. Fix a maximal torus $T \subset B$, so $B = R_u\,.\,T$. Then by the structure theorem of semi-simple groups we can write $\mathfrak{g} = t + \left(\sum_{\alpha>0} \mathfrak{g}_\alpha\right) + \left(\sum_{\alpha<0} \mathfrak{g}_\alpha\right)$ where $t = \operatorname{Lie}(T)$ and $\left(\sum_{\alpha>0} \mathfrak{g}_\alpha\right) = \operatorname{Lie}(R_u\,(B))$. Let χ_α be the character of T, which is associated to $\alpha = (\alpha_i)$ (i.e. if $w \in T$, $y \in \mathfrak{g}_\alpha$ then $\operatorname{Ad}(w)\,(y) = \chi_\alpha(w)\,y$), and let l be a linear functional on the group of characters of T defining the given ordering: i.e.,

$$l(\chi_\alpha) = \sum_i c_i\,\alpha_i > 0 \quad \text{if} \quad \alpha > 0 \quad \text{and} \quad l(\chi_\alpha) < 0 \quad \text{if} \quad \alpha < 0\,.$$

We can always choose l so that all the c_i are integers. If we define a 1-PS $\lambda : \mathbf{G}_m \to T$ by $\lambda(t) = (\ldots, t^{c_i}, \ldots)$, then the weights of X with respect to λ are some subset of $\{l(\alpha) \mid \alpha > 0\}$, hence are positive. Thus X is unstable.

Remark. There are no stable points. One can show that the regular semi-simple elements of $\mathfrak{g}$ have closed orbits of maximal dimension but their stabilizers will be their centralizers, i.e. maximal tori of G, and hence far from finite.

1.16. Chow form. The Chow form is the answer to the problem of describing by an explicit set of numbers a general subvariety $V^r \subset \mathbf{P}^n$. In two cases, the problem has a very easy answer: a hypersurface has its equation F and a linear space L^r has its Plücker coordinates. The Chow form is just a clever combination of these two special cases. Suppose V^r has degree d. There are two ways to proceed

i) If $u = (u_i) \in \mathbf{P}^n$ write H_u for the hyperplance $\sum u_i X_i = 0$. One shows that there is an irreducible polynomial Φ_V such that

$$[V \cap H_u^{(0)} \cap \ldots \cap H_u^{(r)} \neq \varnothing] \Leftrightarrow [\Phi_V(u_i^{(0)}, \ldots, u_i^{(r)}) = 0]$$

Moreover Φ_V is multihomogeneous of degree d in each of the sets of variables $(u_0^{(j)}, \ldots, u_n^{(j)})$, Φ_V is unique up to a scalar, and Φ_V determines V.

ii) If G = Grassmanian of L^{n-r-1}'s in $\mathbf{P}^n$ and $\mathcal{O}_G(1)$ is the ample line bundle on G defined by its Plücker embedding, then the set of $L \in G$ such that $L \cap V \neq \varnothing$ is the divisor D_V of zeroes of some section if $\mathcal{O}_G(d)$ and V and D_V determine each other. (Unfortunately, D_V is almost always a singular divisor.)

These methods give the same result via the identification:

$$\bigoplus_{d=0}^{\infty} \Gamma(G, \mathcal{O}_G(d)) = \left\{ \begin{array}{l} \text{Homogeneous} \\ \text{coordinate} \\ \text{ring of } G \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \text{Subring } W \text{ of } \mathbf{C}[\ldots, U_i^{(j)}, \ldots] \text{ generated} \\ \text{by the Plücker coordinates} \\ P_{i_0, \ldots, i_r} = \det_{(r+1, r+1)} (U_{i_l}^{(j)}),\ i_0 < i_1 < \ldots < i_r \end{array} \right\}$$

Letting W_d be the d^{th} graded piece of W, the identification furnishes an irreducible representation

$$\mathrm{Symm}^d(\Lambda^{r+1}(\mathbf{C}^{n+1})) \twoheadrightarrow W_d \hookrightarrow \bigotimes^{r+1} \mathrm{Symm}^d(\mathbf{C}^{n+1})$$

Thus, although we will usually consider the Chow form as a point of the $SL(n+1)$ representation $\otimes^{r+1}\,\mathrm{Symm}^d(\mathbf{C}^{n+1})$ this form lies in the irreducible piece W_d and can be thought of as defining a divisor on the Grassmanian. For more details on Chow forms, see Samuel [17, Ch. 1 § 9].

1.17. Asymptotic stability. We will say that a variety $V^r \subset \mathbf{P}^n$ is *Chow stable* or simply *stable* if its Chow form is stable for the natural $SL(n+1)$-action. If L is an ample line bundle on V, we say that (V, L) is *asymptotically stable* if

$$\exists\, n_0 \text{ s.t. } \forall n \geqq n_0,\ \Phi_{\Gamma(L^n)}(V) \subset \mathbf{P}^{h^0(L^n)-1} \text{ is stable.}$$

Attention: a stable variety need not be asymptotically stable (nor, of course, vice versa). Indeed, one of the main goals of this exposition is to show that the asymptotically stable curves are exactly the "stable" curves of Deligne and Mumford, and that by using asymptotic stability we can construct $\overline{\mathcal{M}}_g$ as a "quotient" moduli space for these curves.

§ 2. A criterion for $X^r_{\cdot} \subset \mathbf{P}^n$ to be stable

If $f(a)$ is an integer-valued function which is represented by a rational polynomial of degree at most r in n for large n, we will denote by n.l.c. (f) (the normalized leading coefficient of f) the integer e for which $f(n) = e\dfrac{n^r}{r!}$ + lower order terms. (What r is to be taken, will always be clear from the context.)

PROPOSITION 2.1 [1]). *(The "Hilbert-Hilbert-Samuel" Polynomial). Suppose X is a k-variety (not necessarily complete), L is an invertible sheaf on X and $\mathscr{I} \subset \mathcal{O}_X$ is an ideal sheaf such that $Z = \operatorname{Supp} \mathcal{O}_X/\mathscr{I}$ is proper over k. Then there is a polynomial $P(n, m)$ of total degree $\leqq r$, such that, for large m*

$$\chi(L^n/\mathscr{I}^m L^n) = P(n, m) .$$

Proof. We can compactify X and extend L to a line bundle on this compactification, without altering the validity of the theorem so we may as well assume X proper over k. Let $\pi\colon B \to X$ be the blow-up of X along $\mathscr{I}$ (i.e. $B = B_{\mathscr{I}}(X) = \operatorname{Proj}(\mathcal{O}_X \oplus \mathscr{I} \oplus \mathscr{I}^2 \oplus \ldots)$) and let E be the exceptional divisor on B so that $\mathscr{I} . \mathcal{O}_B = \mathcal{O}(-E)$. The well-known theorems of F.A.C. (Serre [18]) for the vanishing of higher cohomology in the relative case imply that when $m \gg 0$:

i) $\pi_*(\mathcal{O}(-mE)) = \mathscr{I}^m$

ii) $R^i\pi_*(\mathcal{O}(-mE)) = (0)$, $i > 0$

Now examine the exact sequence:

$$0 \longrightarrow \mathscr{I}^m L^n \longrightarrow L^n \longrightarrow L^n/\mathscr{I}^m L^n \longrightarrow 0$$

The Hilbert polynomial for $\chi(L^n)$ certainly satisfies the conditions on P. Moreover, in view of i) and ii); we have for $m \gg 0$:

$$\chi(X, \mathscr{I}^m L^n) = \chi\big(B, \pi^* L^n(-mE)\big) = \chi\big(B, (\pi^* L)^{\otimes n} \otimes \mathcal{O}(-E)^{\otimes m}\big)$$

so, a theorem of Snapper [5, 21] guarantees that this last Euler characteristic is also a polynomial of the required type for large m and n. By the additivity of χ we are done.

---

[1]) This result and its geometric interpretation are essentially due to C. P. Ramanujam [16].

DEFINITION 2.2. *In the situation of Proposition* 2.1, *we denote by* $e_L(\mathscr{I})$ *(the multiplicity of* $\mathscr{I}$ *measured via* L*) the integer* n.l.c. $(\chi(L^n/\mathscr{I}^n L^n))$.

EXAMPLES. i) If $\mathscr{I} = 0$ and X is complete, P is the Hilbert polynomial of L. ii) If Z is set-theoretically a point x then P is the Hilbert-Samuel polynomial of $\mathscr{I}$ as an ideal of $\mathcal{O}_{x,X}$ and $e(\mathscr{I})$ is its multiplicity there: in particular, it is independent of L. Note that, in general, $e_L(\mathscr{I})$ depends on the formal completion of X along Z and the pull-backs of $\mathscr{I}, L$ to this formal completion.

2.3. CLASSICAL GEOMETRIC INTERPRETATION. Let $X^r \subset \mathbf{P}^n$ be a projective variety, $L = \mathcal{O}_X(1)$, and Λ be a subspace of $\Gamma(\mathbf{P}^n, \mathcal{O}(1))$. Define L_Λ to be the linear subspace of $\mathbf{P}^n$ given by $s = 0$, $s \in \Lambda$. Define $\mathscr{I}_\Lambda$ to be the ideal sheaf generated by the sections $s \in \Lambda$, i.e. $\mathscr{I}_\Lambda . L$ is the subsheaf of L generated by those sections and $Z = \mathrm{Supp}(\mathcal{O}_X/\mathscr{I}_\Lambda) = X \cap L_\Lambda$ is the set of their base points.

If $p_\Lambda : \mathbf{P}^n - L_\Lambda \to \mathbf{P}(\Lambda) = \mathbf{P}^m$ is the canonical projection, and π is the blow-up of X along $\mathscr{I}_\Lambda$ then there is a unique map q making the following diagram commute:

$$\begin{tikzcd}
 & X - Z \arrow[r, "\mathrm{res}\ p_\Lambda"] \arrow[dl, hook] \arrow[d, hook] & \mathbf{P}^m \\
X & B = B_{\mathscr{I}_\Lambda}(X) \arrow[l, "\pi"'] \arrow[ur, dashed, "q"'] &
\end{tikzcd}$$

Moreover, because sections of $\mathcal{O}_{\mathbf{P}^m}(1)$ pull back to sections of $\mathscr{I}_\Lambda . L$ on X and are blown-up to sections of L twisted by minus the exceptional divisor E,

$$q^*(\mathcal{O}_{\mathbf{P}^m}(1)) = (\pi^* L)(-E). \tag{2.4}$$

Define $p_\Lambda(X)$, the image of X by the projection p_Λ, to be $[\mathrm{cycle}(q(B))]$: that is, $q(B)$ with multiplicity equal to the degree of B over $q(B)$ if these have the same dimension and 0 otherwise. I claim

PROPOSITION 2.5. $e_L(\mathscr{I}_\Lambda) = \deg X - \deg p_\Lambda(X)$.

Proof. If H is the divisor class of a hyperplane section on X, then

$$\deg X = (H^r) = \text{n.l.c.}(\chi(\mathcal{O}_X(n))).$$

By 2.4, q is defined by the linear system of divisors of the form $\pi^{-1}(H) - E$, hence

$$\deg p_A(x) = ((\pi^{-1}(H) - E)^r) = \text{n.l.c.}\ \chi(\pi^*(\mathcal{O}(n)(-nE))) .$$

Finally, from its definition

$$\begin{aligned} e_L(\mathcal{I}_A) &= \text{n.l.c.}\ \chi(\mathcal{O}_X(n)/\mathcal{I}^n\mathcal{O}_X(n)) \\ &= \text{n.l.c.}\ \chi(\mathcal{O}_X(n)) - \text{n.l.c.}\ \chi(\mathcal{I}^n\mathcal{O}_X(n)) \\ &= \deg X - \deg p_A(X) \end{aligned}$$

This proof brings out the geometry even more clearly. If $H_1, \ldots, H_r$ are generic hyperplanes in $\mathbf{P}^r$ then

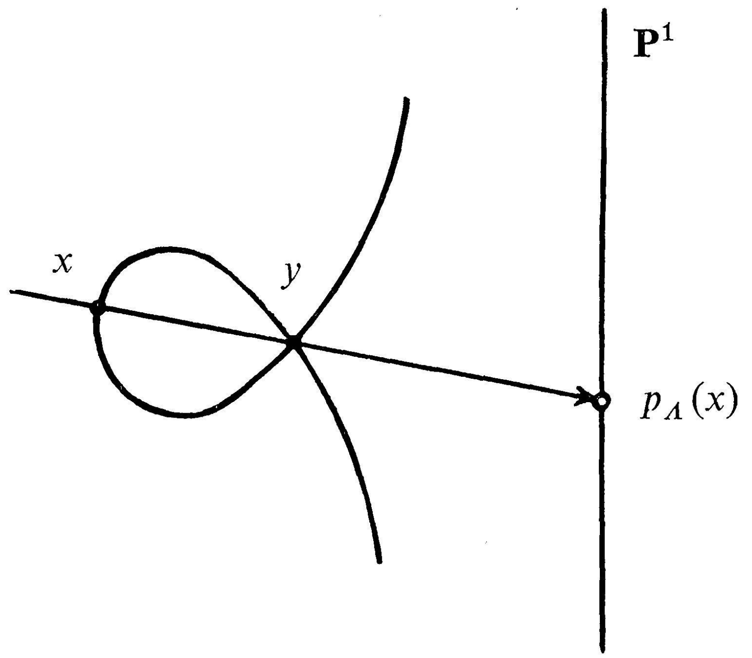
$$\deg(X) = \#(X \cap H_1 \cap \ldots \cap H_r), \quad (\# \text{ denoting cardinality}) .$$

As the H_i specialize to hyperplanes H_i' of the form $s = 0$, $s \in A$ (remaining otherwise generic) the points in this intersection specialize to either:

i) points outside Z: these points correspond to points in the intersection of $\operatorname{Im}(q)$ with r generic hyperplanes on $\mathbf{P}^n$, and each of these is the specialization of $\deg q$ of the original points i.e. $\deg p_A(X)$ points specialize in this way

ii) points in Z: $e_L(\mathcal{I}_A)$ measures the number of points which specialize in this way.

For example, if $X^1 \subset \mathbf{P}^2$ is a curve of degree d, $y = (0, 0, 1)$ is on X and $A = kX_0 + kX_1$, then $|Z| = \{y\}$, $p_A(x_0, x_1, x_2) = (x_0, x_1)$ and the picture is:



Thus $p_A(X) = (a\mathbf{P}^1)$, where a is the degree of the covering p; a generic line meets X in d points and as this line specializes to a non-tangent line through y it meets X at y on $\mathrm{mult}_y(X) = e_L(\mathcal{I}_A)$ points and meets X away from y in $d - e_L(\mathcal{I}_A) = a$ points.

The following technical facts will be useful in calculating the the invariants $e_L(\mathcal{I})$.

Proposition 2.6. a) *If (in the situation of Proposition 2.1) L and $\mathcal{I} . L$ are generated by their sections then* $\left| h^0(L^n/\mathcal{I}^n L^n) - e_L(\mathcal{I}) \frac{n^r}{r!} \right| = O(n^{r-1})$. *(Thus we can calculate $e_L(\mathcal{I})$ from the dimensions of spaces of sections.)*

b) *Suppose, in addition, we are given a diagram*

$$\begin{array}{ccc} X & \underset{\neq}{\supset} & X_0 = f^{-1}(0) \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathrm{Spec}(A) & \ni & 0 \end{array}$$

where f is proper, and a finite dimensional vector space $W \subset \Gamma(X, \mathcal{I} L)$ which

i) *generates $\mathcal{I} . L$*

ii) *defines a closed immersion $X - X_0 \hookrightarrow \mathbf{P}(\hat{W})$*

Then the dimensions of the kernel and cokernel of the map

$(\Gamma(X, L^n)/A$-submodule generated by the image of $W^{\otimes n} \to \Gamma(L^n/\mathcal{I}^n L^n)$

are both $O(n^{r-1})$.

Proof. The idea in a) is to show that $h^i(L^n/\mathcal{I}^n . L^n) = O(n^{r-1})$, $i \geq 1$. We first remark that is a compactification $\bar{X}$ of X over which L extends to a line bundle $\bar{L}$ such that

i) $\bar{L}$ is generated by its sections

ii) some $W \subset \Gamma(X, L)$ which generates $\mathcal{I} . L$ extends to a $\bar{W} \subset \Gamma(\bar{X}, \bar{L})$.

Indeed, on any compactification $\bar{X}$, there exists a coherent sheaf $\bar{\mathcal{F}}$ such that $\bar{\mathcal{F}}|_X \cong L$ and $\bar{\mathcal{F}}$ has properties i) and ii), and the pullback of $\bar{\mathcal{F}}$ to the blow-up $B_{\bar{\mathcal{F}}_1}(\bar{X})$ is a line bundle with these properties: so we might as well replace $\bar{X}$ by $B_{\bar{\mathcal{F}}}(\bar{X})$. Then if we take an ideal sheaf $\bar{\mathcal{I}}$ such that $\bar{W}$ generates $\bar{\mathcal{I}} . \bar{L}$, $\bar{\mathcal{I}} = \mathcal{I} . \mathcal{I}'$ where $\mathcal{I}'$ is supported on $\bar{X} - X$ only, and it suffices to show $h^i(\bar{L}^n/\mathscr{I}^n\bar{L}^n) = O(n^{r-1})$ $i \geqq 1$ since $\bar{L}^n/\mathscr{I}^n\bar{L}^n \cong \bar{L}^n/\mathscr{I}^n\bar{L}^n \oplus \bar{L}^n/\mathscr{I}'^n.\bar{L}^n$ so this bounds $h^i(L^n/\mathscr{I}^nL^n)$. To do this, it suffices, in turn, to bound $h^i(\bar{X}, \bar{L}^n)$ and $h^i(\bar{X}, \bar{\mathscr{I}}^n . \bar{L}^n) = h^i(B_{\bar{\mathscr{I}}}(\bar{X}), \bar{L}(-\bar{E})^{\otimes n})$ (where E is the exceptional divisor on $B_{\bar{\mathscr{I}}}(\bar{X})$). These bounds follow from:

Lemma 2.7. *If X^r is proper over k and L is a line bundle on X generated by its sections, then* $h^i(L^{\otimes n}) = O(n^{r-1})$, $i \geqq 1$.

Proof. Let X_0 be the image of X in $\mathbf{P}^n$ under the map given by the sections of L. Then $L = \pi^*(\mathcal{O}_{X_0}(1))$ and

$$\begin{aligned} H^i(X, L^{\otimes n}) &= H^i(X, \pi^*(\mathcal{O}_{X_0}(n))) \\ &\cong H^0(X_0, (R^i\pi_*\mathcal{O}_{x_0}) \otimes \mathcal{O}_{X_0}(n)) \\ &\quad \text{for } n \text{ large.} \end{aligned}$$

The last isomorphism follows from first applying the Leray spectral sequence, and then noting that all the terms involving higher cohomology groups vanish for large n, by the ampleness of $\mathcal{O}_{X_0}(1)$. But if $p \in \operatorname{Supp} R^i\pi_*\mathcal{O}_{X_0}$ for $i \geqq 1$, the fibre $\pi^{-1}(p)$ has positive dimension, hence $\dim \operatorname{Supp} R^i\pi_*\mathcal{O}_{X_0}$ $\leqq r-1$ which gives the desired $O(n^{r-1})$ bound on the dimension of the last space.

A suitable compactification and an argument like that in the proof of a), reduce the part of the statement of b) about the cokernel to bounding an $h^1(\mathscr{I}^n . L^n)$ and this is accompanied as in a) by a blow-up and the lemma. The procedure for dealing with the kernel is somewhat different: What we want to control is the dimension

$$(H^0(\mathscr{I}^nL^n)/A\text{-submodule generated by the image of } W^{\otimes n})$$

That is to say, for $n \gg 0$, the dimension of:

$$(H^0(B(X), \pi^*L^n(-nE))/A\text{-submodule generated by image of } W^{\otimes n})$$

Let $B = B_{\mathscr{I}}(X)$ and q be the proper, birational map $B \xrightarrow{q} B' \subset \mathbf{P}^n \times \operatorname{Spec} A$ induced by W. Then $q^*(\mathcal{O}_{B'}(1)) = \pi^*L(-E)$ and for large n, we have

$$\begin{array}{ccc} H^0(B, L^n(-nE)) & \cong & H^0(B', q_*(\mathcal{O}_B) \otimes \mathcal{O}_{B'}(n)) \\ \cup\!\!\uparrow & & \uparrow \\ \left[\begin{array}{l} A\text{-submodule} \\ \text{generated by} \\ \text{the image of } W^{\otimes n} \end{array}\right] & \cong & H^0(B', \mathcal{O}_{B'}(n)) \end{array}$$

The cokernel of the inclusion on the right is just $H^0(B', q_*(\mathcal{O}_B)/\mathcal{O}_{B'}(n))$. But the support of this last sheaf is proper over $0 \in \operatorname{Spec} A$, hence of dimension less than r, so a final application of the lemma completes the proof.

2.8. Fix : $X^r \subset \mathbf{P}^n$ a projective variety,

$X_0, \ldots, X_n$ coordinates on $\mathbf{P}^n$,

Φ_X the Chow form of X,

$$\lambda(t) = \begin{bmatrix} t^{\rho_0} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & t^{\rho_n} \end{bmatrix} \cdot t^{-k}, \quad \rho_0 \geqq \rho_1 \geqq \ldots \geqq \rho_n \geqq 0,$$

k chosen so that this is a 1-PS of $SL(n+1)$, i.e. $k = -\sum \rho_i/n+1$.

We define an ideal sheaf $\mathscr{I} \subset \mathcal{O}_{X \times A^1}$ by

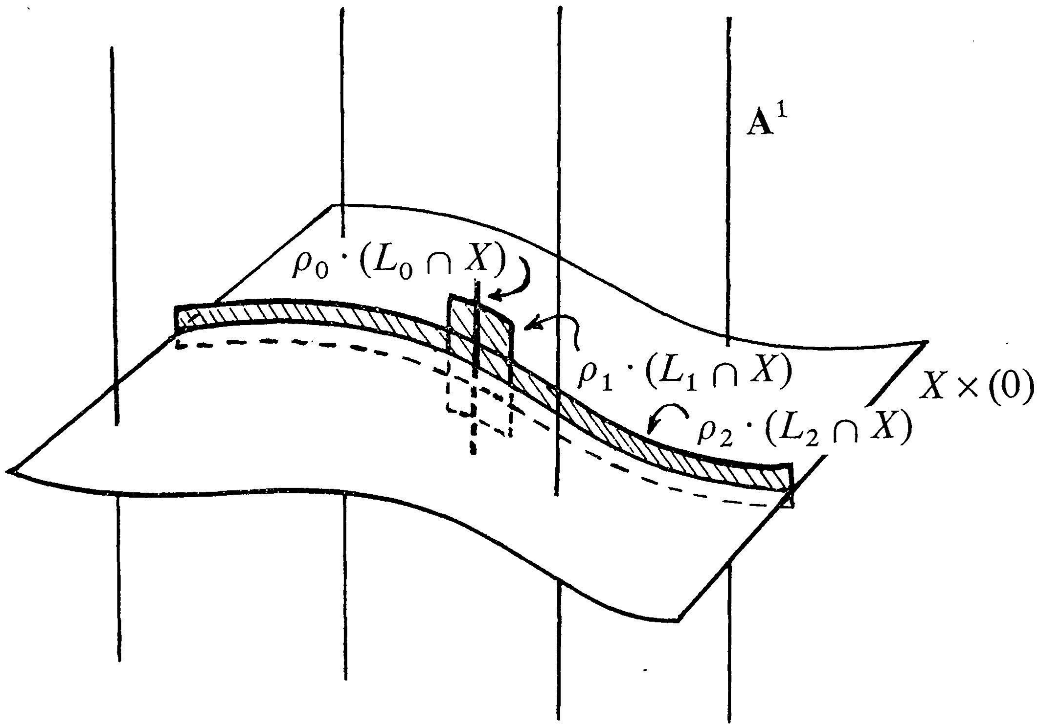
$$\mathscr{I} \cdot [\mathcal{O}_X(1) \otimes \mathcal{O}_{\mathbf{A}^1}] = \text{subsheaf generated by } \{ t^{\rho_i} X_i \}, \; i = 0, \ldots, n.$$

Remarks. i) From an examination of the generators of $\mathscr{I}$, one sees that the support of the subscheme $Z = \mathcal{O}_{X \times \mathbf{A}^1}/\mathscr{I}$ is concentrated over $0 \in \mathbf{A}^1$; if we normalize the ρ_i so that $\rho_n = 0$ then the support of $\mathscr{I}$ also lies over the section $X_n = 0$ in X.

ii) Consider the weighted flag:

$$\begin{array}{ccccc} (X_1 = \ldots = X_n = 0) & \subset (X_2 = \ldots = X_n = 0) & \subset \ldots \subset & (X_n = 0) \\ \| & \| & & \| \\ L_0 & L_1 & & L_{n-1} \\ \text{weight } \rho_0 & \text{weight } \rho_1 & & \text{weight } \rho_{n-1} \end{array}$$

The subscheme Z looks roughly like a union of ρ_i^{th}-order normal neighborhoods of $L_i \cap X$. It is easily seen to depend only on the weighted flag and not on the splitting defined by λ.



iii) Roughly speaking, $e_{\mathcal{O}_{\mathbf{A}^1} \otimes \mathcal{O}_X(1)}(\mathcal{I})$, which we will denote $e(\mathcal{I})$ measures the degree of contact of this weighted flag with X[1]. The multiplicity of $\mathcal{I}$ can be expected to get bigger, for example, if L_0 becomes a more singular point of X or if L_{n-1} oscillates to X to higher degree. The main theorem of this chapter makes this more precise:

THEOREM 2.9. *In the situation of* 2.8, Φ_X *is stable (resp.: semi-stable) with respect to* λ *if and only if:*

$$e(\mathcal{I}) < \frac{(r+1)\deg X}{n+1} \cdot \sum_{i=0}^{n} \rho_i$$

$$\left(\text{resp.: } e(\mathcal{I}) \leq \frac{(r+1)\deg X}{n+1} \cdot \sum_{i=0}^{n} \rho_i\right)$$

Proof. We begin with a definition.

DEFINITION 2.10. *If* $\mu\colon \mathbf{G}_m \to GL(W)$ *is a representation of* $\mathbf{G}_m$ *and* W_i *is the eigenspace where* $\mathbf{G}_m$ *acts by the character* t^i, *then the* μ*-weight of* W *is* $\sum_{i=-\infty}^{\infty} i \,.\, \dim W_i$. *If* $w \in W_i$ *then we say* i *is the* μ*-weight of* w.

[1]) It seems to be a general fact of life that one must go up to some $(r+1)$ dimensional variety—here $X \times \mathbf{A}^1$—to measure such a contact on an r-dimensional variety.

1) THE LIMIT CYCLE. If $X^{\lambda(t)}$ is the image of X by $\lambda(t)$, then taking $\lim_{t \to 0} X^{\lambda(t)}$ gives a scheme $X^{\lambda(0)}$ and an underlying cycle $\tilde{X}$, both of which are fixed by λ. Moreover, $\Phi_{X^{\lambda(t)}} = (\Phi_X)^{\lambda(t)}$ so if $\Phi_X = \sum_{i=a}^{b} \Phi_{X,i}$ where $\Phi_{X,i}$ is the component of Φ_X in the i^{th} weight space; then

$$\begin{aligned} \Phi_{X^{\lambda(t)}} &= \sum_{i=a}^{b} t^i \, \Phi_{X,i} \\ &= t^a \left[\Phi_{X,a} + t \text{ (other terms)} \right] \end{aligned}$$

Hence, $\Phi_{\tilde{X}} = \Phi_{X,a}$ and a is the λ-weight of $\Phi_{\tilde{X}}$. By definition, Φ_X is stable (resp: semi-stable) with respect to λ if and only if $a < 0$ (resp: $a \leqq 0$) or equivalently if and only if the λ-weight of $\Phi_{\tilde{X}}$ is < 0 (resp: $\leqq 0$).

2) The next step is to connect this weight with a Hilbert polynomial; this is done by:

PROPOSITION 2.11. *Let* $V^r \subset \mathbf{P}$ *be fixed by a* 1-PS λ *of* $SL(n+1)$, *let* I *be the homogeneous ideal of* V *and let* $R_n = (k[x_0, \ldots, X_n]/I)_n$ (i.e. $V = \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty} R_n \right)$). *Let* a_V *be the* λ*-weight of* Φ_V *and* r_n^V *be the* λ*-weight of* R_n. *Then for large* n, r_n^V *is represented by a polynomial in* n *of degree at most* $(r+1)$ *with* n.l.c. a_V.

Proof. a) Assume V is linear. In suitable coordinates, we can write $V = V(X_{r+1}, \ldots, X_n)$ and $\lambda(t) = \begin{bmatrix} t^{a_0} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & t^{a_n} \end{bmatrix}$. Then in the notation of 1.16, the Chow form of V is the monomial

$$\Phi_V = \det(U_i^{(j)}), \quad i, j = 0, \ldots, n .$$

Hence $\Phi_{\tilde{V}} = \Phi_V$ and has weight $\sum_{i=0}^{r} a_i$. On the other hand the λ-weight of R_n depends only on $a_0 \ldots a_r$, is symmetric in these weights, and is linear in the vector $(a_0, \ldots, a_r)$, hence depends only on $\sum_{i=0}^{r} a_i$. By considering the case $a_0 = \ldots = a_r$ we see that

$$r_n^V = \frac{n}{r+1} \left(\sum_{i=0}^{r} a_i \right) \dim R_n = a_V \cdot \frac{n}{r+1} \cdot \binom{n}{r}$$

which is certainly of the form claimed.

b) V is a positive cycle of linear spaces. Here it is more convenient to consider the ideal I instead of V. By noetherian induction, we can suppose the claim proven for all λ-fixed ideals $I' \supsetneqq I$. Then if $V = \sum a_i L_i$, let J_1 be the ideal of L_1, and choose an $a \in k[X] - I$ which is a λ-eigenvector of weight, say, w and such that $J_1 \, a \subset I$. Now look at the exact sequence:

$$0 \to a + I/I \to k[x]/I \to k[x]/I + a \to 0$$

The claim is true for $I + a$ by the noetherian induction. If $I' = \{f \mid af \in I\} \supset J_1 \supsetneqq I$, then via the shift of weights by w, $a + I/I \cong k[x]/I'$; but this shift changes the λ-weight by an amount $w \,.\, \dim [(k[x]/I')_n]) = O(n^r)$, hence does not affect the leading coefficient of the λ-weight. The claim for I', which also follows from the noetherian induction, thus proves the claim for I.

c) Reduction to case b). Recall the Borel fixed point theorem: if G is a connected solvable algebraic group acting on a projective variety W, then there is a fixed point on $\overline{O^G(y)}$ for every $y \in W$. Let $[V]$ be the associated point of V in $\mathrm{Hilb}_{\mathbf{P}^n}$ and consider the orbit of $[V]$ under the action of a maximal torus $T \subset SL(n+1)$ containing $\lambda(t)$. Let $[V_0]$ be a T-invariant point in $\overline{O^T([V])}$. Then V_0 is a sum of linear spaces, since these are the only T-invariant subvarieties of $\mathbf{P}^n$. If we decompose Φ_V by $\Phi_V = \sum_\alpha \Phi_V^\alpha$, where α runs over the characters of T and Φ_V^α is the part of Φ_V on which T acts with weight α, then for any $\tau \in T$, $\Phi_V^\tau = \sum_\alpha c_\alpha^\tau \Phi_V^\alpha$ for suitable constants c_α^τ. Since Φ_{V_0} is both T-invariant and a limit of forms Φ_V^τ, $\tau \in T$, $\Phi_{V_0} = \Phi^\alpha$ for some α. Moreover since V is a λ-invariant point, all the characters α appearing in the decomposition of Φ_V must have the same value on λ, hence the λ-weight of Φ_{V_0} is the λ-weight of Φ_V.

It remains only to compare the homogeneous coordinate rings. Now V and V_0 are members of a flat family V_t, $t \in S$ for some connected parameter space S, so that if $n \gg 0$, $H^0(V_t, \mathcal{O}_{V_t}(n))$ are the fibres of a vector bundle over S. This means that the λ-action on these fibres varies continuously, hence that the λ-weights of all the fibres are equal. Now the claim for V follows from b).

REMARK. The relation between Chow forms and Hilbert points in c) is really much more general: in fact, Knudsen [12] has shown that there is a canonical isomorphism of 1-dimensional vector spaces $k \cdot \Phi_V \cong [(r+1)^{\mathrm{st}}$ "differences"—formed via $\otimes$—of successive spaces in the sequence $\Lambda^{\dim R_n} R_n]$, and it is possible to base the whole proof of 2.11 on this.

3) Next we will see how to obtain $X^{\lambda(0)}$ by blowing up $\mathscr{I}$. Consider the map

$$\Lambda_1 : \mathbf{G}_m \times X \to \mathbf{P}$$
$$(t, X) \mapsto \lambda(t)(x) .$$

If the embedding of X is defined by $s_0, \ldots, s_n \in \Gamma[X, \mathcal{O}_X(1)]$ and the action of $\lambda(t)$ is by $(a_0, \ldots, a_n) \mapsto (t^{r_0} a_0, \ldots, t^{r_n} a_n)$ with $r_0 \geqq r_1 \geqq \ldots \geqq r_n$ and $\sum_{i=0}^{n} r_i = 0$ (i.e. $(0, \ldots, 0, 1)$ is an attractive fixed point and $(1, 0, \ldots, 0)$ is a repulsive fixed point), then $\Lambda^*_1(X_1) = t^{r_i} s_i$. Now $t^{-\gamma}$ is a unit on $\mathbf{G}_m \times X$, so changing the identification $\Lambda_1^*(\mathcal{O}_{\mathbf{P}^n}(1)) \cong \mathcal{O}_{\mathbf{G}_m} \otimes \mathcal{O}_X(1)$ by this unit we can assume $\Lambda_1^*(X_1) = t^{\rho_i} s_i$ where $\rho_i = r_i - \gamma$ is normalized as in 2.8 so that $\rho_n \geqq 0$. Then Λ_1 "extends" to a rational map $\mathbf{A}^1 \times X \to \mathbf{P}^n$ which is defined by the section $\{ t^{\rho_i} s_i \} \in \Gamma(\mathbf{A}^1 \times X, p_2^* \mathcal{O}_X(1)$. $\mathscr{I}$ is just the ideal sheaf these generate in $\mathcal{O}_{\mathbf{A}^1 \times X}$ and Z is just the set of base points of the rational map. Blowing up along $\mathscr{I}$ gives the picture

$$\begin{tikzcd}
\begin{matrix} E \\ \text{exceptional} \\ \text{divisor} \end{matrix} \arrow[rr, hook] & & B = B_{\mathscr{I}}(\mathbf{A}^1 \times X) \arrow[dl, "\pi"'] \arrow[dr, "\Lambda"] & & \\
& \mathbf{A}^1 \times X \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & & \mathbf{A}^1 \times \mathbf{P}^n \arrow[dl, "p_1"] & \\
X & & \mathbf{A}^1 & &
\end{tikzcd}$$

where the morphism Λ is defined by the sections $\{ t^{\rho_i} s_i \}$ in $\Gamma[B, (p_2 \pi)^* (\mathcal{O}(1))(-E)]$. Now $\mathrm{Im}(\Lambda)$ is the closed subscheme of $\mathbf{A}^1 \times \mathbf{P}^n$ given by $\mathrm{Proj}(\bigoplus_{m=0}^{m} R_m)$ where

$$(2.12) \qquad R_m = \begin{bmatrix} k\,[t]\text{-submodule of } \Gamma\left(X, \mathcal{O}\,(m)\right) \otimes_k k\,[t] \\ \text{generated by } m^{\text{th}} \text{ degree monomials in } \{\, t^{\rho_i} s_i \,\} \end{bmatrix}$$

In fact, Im Λ is *flat* over $\mathbf{A}^1$, because of:

LEMMA 2.13. *Let S be a non-singular curve, X flat over S and $f : X \to Y$ be a proper map over S. Then the scheme $(f(X), \mathcal{O}_Y/\ker f^*)$ is flat over S.*

Proof. We may as well suppose $S = \operatorname{Spec} R$; and then this amounts to showing the $\mathcal{O}_Y/\ker f^*$ has no R-torsion: if $a \in \mathcal{O}_Y/\ker f^*$, $r \in R$, then $r \,.\, a = 0 \Rightarrow r \,.\, f^*\, a = 0 \Rightarrow f^*\, a = 0 \Rightarrow a = 0$.

In particular, we see that $X^{\lambda(0)}$ is the fibre of Im Λ over $t = 0$, i.e. $X^{\lambda(0)} = \operatorname{Proj}\,\Big(\bigoplus\limits_{m=0}^{m} R_m/tR_m \Big)$.

4) The proof is completed by making precise the relation between $\mathcal{I}$ and the λ-weight of $\Phi_{\tilde{X}}$. One must be careful however because there are two $\mathbf{G}_m$-actions on R_m/tR_m, that given by the identification $R_1/tR_1 = \oplus\, (t^{r_i} s_i)\, k$, which is just λ, and that given by the identification $R_1/tR_1 = \oplus\, (t^{\rho_i} s_i)\, k$; call this action μ. The weights of μ on R_m/tR_m are just those of λ translated by $m\gamma$. By Proposition 2.11

$$\begin{aligned} \lambda\text{-weight of } \Phi_{\tilde{X}} &= \text{n.l.c. } (\lambda\text{-weight of } R_m/tR_m) \\ &= \text{n.l.c. } (\mu\text{-weight of } R_m/tR_m + \gamma m \dim\,(R_m/tR_m)) \\ &= \text{n.l.c. } (\mu\text{-weight of } R_m/tR_m) - \left(\frac{r+1 \deg X}{n+1} \sum_{i=0}^{n} \rho_i \right) \end{aligned}$$

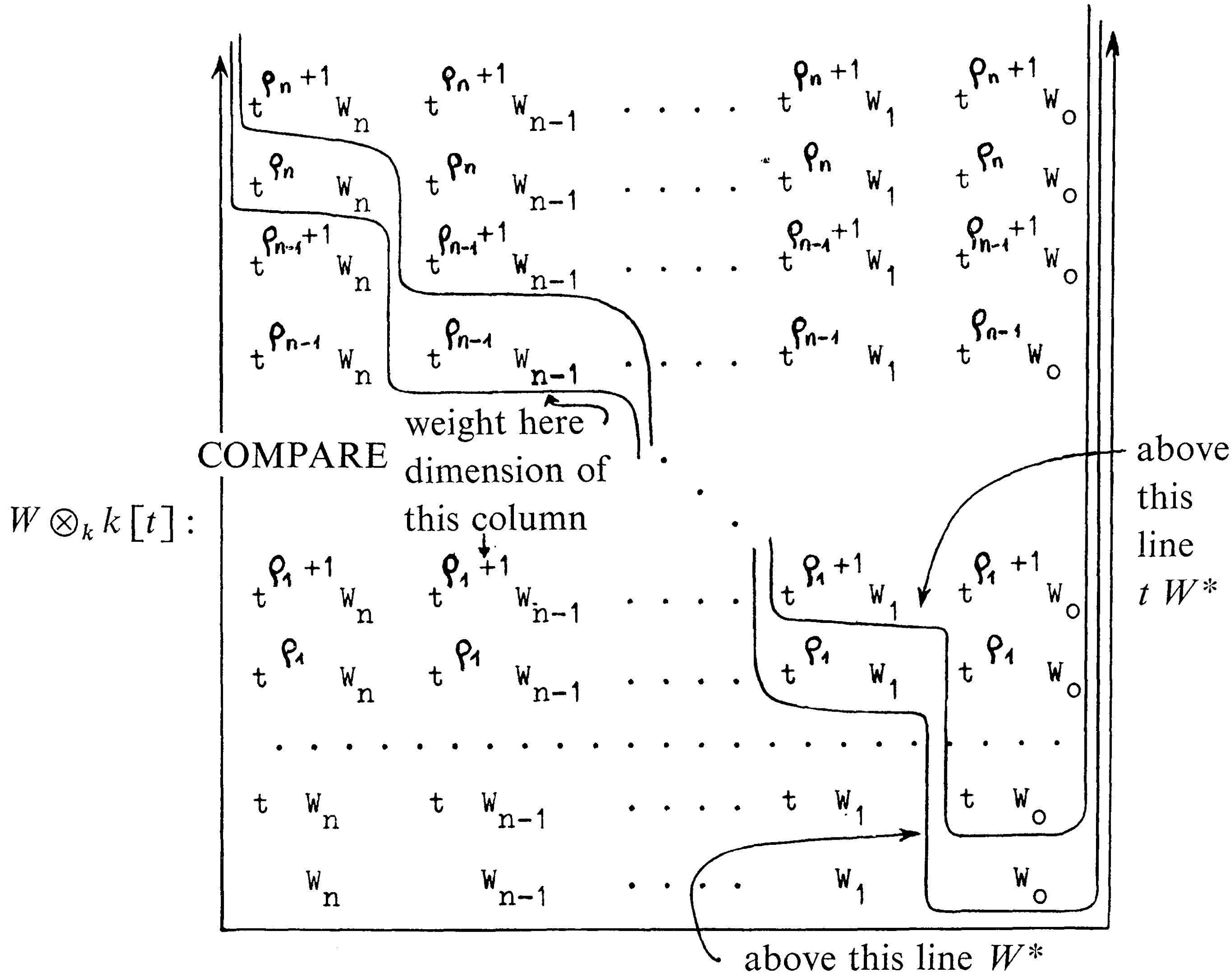
using $\gamma = -\dfrac{1}{n+1} \sum \rho_i$ and

$$\begin{aligned} \dim\,(R_m/tR_m) &= (\deg X_{\lambda(0)})\, \frac{m^r}{r\,!} + \text{lower terms} \\ &= \frac{(\deg X)\, m^r}{r\,!} + \text{lower terms.} \end{aligned}$$

A droll lemma allows us to re-express the μ-weight of R_m/tR_m.

LEMMA 2.14. *Let W be a k-vector space and let $\mathbf{G}_m$ act by μ on W with weights $\rho_n \geqq \rho_{n-1} \dots \geqq \rho_0 = 0$. Let W_i be the eigenspace of weight ρ_i and let W^* be the $k\,[t]$-submodule of $W \otimes k\,[t]$ generated by $\oplus\, t^{\rho_i} W_i$. Then $\dim\,(k\,[t] \otimes W/W^*) = \mu$-weight of W^*/tW^*.*

Proof by Diagram :



Recalling the definition of R_m (2.12), and applying this to the μ-action on R_m/tR_m, we see that the μ-weight of R_m/tR_m is just: $\dim\big(\Gamma\ (X,\ \mathcal{O}\ (m)) \otimes_k k\ [t]/R_m\big)$. But the sections $\{\, t^{\rho_i} s_i \,\}$ whose m^{th} tensor powers generate R_m, also generate $\mathcal{I}\ .\ p_2^*\ (\mathcal{O}_{X(1)})$ so by a) and b) of Proposition 2.6, this last dimension can be used to calculate $e\ (\mathcal{I})$. Putting all this together, we see that:

$$
\begin{aligned}
&\Phi_X \text{ is stable with respect to } \lambda \\
&\Leftrightarrow \lambda\text{-weight of } \Phi_X < 0 \\
&\Leftrightarrow e_L(\mathcal{I}) - \frac{(r+1)}{(n+1)} \deg X \sum_{i=0}^{n} \rho_i < 0
\end{aligned}
$$

which, with the analogous statement for semi-stability, is our theorem.

2.15. Interpretation via reduced degree. If $X^r \subset \mathbf{P}^n$ is a variety, its reduced degree is defined to be:

$$\text{red. deg}\ (X) = \frac{\deg X}{n+1-r}$$

A very old theorem says that if X is not contained in any hyperplane then red. deg $(X) \geqq 1$. Reduced degree measures, in some sense, how complicatedly X sits in $\mathbf{P}^n$, and there are classical classifications of varieties with small reduced degree. For example if X has reduced degree 1 and is not contained in any hyperplane then X is either

a) a quadric hypersurface

b) the Veronese surface in $\mathbf{P}^5$ or a cone over it

c) a rational scroll: $X = \mathbf{P}\left(\bigoplus_{i=0}^{r} \mathcal{O}_{\mathbf{P}^1}(n_i)\right) \subset \mathbf{P}^N$, $n_i > 0$

where $N = \sum_{i=0}^{r} (n_i+1) - 1$, or a cone over it. (This is called a scroll because the fibres $\mathbf{P}^{r-1}$ of X over $\mathbf{P}_1$ are linearly embedded.)

Some other facts about reduced degree are:

i) canonical curves, K3-surfaces and Fano 3-folds have red. deg $= 2$;

ii) all non-ruled surfaces and all special curves have red. deg $\geqq 2$. (For special curves, this is just a restatement of Clifford's theorem.)

iii) for ample L on X^r, the embedding by $L^{\otimes r}$ has reduced degree asymptotic to $r\,!$ as $n \to \infty$;

iv) red-deg is preserved under taking of proper hyperplane sections.

It would be very interesting to know whether almost all 3-folds (in a sense similar to that of ii) for surfaces) have red. deg $\geqq 2 + \varepsilon$. The following definition is introduced only tentatively as a means of linking the present ideas to older ideas (e.g. Albanese's method to simplify singularities of varieties):

2.16. Definition. *A variety $X^r \subset \mathbf{P}^n$ is linearly stable (resp. linearly semi-stable) if, whenever $L^{n-m-1} \subset \mathbf{P}^n$ is a linear space such that the image cycle $p_L(X)$ of X under the projection $p_L : \mathbf{P}^n - L \to \mathbf{P}^m$ has dimension r, then* red deg $p_L(X) >$ red deg X (*resp.* red-deg $p_L(X) \geqq$ red deg X).

Attention: p_L is allowed to be finite to 1, and which case $p_L(X)$ must be taken to be the image cycle. Linear stability is a property of the linear system embedding X; if $X^r \subset \mathbf{P}^n$ is embedded by $\Gamma(X, L)$, then X linearly stable means that for all subspaces $\Lambda \subset \Gamma(X, L)$

$$\frac{\deg p_L(X)}{\dim \Lambda - r} > \frac{\deg X}{n+1-r}$$

or equivalently, by applying Proposition 2.5,

$$e(\mathscr{I}_\Lambda) < \frac{\deg X}{n+1-r} (\operatorname{codim} \Lambda)$$

EXAMPLES. i) when X is a curve of genus 0, it is linearly semi-stable but not stable. When $g \geqq 1$, Clifford's theorem shows that X is linearly stable whenever it is embedded by a complete non-special linear system (see § 4 below).

ii) $\mathbf{P}^2$ is linearly unstable when embedded by $\mathcal{O}(n)$, $n \geqq 3$ because it projects to the Veronese surface. In view of the next proposition, a very interesting problem is that of finding large classes of linearly (semi)-stable surfaces.
(It may, however, turn out that linear stability is really too strong, or unpredictable, a property for surfaces in which case this Proposition is not very interesting !)

PROPOSITION 2.17. *Fix $X^r \subset \mathbf{P}^n$, let C be any smooth curve and let L be an ample line bundle on C. Let $\Phi_i : C \times X \to \mathbf{P}^{N(i)}$ be the embedding defined by $\{S_j \otimes X_l\}$ where $\{S_j\}$ is a basis of $\Gamma(L^{\otimes i})$ and $X_l \in \Gamma(X, \mathcal{O}_X(1))$ are the homogeneous coordinates. If $\Phi_i(C \times X)$ is linearly semi-stable for all large i, then X^r is Chow-semi-stable.*

Proof. Choose a 1-PS: $\lambda(t) = \begin{bmatrix} t^{\rho_0} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & t^{\rho_n} \end{bmatrix} t^{-\frac{\Sigma \rho_i}{n+1}}$
as in (2.8).
Choose a point $p \in C$ an isomorphism $L_p \cong \mathcal{O}_p$ and an i large enough that $L^{\otimes i}$ is very ample and $L^{\otimes i}(-\rho_0 p)$ is non-special. Then the map

$$\bigoplus_{l=1}^{n} \Gamma(C, L^{\otimes i}) \,.\, X_l \xrightarrow{\Phi_i} \bigoplus_{l=0}^{n} [\mathcal{O}_{p,C}/\mathscr{M}_{p,C}^{\rho_0}] \,.\, X_i$$

is surjective. Let Λ^i be the inverse image of $\bigoplus_{l=0}^{n} [(\mathscr{M}_{p,C}^{\rho_l}/\mathscr{M}_{p,C}^{\rho_0}) \cdot X_l]$ under this map and let $\mathscr{I}_\Lambda^i \subset \mathcal{O}_{C \times X}$ be the induced ideal. Since all the $L^{\otimes i}$ are trivial near p and $\mathscr{I}_\Lambda^i$ has support on the fibre of $X \times C$ over P, the ideals

$\mathscr{I}_A^i$ are independent of i; we denote this ideal by $\mathscr{I}_A$. The hypothesis says that for large i

$$e(\mathscr{I}_A) \leqq \frac{\deg(C \times X)}{(n+1)(h^0(L^i) - r - 1)} \operatorname{codim} A$$
$$= \frac{(r+1)\deg X \deg L^{\otimes i}}{(n+1)(\deg L^{\otimes i} - g + 1) - r - 1} \cdot \sum_{l=0}^{n} \rho_l$$

and letting $i \to \infty$,

$$e(\mathscr{I}_A) \leqq \frac{(r+1)\deg X}{n+1} \sum_{l=0}^{n} \rho_l$$

But $C \times X$ along $p \times X$ is formally isomorphic to $\mathbf{A}^1 \times X$ along $0 \times X$ with corresponding $\mathscr{I}_A'$ s, so by Theorem 2.9., X is Chow-semi-stable.

§ 3. Effect of Singular Points on Stability

We begin with an application of Theorem 2.9.

Proposition 3.1. *Let $X^1 \subset \mathbf{P}^n$ be a curve with no embedded components such that* $\deg X/n+1 < 8/7$. *If X is Chow-semi-stable, then X has at most ordinary double points.*

Remarks. i) When $n = 2$, $\deg X/n+1 < 8/7 \Leftrightarrow \deg X < 4$ and the proposition confirms what we have seen in 1.10 and 1.11

ii) Suppose L is ample on X^1 and $X_m \subset \mathbf{P}^{N(m)}$ is the embedding of X defined by $\Gamma(X, L^{\otimes m})$. By Riemann-Roch, $\deg X_m/N(m) \to 1$ as $m \to \infty$, hence:

Corollary 3.2. *An asymptotically stable curve X has at most ordinary double points.*

In particular, if $X \subset \mathbf{P}^2$ has degree $\geqq 4$ and has one ordinary cusp, then, in $\mathbf{P}^2$, X is stable but when re-embedded in high enough space, X is unstable! The fact that this surprising flip happens was discovered by D. Gieseker and came as an amazing revelation to me, as I had previously assumed without proof the opposite.

iii) We will see in Proposition 3.14 that the constant 8/7 is best possible.

Proof of 3.1. We note first that a semi-stable X of any dimension cannot be contained in a hyperplane: if $X \subset V(X_0)$, then X has only positive weights with respect to the 1-PS

$$\lambda(t) = \begin{bmatrix} t^{-n} & & & & 0 \\ & t & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & t \end{bmatrix}$$

The plan is clear: by Theorem 2.9, it suffices to show that if x is a bad singularity of X, then there is a 1-PS.

$$\lambda(t) = \begin{bmatrix} t^{\rho_0} & & & 0 \\ & \cdot\cdot & & \\ & & \cdot & \\ 0 & & & t^{\rho_n} \end{bmatrix}$$

such that

$$e(\mathscr{I}) \geqq \frac{16}{7} \sum_{i=0}^{n} \rho_i > \frac{\deg X \,.\, (r+1)}{(n+1)} \sum_{i=0}^{n} \rho_i \,.$$

First, if $x \in X$ has multiplicity at least three, then take coordinates $(X_0, \ldots, X_n)$ so that $x = (1, 0, \ldots, 0)$ and let $\lambda(t) = \begin{bmatrix} t & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 1 \end{bmatrix}$ Then $\mathscr{I} \,.\, \mathcal{O}_{\mathbf{A}1 \times X}(1)$ is generated by $\{ tX_0, X_1, \ldots, X_n \}$. Since $\{ X_1, \ldots, X_n \}$ generate $\mathscr{M}_{x,X}$ and X_0 is a unit at x, $\mathscr{I} = (t, \mathscr{M}_x)\, \mathcal{O}_{\mathbf{A}1 \times X}$, i.e. $\mathscr{I}$ is the maximal ideal of $(0, x)$ on $\mathbf{A}^1 \times X$. Therefore, $e(\mathscr{I}) = \mathrm{mult}_{(0,x)}(\mathbf{A}^1 \times X) = \mathrm{mult}_x X \geqq 3$, which does what we want since $16/7 \sum_{i=0}^{n} \rho_i = 16/7 < 3$.

Now if $x \in V$ is a non-ordinary double point—i.e. a double point whose tangent cone is reduced to a single line—then $\dim(\mathscr{M}_{x,X}/\mathscr{M}^2_{x,X}) = 2$ and $\mathscr{M}_{x,X} \underset{\neq}{\supsetneq} I \underset{\neq}{\supsetneq} \mathscr{M}^2_{x,X}$ where I is the ideal of the tangent cone at x. Choose coordinates $(X_0, \ldots, X_n)$ such that

i) $X_0(x) \neq 0$

ii) $v = X_1/X_0$ and $u = X_2/X_0$ span $\mathscr{M}_{x,X}/\mathscr{M}^2_{x,X}$

iii) $u \in I$ so that $u^2 \in \mathscr{M}^3_{x,X}$.

iv) $X_3/X_0, \ldots, X_n/X_0 \in \mathscr{M}^2_{x,X}$

Then if $\lambda(t) = \begin{bmatrix} t^4 & & & & & & & 0 \\ & t^2 & & & & & & \\ & & t & & & & & \\ & & & 1 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ 0 & & & & & & \cdot & \\ & & & & & & & 1 \end{bmatrix}$ the associated ideal is

$\mathscr{I} = (t^4 X_0, t^2 X_1, t X_2, X_3, \ldots, X_n)$. But $\mathcal{O}_{\mathbf{A}^1 \times X}/\mathscr{I}$ is supported only at the point $(0, x)$ hence $e(\mathscr{I})$ is again Hilbert-Samuel multiplicity and is at least equal to the multiplicity of the possibly larger ideal $\mathscr{I}' = (t^4, t^2 v, tu, \mathscr{M}^2_{x,X})$. If I is the ideal $(t^4, \mathscr{M}^2_{x,X})$, then since

$$
\begin{aligned}
(t^2 v)^2 &= t^4 v^2 \in I^2 \\
(tu)^4 &= t^4 (u^2)^2 \in t^4 (\mathscr{M}^3_{x,X})^2 \subset I^4 \quad \text{by iii)}
\end{aligned}
$$

$\mathscr{I}'$ is integral over I. Hence

$$
e(\mathscr{I}) \geqq e(\mathscr{I}') = e(I) = (4)\,.\,(2)\,.\,e(\mathscr{M}_{x,X}) = 16 = \frac{16}{7} \sum_{i=0}^{n} \rho_i
$$

as required.

The attempt to systematize this theorem leads to a numerical measure of the degree of singularity of a point. The results that follow are part of a joint investigation of this concept by D. Eisenbud and myself. Full proofs will appear later. Many of these results have also been discovered independently by Jayant Shah.

DEFINITION 3.3. *If $\mathcal{O}$ is an equi-characteristic* [1] *local ring of dimension r, and $k \geqq 0$ is an integer, then we define $e_k(\mathcal{O})$, the k^{th} flat multiplicity of $\mathcal{O}$, by*

$$
\begin{aligned}
e_0(\mathcal{O}) &= \sup \left\{ \frac{e(I)}{r!\,\operatorname{col}(I)} \;\middle|\; I \text{ of finite colength in } \mathcal{O} \right\} \\
e_k(\mathcal{O}) &= e_0(\mathcal{O}[[t_1, \ldots, t_k]])
\end{aligned}
$$

It is obvious that if $\hat{\mathcal{O}}$ is the completion of $\mathcal{O}$, then $e_k(\hat{\mathcal{O}}) = e_k(\mathcal{O})$.

PROPOSITION 3.4. $e_k(\mathcal{O}) \geqq \max\big(1, e(\mathcal{O})/(r+k)\,!\big)$.

[1]) The hypothesis on $\mathcal{O}$ can be avoided, and the proof simplified, by a use of the associated graded ring instead of the Borel fixed point theorem (D. Eisenbud).

Proof. The second bound is obvious. To get the first note that if J is any ideal of finite colength then $e(J^n) = n^r e(J)$ and $\operatorname{col}(J^n) = \dfrac{e(J)\, n^r}{r!} + O(n^{r-1})$, hence

$$\frac{e(J^n)}{r!\operatorname{col}(J^n)} \to 1 \quad \text{as} \quad n \to \infty .$$

To get an upper bound on e_k we first obtain another lower bound!

PROPOSITION 3.5. $e_0(\mathcal{O}) \geqq e_0(\mathcal{O}[[t]])$; *moreover if* $r = \dim \mathcal{O} > 0$ *and there is equality, then the sup defining* $e_0(\mathcal{O}[[t]])$ *is not attained. Hence*

$$e_0(\mathcal{O}) \geqq e_1(\mathcal{O}) \geqq e_2(\mathcal{O}) \geqq \dots\dots\dots \geqq 1 .$$

Proof. We begin by giving a lemma which is useful in the applications of e_0 as well.

LEMMA 3.6. *Let* $\mathcal{I}$ *be the set of ideals of* $\mathcal{O}[[t]]$ *of the form* $I = \bigoplus_{i=0}^{\infty} I_i t^i$, *where* I_i *is an increasing sequence of ideals of finite colength in* $\mathcal{O}$ *such that* $I_N = \mathcal{O}$ *for some* N. *Then*

$$e_0(\mathcal{O}[[t]]) = \sup_{I \in \mathcal{I}} \frac{e(I)}{r!\operatorname{col}(I)}$$

Proof. For any equi-characteristic local ring R, let Hilb^n_R be the subscheme of the Grassmanian of codimension n subspaces of $R/\mathcal{M}^n_R$ parametrizing those subspaces which are ideals: since any ideal in R of colength n contains $\mathcal{M}^n_R$, Hilb^n_R parameterizes these ideals. Let $e\colon \mathrm{Hilb}^n_R \to \mathbf{Z}$ be the map assigning to an ideal its multiplicity. By results of Teissier and Lejeune [23], e is upper-semi-continuous.

The natural $\mathbf{G}_m$-action on $\mathcal{O}[[t]]$ by $t \to \lambda t$ induces a $\mathbf{G}_m$-action on $\mathrm{Hilb}^n_{\mathcal{O}[[t]]}$. By the Borel fixed point theorem, there is, for every I, an ideal fixed by this action in $O^{\mathbf{G}_m}(I)$. Such an ideal must, by the upper-semi-continuity of multiplicity have multiplicity at least as large as $e(I)$. Thus, to compute $e_0(\mathcal{O}[[t]])$ it suffices to look at $\mathbf{G}_m$-fixed ideals of finite colength and $\mathcal{I}$ is just the set of such ideals.

Fix $I = \bigoplus_{i=0}^{\infty} I_i t^i$, where $I_0 \subset I_1 \subset \dots \subset I_N = \mathcal{O}$ is an increasing sequence of ideals in $\mathcal{O}$. Clearly $\operatorname{col}(I) = \sum_{i=0}^{N-1} \operatorname{col}(I_i)$. To bound $e(I)$ we note that

$$
\begin{aligned}
I^n \supset (I_0^n) &\oplus (I_0^{n-1} I_1 t) \oplus (I_0^{n-2} I_1^2 t^2) \oplus \ldots \oplus (I_0 I_1^{n-1} t^{n-1}) \oplus \\
&\oplus (I_1^n t^n) \oplus (I_1^{n-1} I_2 t^{n+1}) \oplus \ldots\ldots\ldots \oplus (I_{N-2} I_{N-1}^{n-1} t^{(N-1)n-1}) \oplus \\
(3.7) \qquad &\oplus (I_{N-1}^n t^{(N-1)n}) \oplus (I_{N-1}^{n-1} t^{(N-1)n+1}) \oplus \ldots\ldots \oplus (I_{N-1} t^{Nn-1}) \\
&\oplus (\mathcal{O} t^{Nn}) \oplus \ldots \\
\Rightarrow I^n \supset (I_0^n &\oplus I_0^n t \oplus \ldots \oplus I_0^n t^{n-1}) \oplus (I_1^n t^n \oplus \ldots \oplus I_1^n t^{2n-1}) \oplus \ldots\ldots \\
&\oplus (I_{N-1}^n t^{(N-1)n} \oplus I_{N-1}^{n-1} t^{(N-1)n+1} \oplus \ldots . \oplus I_{N-1} t^{Nn-1}) \\
&\oplus \mathcal{O} t^{Nn} \oplus \ldots
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\operatorname{col}(I^n) &\leq \sum_{i=0}^{N-2} n \operatorname{col}(I_i^n) + \sum_{j=1}^{n} \operatorname{col}(I_{n-1}^j) \\
&= \frac{n^{r+1}}{r!} \sum_{i=0}^{N-2} e(I_i) + \frac{n^{r+1}}{(r+1)!} e(I_{N-1}) + O(n^r)
\end{aligned}
$$

(We have evaluated the second sum by "integration"!)
Finally

$$
\frac{e(I)}{(r+1)! \operatorname{col}(I)} \leq \frac{(r+1) \sum_{i=0}^{N-2} e(I_i) + e(I_{N-1})}{(r+1)! \sum_{i=0}^{N-1} \operatorname{col}(I_i)} \leq \frac{\sum_{i=0}^{N-1} e(I_i)}{r! \sum_{i=0}^{N-1} \operatorname{col}(I_i)},
$$

with strict inequality if $r > 0$

$$
\leq \max_i \frac{e(I_i)}{r! \operatorname{col}(I_i)} \leq e_0(\mathcal{O}) .
$$

Corollary 3.8. *If $\mathcal{O}$ is regular, $e_0(\mathcal{O}) = 1$ and if $r > 1$, the defining sup is not attained.*

Corollary 3.9. (Lech [1]). *For all $\mathcal{O}$ and all $I \subset \mathcal{O}$, $e(I) \leq r!\, e(\mathcal{O}) \operatorname{col}(I)$, hence $e_0(\mathcal{O}) \leq e(\mathcal{O})$.*

Proof. None of the quantities involved change if we complete $\mathcal{O}$. But after doing this, we can write $\mathcal{O}$ as a finite module over $\mathcal{O}_0 = k[[t_1, \ldots, t_r]]$ so that:

(*) There is a sub $\mathcal{O}_0$-module $\mathcal{O}_0^{e(\mathcal{O})} \subset \mathcal{O}$ such that the quotient $\mathcal{O}/\mathcal{O}_0$ is an $\mathcal{O}_0$-torsion module M.

[1]) Cf. [13], Theorem 3.

Let $I_0 = I \cap \mathcal{O}_0$. Then $\operatorname{col}(I) \geqq \operatorname{col}(I_0)$ and

$$\begin{aligned}\dim(\mathcal{O}/I^n) &\leqq \dim \mathcal{O}/I_0^n\,\mathcal{O}\\ &\leqq \dim(M/I_0^n\,M) + \dim(\mathcal{O}_0^{e(\mathcal{O})}/I_0^n\,\mathcal{O}^{e(\mathcal{O})})\end{aligned}$$

Condition (*) implies that $\dim(M/I_0^n\,M)$ is represented by a polynomial of degree less than r, hence

$$\begin{aligned}e(I) &\leqq e(\mathcal{O})\,e(I_0)\\ &\leqq r\,!\,e(\mathcal{O})\operatorname{col}(I_0) \text{ by Corollary 3.8}\\ &\leqq r\,!\,e(\mathcal{O})\operatorname{col}(I)\end{aligned}$$

We state two other useful properties of e_k:

PROPOSITION 3.10. i) *If* $\mathcal{O}$ *and* $\mathcal{O}'$ *are local domains with the same fraction field and* $\mathcal{O}'$ *is integral over* $\mathcal{O}$, *then* $e_k(\mathcal{O}') \leqq e_k(\mathcal{O})$.

ii) *If* $\mathcal{O} = (k[[t]] + \mathcal{P})$ *is an augmented* $k[[t]]$*-algebra, let* $\mathcal{O}_\eta = \mathcal{O}_{\mathcal{P}}$, *a local ring with residue field* $k((t))$ *and let* $\mathcal{O}_s = \mathcal{O}/t\mathcal{O}$ *be its specialization over* k; *then* $e_k(\mathcal{O}_\eta) \leqq e_k(\mathcal{O}_s)$.

We come now to the main definitions.

DEFINITION 3.11. $\mathcal{O}$ *is semi-stable if* $e_1(\mathcal{O}) = 1$; $\mathcal{O}$ *is stable if, in addition, the defining sup is not attained.*

This terminology is justified by the following proposition which shows that the semi-stability of the local rings on a variety X is just the local impact of the global condition of asymptotic semi-stability for X.

PROPOSITION 3.12. *Fix a variety* X^r, *an ample line bundle* $L = \mathcal{O}_X(D)$ *on* X, *and* $p \in X$. *Then if* $\mathcal{O}_{p,X}$ *is unstable,* (X, L) *is asymptotically unstable.*

Proof. Choose an ideal $I \subset \mathcal{O}_{p,X}[[t]]$ such that

i) $e(I) = (1+\varepsilon)(r+1)\,!\operatorname{col}(I)$, $\varepsilon > 0$

ii) $I = \bigoplus_{i=0}^{\infty} I_i t^i$, $I_0 \subset I_1 \subset \dots \subset I_N = \mathcal{O}_{p,X}$ a sequence of ideals of finite colength. (This is possible because of Lemma 3.6).

Let Φ_m denote the projective embedding of X by $\Gamma(X, L^{\otimes m})$. Choose m large enough that

a) for all $Q \in X$, $\Gamma(X^r, L^m) \xrightarrow{\psi} \Gamma(X, L^m/I_0 \mathscr{M}_{Q,X} \,.\, L^m)$ is surjective

b) L^m is very ample

c) $h^0(X, L^m) > \dfrac{1}{1+\varepsilon} \dfrac{m^r(D^r)}{r!} = \dfrac{1}{1+\varepsilon} \dfrac{\deg \Phi_m(X)}{r!}$

(That the last condition can always be realized is a consequence of Riemann-Roch for X.)
Next choose a basis $X_{i,j}$, $0 \leqq i \leqq N$, of $\Gamma(X, L^m)$ such that

$$\begin{aligned} &X_{0,j} \text{ is a basis of } \psi^{-1}(I_0)\,, \\ &X_{1,j} \text{ is a basis of } \psi^{-1}(I_1)/\psi^{-1}(I_0)\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &X_{N,j} \text{ is a basis of } \Gamma(X, L^m)/\psi^{-1}(I_{N-1})\,, \end{aligned}$$

Finally, let λ be the 1-PS which multiplies $X_{i,j}$ by t^i : i.e. in the form of (2.8) $\rho^{(i,j)} = i$; then by assumption (a) the ideal $\mathscr{I}$ corresponding to λ in (2.8) is just I and is supported at the single point $(0, p) \in \mathbf{A}^1 \times X$. Moreover, by condition a)

$$\begin{aligned} \sum_{i,j} \rho^{(i,j)} &= N \dim(\mathscr{O}/I_{N-1}) + (N-1)\dim(I_{N-1}/I_{N-2}) \\ &\quad + \ldots + 2\dim(I_2/I_1) + \dim I_1/I_0 = \operatorname{col}(I) \end{aligned}$$

(This is Lemma 2.14 again). Hence,

$$\begin{aligned} e(\mathscr{I}) &= e(I) \\ &= (1+\varepsilon)\,.\,(r+1)\,!\operatorname{col}(I) \\ &> (1+\varepsilon)\,.\,(r+1)\,.\,\frac{\deg \Phi_m(X)}{(1+\varepsilon)\,h^0(L^m)} \cdot \sum_{i,j} \rho^{(i,j)} \\ &= \frac{(r+1)\deg \Phi_m(X)}{h^0(L^m)} \cdot \sum_{i,j} \rho^{(i,j)} \end{aligned}$$

By Theorem 2.9, $\Phi_m(X)$ is unstable.
Restating Corollary 3.7 gives us a trivial class of stable points:

PROPOSITION 3.13. *If $\mathscr{O}$ is regular and of positive dimension it is stable.*

The next step is to pindown the meaning of semi-stability for small dimensional local rings. For dimension 1, we can be quite explicit:

PROPOSITION 3.14. *If* $\dim \mathcal{O} = 1$ *and* $\mathcal{O}$ *is Cohen-Macauley (i.e.* Spec $\mathcal{O}$ *has no embedded components), then:*

i) $\mathcal{O}$ *stable* $\Leftrightarrow \mathcal{O}$ *regular* $\Leftrightarrow e(\mathcal{O}) = e_0(\mathcal{O}) = e_1(\mathcal{O}) = \ldots = 1$.

ii) $\mathcal{O}$ *semi-stable but not stable* $\Leftrightarrow \mathcal{O}$ *an ordinary double point* $\Leftrightarrow e(\mathcal{O}) = e_0(\mathcal{O}) = 2$, $e_1(\mathcal{O}) = e_2(\mathcal{O}) = \ldots = 1$.

iii) $\mathcal{O}$ *a higher double point* $\Rightarrow e_1(\mathcal{O}) \geqq 8/7$.

iv) $\mathcal{O}$ *a triple point or higher multiplicity* $\Rightarrow e_1(\mathcal{O}) \geqq 3/2$.

Proof. If $\mathcal{O}$ is a triple or higher point, so is $\mathcal{O}[[t]]$, hence $e(\mathcal{O}[[t]]) \geqq 3$, and by Proposition 3.4, $e_1(\mathcal{O}) = e_0(\mathcal{O}[[t]]) \geqq 3/2$.

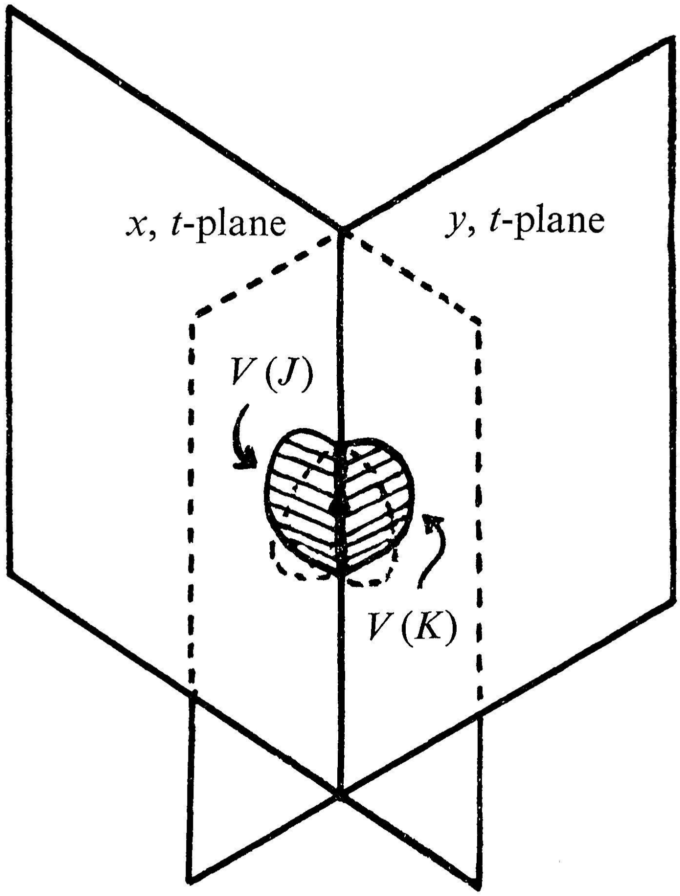
As for Cohen-Macaulay double points, when char. $\neq 2$ these are all of the form $\hat{\mathcal{O}} = k[[x, y]]/(x^2 - y^n)$, $2 \leqq n \leqq \infty$. (Think of $\hat{\mathcal{O}}$ as a quadratic free $k[[y]]$-algebra; the argument can be readily adapted to char. 2 also). If $n \geqq 3$, then in $k[[x, y, t]]/(x^2 - y^n)$, take $I = (x^2, xy, y^2, xt, yt^2, t^4)$. (This, of course, is the ideal of Proposition 3.1 again). I has complementary basis $(1, x, y, t, yt, t^2, t^3)$, hence $\mathrm{col}(I) = 7$. I claim $e(I) = 16$, which will imply iii). We first note that I is integral over (y^2, t^4). We compute the multiplicity of (y^2, t^4) as

$$\begin{aligned} &\textit{intersection-multiplicity at } \mathcal{M}\ \big((\mathrm{Spec}\ \mathcal{O}) \,.\, (y^2=0) \,.\, (t^4=0)\big) \\ &= 8 \,.\, \textit{intersection-multiplicity}\ \big((\mathrm{Spec}\ \mathcal{O}) \,.\, (y=0) \,.\, (t=0)\big) \\ &= 16 \end{aligned}$$

since $\mathcal{O}$ is a double point.

When $\mathcal{O}$ is an ordinary double point, I claim $e_0(\mathcal{O}[[t]]) = 1$. Since this value is attained by the maximal ideal $\mathcal{M}$: $\dfrac{e(\mathcal{M})}{2!\,\mathrm{col}(\mathcal{M})} = \dfrac{2}{2} = 1$, this will prove ii), hence i) in view of Proposition 3.13.

In general, if $\mathcal{O} = k[[x, y]]/(x \,.\, y)$, an ideal $I \subset \mathcal{O}[[t]]$ corresponds to a pair of ideals $J \subset k[[x, t]]$ and $K \subset k[[y, t]]$ such that $J + (x)/(x)$ and $K + (y)/(y)$ have the same image, say (t^n), in $k[[t]]$. A rough picture is given below: the condition on the two ideals ensures that they glue along the intersection of the two planes.



In this situation, $\operatorname{col}(I) = \operatorname{col}(J) + \operatorname{col}(K) - n$, and $e(I) = e(J) + e(K)$, so the inequality $e(I)/2 \,.\, \operatorname{col}(I) \leqq 1$ follows from:

LEMMA 3.15. *If* $I \subset k[[x, y]]$ *and* $I + (x) = (x, y^a)$, *then* $e(I) \leqq 2\operatorname{col}(I) - a$.

Proof. By applying Lemma 3.6, we can reduce to the case where I is generated by monomials:

$$I = \bigoplus_{l=0}^{\infty} (y^{r_l} \,.\, x^l) \,.\, k[[y]], \text{ with } a = r_0 \geqq r_1 \geqq \dots \geqq r_N = 0.$$

Then as in (3.7):

$$I^n \supset (y^{nr_0})\, k \oplus (y^{(n-1)r_0 + r_1} x)\, k \oplus (y^{(n-2)r_0 + 2r_1} x^2)\, k \oplus \dots$$
$$\oplus (y^{nr_1} x^n)\, k \oplus (y^{(n-1)r_1 + r_2} x^{n+1})\, k \oplus \dots\dots \oplus (y^{nr_2} x^{2n})\, k \oplus \dots\dots\dots\dots$$

$$\Rightarrow \operatorname{col}(I^n) \leqq \frac{n(n+1)}{2} r_0 + n^2 r_1 + n^2 r_2 + \dots\dots + n^2 r_{N-1}$$

$$\Rightarrow \frac{e(I)}{2} \leqq \frac{r_0}{2} + r_1 + \dots\dots + r_{N-1} = \operatorname{col}(I) - \frac{a}{2}.$$

REMARK. If $I \subset \mathcal{O}[[t]]$ is of the form of Lemma 3.6, the expansion (3.7) for I^n, which we have used again here, can be used to give even better bounds for $e(I)$. To get these however, requires the more involved theory of mixed multiplicities which will be discussed in § 4.

The meaning of semi-stability for two dimensional singularities is not yet completely worked out, but what follows gives a good overview of the situation.

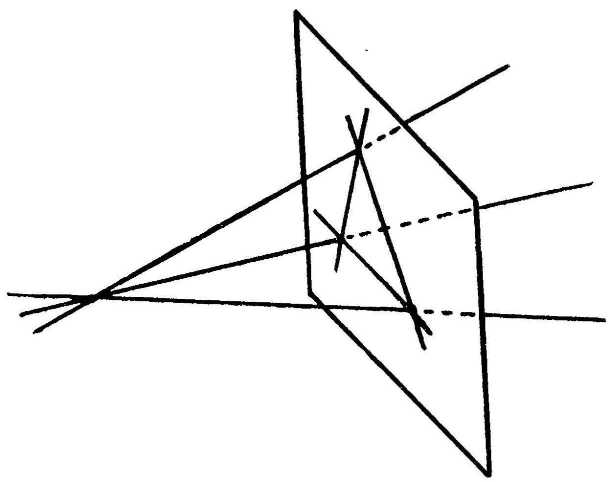
DEFINITION 3.16. *If* $\mathcal{O}$ *is a normal 2-dimensional local ring,* x *is the closed point of* Spec $\mathcal{O}$, *and* $X^* \xrightarrow{\pi}$ Spec $\mathcal{O}$ *is a resolution of* $\mathcal{O}$ *(i.e.* π *is proper and birational), then we define*

i) *big genus of* $\mathcal{O}$ = $\dim R^1\pi_*(\mathcal{O}_{X^*})$
($R^1\pi_*$ *is a torsion* $\mathcal{O}$*-module supported at* x)

ii) *little genus of* $\mathcal{O}$ = $\sup_Z (p_a(\mathcal{O}_Z))$, *where* Z *runs over the effective cycles on* $\pi^{-1}(x)$.

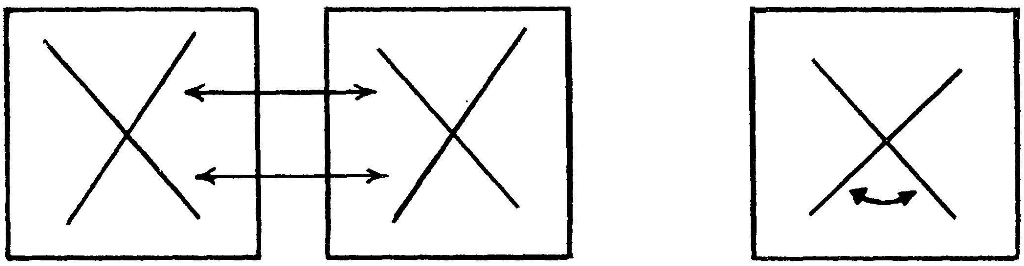
Wagreich [24] has shown that big genus $\geqq$ little genus—hence the names—and Artin [3] has shown that if the little genus is zero then so is the big genus. (But when little genus = 1, big genus may be > 1). We call $\mathcal{O}$: rational (resp. strongly elliptic) if its big genus is 0 (resp. 1), and weakly elliptic if its little genus is 1.

If there is to be any hope of constructing compact moduli spaces for semi-stable surfaces, the non-normal singularity $xyz = 0$ must be semi-stable—in fact, it is. But $xyz = 0$ is the cone over a plane triangle so the triple point on it is really a degenerate "elliptic" singularity. In fact, $xyz = 0$ is a limit of the family of non-singular cubics $xyz + t(x^3+y^3+z^3) = 0$. Similarly, the standard singularities A_{n-1}: $xy = z^n$ and D_n: $x^2=y^2z+z^n$ have non-normal limits $xy = 0$ and $x^2 = y^2z$ respectively as $n \to \infty$. We can summarize these considerations in the heuristic conjecture: the semi-stable singularities of surfaces will be a limited class of rational and strongly elliptic normal singularities and their non-normal limits.



We now list without proof some classes of semi-stable singularities.

3.17. ELLIPTIC POLYGONAL CONES. In $\mathbf{P}^{n-1}$ take a generic n-gon $\bigcup_{i=0}^{n} \overline{p_i p_{i+1}}$ $(p_0 = p_{n+1})$ and take the cone in $\mathbf{C}^n$ over it. This is a union of n-planes crossing normally in pairs and meeting at an n-fold point at the origin. We also allow the degenerate cases $n = 2$ (local equation $x^2 = y^2 z^2$) and $n = 1$ (local equation $x^2 = y^2 (y + z^2)$) which correspond respectively, to glueing two planes to each other along a pair of transversal lines, and to glueing a pair of transversal lines in a plane together as shown below.



$n = 2$ $\qquad\qquad\qquad\qquad$ $n = 1$

PROPOSITION 3.18. *Elliptic polygonal n-cones are semi-stable if and only if* $1 \leqq n \leqq 6$. *Moreover, all small deformations of these singularities are semi-stable.*

Examples of such singularities are:

i) Cone over a smooth elliptic curve with generic j in $\mathbf{P}^n$, $3 \leqq n \leqq 5$. (In fact, I expect this holds for arbitrary j). These are also called the simple elliptic (Saito) or parabolic (Arnold) singularities, and may be described as $\bigoplus_{m=0}^{\infty} \Gamma(E, L^m)$ where E is an elliptic curve and L is a line bundle of positive degree n: with this description, they are also defined for $n = 1, 2$. For small n, these have the form

$$x^2 + y^3 + z^6 + a(y^2 z^2) = 0 \qquad (n = 1),$$
$$x^2 + y^4 + z^4 + a(y^2 z^2) = 0 \qquad (n = 2),$$
$$x^3 + y^3 + z^3 + a(xyz) = 0 \qquad (n = 3).$$

ii) The hyperbolic singularities of Arnold:

$$xyz + x^n + y^m + z^p = 0 \qquad \frac{1}{n} + \frac{1}{m} + \frac{1}{p} < 1.$$

iii) Rational double points.

iv) Pinch points: these have local equation $x^2 = y^2 z$.

3.18. RATIONAL POLYGONAL CONES. In $\mathbf{P}^{n-1}$ take $(n-1)$ generic line segments $\overline{P_0 P_1} \cup \overline{P_1 P_2} \dots \cup \overline{P_{n-1} P_n}$ and in $\mathbf{C}^n$ take the cone over them: one obtains $(n-2)$ planes crossing normally in $(n-1)$ lines.

PROPOSITION 3.19. *Rational polygonal n-cones are semi-stable if and only if* $2 \leqq n \leqq 6$. *Hence, all small deformations of these singularities are semi-stable.*

A typical singularity which arises in this way is the cone over a rational normal curve in $\mathbf{P}^{n-1}$, $2 \leqq n \leqq 6$.

By applying the semi-stability condition to the ideal $I = \bigoplus_{j=0}^{i} t^{i-j} \cdot (\widetilde{I^j}) \subset \mathcal{O}\,[[t]]$, where I is an ideal in $\mathcal{O}$ and $\sim$ denotes integral closure in $\mathcal{O}$, one can prove the following necessary condition for semi-stability:

PROPOSITION 3.19. *If $\mathcal{O}^r$ is semi-stable, $I \subset \mathcal{O}$ and $P(i) = \dim(\mathcal{O}/(\widetilde{I^i}))$, then*

$$P(1) + \dots\dots + P(i) \geqq \frac{e(I)\, i^{r+1}}{(r+1)!}\ .$$

When $r = 2$, and $\mathcal{O}$ is Cohen-Macaulay this reduces us to *ten* basic types of singularities. In the first few cases we have listed the singularities of this type which are actually semi-stable.

1) Regular points: always stable.

2) Double coverings of $\mathbf{C}^2$ with branch curve of multiplicity $\leqq 4$: semi-stable here are,

   a) rational double points and their non-normal limits $xy = 0$, $x = y^2 z$,

   b) hyperbolic double points,

   c) parabolic double points.

3) Triple points in $\mathbf{C}^3$: Semi-stable are,

   a) cones over non-singular elliptic curves,

   b) hyperbolic triple points.

4-5) Triple and quadruple points in $\mathbf{C}^4$.

6-7) Quadruple and quintuple points in $\mathbf{C}^5$.

8-9) Quintuple and sextuple points in $\mathbf{C}^6$.

10) Sextuple points in $\mathbf{C}^7$.

REMARK. With Eisenbud, we made some computations by computor to eliminate cases; the computer came up with some amusing examples. For instance it found an ideal I in $k[[x, y, z, t]]/(x^2+y^3+z^7)$ with col (I) $= 63{,}398$, mult $(I) = 381{,}024$, showing that $e_0 \geqq 1.000167$, hence that the singularity $x^2 + y^3 + z^7 = 0$ is unstable.

Further restrictions, confirming the heuristic conjecture, on what singularities are semi-stable are provided by:

PROPOSITION 3.20. *If $\mathcal{O}$ is normal and semi-stable then $\mathcal{O}$ is rational or weakly elliptic. Moreover, there are no cuspidal curves, i.e. generically all singular curves are ordinary.*

We omit the proof except to note that the last statement comes from the observation that for large n the choices $I_n = (T^9, u^{9n}, v^{9n})^{\sim}$ show that $e_2(k[[T^2, T^3]]) \geqq 1 + 22/221$!

Now suppose $\mathcal{O}$ is not Cohen-Macaulay. We can create a slew of stable $\mathcal{O}$'s using i) of Proposition 3.10. For example if $k[[x, y]]$ $\supset \mathcal{O} \supset k[[x, xy, y^2]]$, then $\mathcal{O}$ is semi-stable since the ring on the right which is the pinch point is semi-stable; a typical example is $\mathcal{O} = k[[x, xy, y^2, y^3]]$, a very partial pinch in which only the y-tangent has been removed. Fortunately most of these points cannot appear as singularities of varieties on boundary of moduli spaces as they have no smooth deformations. More precisely, (cf. [27]):

THEOREM 3.21. *If $\mathcal{O}$ is a 2-dimensional local ring which is not Cohen-Macauley such that $\mathcal{O} = \mathcal{O}'/t\mathcal{O}'$ where $\mathcal{O}'$ is a normal 3-dimensional local ring; let $\mathcal{O}_{\text{norm}}$ be its normalization and $\tilde{\mathcal{O}} = \{ a \in \mathcal{O}_{\text{norm}} \mid$ for some $n, \mathcal{M}_{\mathcal{O}}^n a \subset \mathcal{O} \}$.*

Then i) *$\tilde{\mathcal{O}}$ is a local ring*

ii) *If in addition $\mathcal{O}$ has characteristic 0, then*

$$\dim(\tilde{\mathcal{O}}/\mathcal{O}) \leqq \textit{big genus of } \tilde{\mathcal{O}}.$$

REMARK. If, as seems likely, in view of Proposition 3.20 the big genus of the Cohen-Macaulay ring $\tilde{\mathcal{O}}$ is 0 or 1, this means that $\mathcal{O}$ must be nearly Cohen-Macauley.

We conclude this section by outlining an as yet completely uninvestigated approach to deciding which singularities should be allowed on the objects of a moduli space.

DEFINITION 3.22. *$\mathcal{O}^r$ is an insignificant limit singularity if, whenever $\mathcal{O}'$ is an $(r+1)$ dimensional local ring such that $\mathcal{O} = \mathcal{O}'/t\mathcal{O}'$ for some $t \in \mathcal{O}'$, $\pi: X \to \mathrm{Spec}\, \mathcal{O}'$ is a resolution of* Spec $\mathcal{O}'$ and $E \subset X$ *is an exceptional divisor (i.e.* $\dim \pi(E) < \dim E$), *then* E *is birationally ruled, that is, the function field of* E *is a purely transcendental extension of a proper subfield. Equivalently, setting* $\mathcal{O}/\mathcal{M}_{\mathcal{O}} = k$, *this says that whenever* R *is a discrete rank* 1 *valuation ring containing* $\mathcal{O}'$ *with* tr . $\mathrm{deg.}_k R/\mathcal{M}_R = r$, *then* $R/\mathcal{M}_R = K(t)$, *for some* K *such that* $\mathrm{tr.\,deg.}_k K = r - 1$.

EXAMPLES. 1) $xy = 0$ is insignificant because on deforming this only A_n singularities arise.

2) $x^2 + y^3 = 0$ is significant because the deformation $t^6 = x^2 + y^3$ blows up to a non-singular elliptic curve with $(E^2) = -1$. Similarly I can show that all higher plane curve singularities are significant.

3) $x^3 + y^3 + y^4 = 0$ is significant because $t^{12} = x^3 + y^3 + y^4$ blows up to a 3-fold containing a K3 surface.

4) Jayant Shah [26] has proven that rational double points and Arnold's parabolic and hyperbolic singularities are insignificant. As a limiting case, normal crossings $xyz = 0$ is insignificant.

REMARKS. 1) Why should birational ruling of exceptional divisors be the right criterion for insignificance? The reason is that all exceptional divisors which arise from blow-ups of non-singular points are birationally ruled and all birationally ruled varieties arise in this way. So on the one hand, such exceptional divisors must be permitted, and on the other, the examples suggest that sufficiently tame singularities cannot "swallow" anything else.

2) The examples suggest that $\mathcal{O}$ semi-stable and $\mathcal{O}$ insignificant are closely related. For instance, perhaps these are the same when embedding-dim $\mathcal{O}$ $= 1$. In dim 2 for example, after hyperbolic and parabolic singularities in the Dolgacev-Arnold list [2, 7] of 2-dimensional singularities come 31 special singularities. These are all unstable and in a recent letter to me Dolgacev remarks that all of these have deformations which blow up to K3 surfaces as in Example 3. If semi-stability and insignificance turn out to be roughly the same in arbitrary dimension, we would have a very powerful tool to apply to moduli problems.

§ 4. Asymptotic Stability of Canonically Polarized Curves

The chief difficulty of using the numerical criterion of Theorem 2.9 to prove the stability of a projective variety is that it is necessary to look inside $\mathcal{O}_{X \times \mathbf{A}^1}$ to compute the multiplicity $e_L(\mathscr{I})$. To circumvent this difficulty, we will construct an upper bound on $e_L(\mathscr{I})$ in terms of data on X alone. For curves, this bound involves only the multiplicities of ideals $\mathscr{I} \subset \mathcal{O}_X$, but for higher dimensional varieties—in particular, surfaces—it requires a theory of mixed multiplicities, i.e. multiplicities for several ideals simultaneously. To motivate the global theory, we will first describe what happens in the local case. Here the basic ideas were introduced by Teissier and Rissler [22]. Recall that if $\mathcal{O}$ is a local ring of dimension r with infinite residue field and I is an ideal of finite colength in it then whenever $f_1, \ldots f_r$ are sufficiently generic elements of I, $e(I) = e((f_1, \ldots, f_r))$. This suggests

Definition 4.1. *If $\mathcal{O}^r$ is a local ring and $I_1, \ldots, I_r$ are ideals of finite colength in $\mathcal{O}$, the mixed multiplicity of the I_i is defined by*

$$e(I_1, \ldots, I_r) = e((f_1, \ldots, f_r))$$

where $f_i \in I_i$ is a sufficiently generic element. (The set of integers $e((f_1, \ldots, f_r))$ has some minimal element and a choice $(f_1, \ldots, f_r)$ is sufficiently generic if the minimum is attained for these f_i.)

The basic property of these multiplicities is:

Proposition 4.2. *Let $I_1, \ldots, I_k$ be ideals of finite colength of a local ring $\mathcal{O}^r$ and let*

$$P_r(m_1, \ldots, m_k) = \sum_{\substack{\Sigma r_i = r \\ r_i \geqq 0}} \frac{1}{\prod (r_i!)} \cdot e(I_1^{[r_1]}, \ldots, I_k^{[r_k]}) \cdot m_1^{r_1} \ldots m_k^{r_k}$$

where $I_i^{[r_i]}$ indicates that I_i appears r_i times. Then

i) $$\left| \dim (\mathcal{O} / \prod_{i=1}^{k} I_i^{m_i}) - P_r(m_1, \ldots, m_k) \right| = 0((\textstyle\sum m_i)^{r-1})$$

ii) *There exists a polynomial of total degree r*

$$P(m_1, \ldots, m_k) = P_r(m_1, \ldots, m_k) + \text{lower order terms}$$

and an N_0 such that if $m_i \geqq N_0$ for all i, then

$$\dim (\mathcal{O} / \prod I_i^{m_i}) = P(m_1, \ldots, m_k).$$

Proof. See Teissier and Rissler [22].
Using this we obtain the estimate:

PROPOSITION 4.3. *Let* $I \subset \mathcal{O}\,[[t]]$ *be an ideal of finite codimension and let* $I_k = \{ a \in \mathcal{O} \mid at^k \in I \}$; *then* $I_0 \subseteq I_1 \subseteq \ldots \subseteq I_N = \mathcal{O}, N \gg 0$. *Then for all sequences* $0 = r_0 < r_1 < \ldots < r_l = N$,

$$e(I) \leqq \sum_{k=0}^{l-1} (r_{k+1} - r_k) \sum_{j=0}^{r} e(I_{r_k}^{[j]}, I_{r_{k+1}}^{[r-j]}) \,.$$

Proof. Since $I \supset \oplus\, t^{r_i} I_{r_i}$

$$\begin{aligned} I^n \supset\ & I_{r_0}^n (\mathcal{O} + t\mathcal{O} + \ldots + t^{r_1 - 1}\mathcal{O}) + I_{r_0}^{n-1} I_{r_1} (t^{r_1}\mathcal{O} + t^{r_1+1}\mathcal{O} + \ldots + t^{2r_1 - 1}\mathcal{O}) \\ & + \ldots + I_{r_0} I_{r_1}^{n-1} (t^{(n-1)r_1}\mathcal{O} + \ldots + t^{nr_1 - 1}\mathcal{O}) \\ & + I_{r_1}^n (t^{nr_1}\mathcal{O} + \ldots + t^{(n-1)r_1 + r_2 - 1}\mathcal{O}) + I_{r_1}^{n-1} I_{r_2} (t^{(n-1)r_1 + r_2}\mathcal{O} + \ldots) \\ & + \ldots + I_{r_{l-1}}^n (t^{nr_{l-1}}\mathcal{O} + \ldots) + I_{r_{l-1}}^{n-1} (t^{(n-1)r_{l-1} + r_l}\mathcal{O} + \ldots) \\ & + \ldots + t^{nr_l}\mathcal{O}\,[[t]] \,. \end{aligned}$$

whence

$$\dim (\mathcal{O}\,[[t]]/I^n) \leqq \sum_{k=0}^{l} (r_{k+1} - r_k) \sum_{i=0}^{n-1} \dim \left(\mathcal{O}/(I_{r_k}^{n-i} \,.\, I_{r_{k+1}}^{i})\right)$$

$$(4.4) \quad = \sum_{k=0}^{l} (r_{k+1} - r_k) \sum_{i=0}^{n-1} \left[\sum_{j=0}^{r} \frac{1}{j\,!\,(r-j)\,!}\, e\,(I_{r_k}^{[r-j]}, I_{r_{k+1}}^{[j]})\,(n-i)^{r-j} i^j + R_i \right]$$

By Proposition 4.2 i) each remainder terms R_i is $O\,(n^{r-1})$. Indeed, ii) of 4.2 says that except when i or $n - i < N_0$, the R_i are all represented by a polynomial of degree $r - 1$ so that we can obtain a uniform $O\,(n^{r-1})$ estimate for the R_i; hence $\sum_{i=0}^{n-1} R_i = O\,(n^r)$.

But the n.l.c. of the $(r+1)^{\mathrm{st}}$ degree polynomial representing $\dim\,(\mathcal{O}\,[[t]]/I^n)$ is by definition $e\,(I)$; so evaluating the n.l.c. of the sum in (4.4) using the lemma below, gives the proposition.

LEMMA 4.5. $$\frac{j\,!\,(r-j)\,!}{(r+1)\,!}\, n^{r+1} = \sum_{i=0}^{n-1} (n-i)^{r-j}\, i^j + O\,(n^r)$$

Proof. We can reexpress the left hand side in terms of the β-function as

$$\frac{j\,!\,(r-j)\,!}{(r+1)\,!}\, n^{r+1} = \beta\,(j, r-j)\, n^{r+1} = \left(\int_0^1 t^j (1-t)^{r-j}\, dt \right) n^{r+1} ,$$

and the right hand side is just another expression for n^{r+1} times this integral as a Riemann sum plus error term.

To globalize these ideas we combine them with some results of Snapper [5, 21].

DEFINITION 4.6. *Let X^r be a variety, L be a line bundle on X and $\mathcal{I}_1, \ldots, \mathcal{I}_r$ be ideals on $\mathcal{O}_X$ such that* $\operatorname{supp}(\mathcal{O}_X/\mathcal{I}_i)$ *is proper. Choose a compactification $\bar{X}$ of X on which L extends to a line bundle $\bar{L}$ and let $\pi: \bar{B} \to \bar{X}$ be the blowing up of $\bar{X}$ along $\prod \mathcal{I}_i$ so that $\pi^{-1}(\mathcal{I}_i) = \mathcal{O}_{\bar{B}}(-E_i)$. Let $\pi^* L = \mathcal{O}_{\bar{B}}(D)$. We define*

$$e_L(\mathcal{I}_1, \ldots, \mathcal{I}_r) = (D^r) - ((D-E_1) \cdot \cdots\cdots \cdot (D-E_r)).$$

We omit the check that this definition is independent of the choice of $\bar{X}$ and $\bar{L}$.

4.7. CLASSICAL GEOMETRIC INTERPRETATION. Suppose X is a projective variety, $L = \mathcal{O}_X(1)$ and $\mathcal{I}_i \cdot L$ is generated by a space of sections $W_i \subset \Gamma(\mathbf{P}^n, \mathcal{O}(1))$. If $H_1, \ldots, H_r$ are generic hyperplanes of $\mathbf{P}^n$, then $\#(H_1 \cap \ldots \cap H_r \cap X) = \deg X$. One sees by an argument like that of Proposition 2.5, that as the H_i specialize to hyperplanes defined by elements of W_i but otherwise generic, the number of points in $H_1 \cap \ldots \cap H_r \cap X$ which specialize to a point in one of the W_i's is just $e_L(\mathcal{I}_1, \ldots, \mathcal{I}_r)$.

We can globalize Proposition 4.2 to give an interpretation of the mixed multiplicity by Hilbert polynomials.

PROPOSITION 4.8. i) *Let X^r be a variety, $L_1, \ldots, L_n$ be line bundles on X and $\mathcal{I}_1, \ldots, \mathcal{I}_l$ be ideals in $\mathcal{O}_X$ such that* $\operatorname{supp}(\mathcal{O}_X/\mathcal{I}_i)$ *is proper for all i. Then there is a polynomial $P(n, m)$ of total degree r and an M_0 such that if $m_j \geqq M_0$ for all j then*

$$\chi\Big(X, \bigotimes_{i=1}^{k} L_i^{n_i} \Big/ \prod_{j=1}^{l} \mathcal{I}_j^{m_j} \cdot \bigotimes_{i=1}^{k} L_i^{n_i}\Big) = P(n, m).$$

Now suppose all the line bundles are the same, say L and let

$$P_r(m_1, \ldots, m_l) = \sum_{\substack{\Sigma r_i = r \\ r_i \geqq 0}} \frac{1}{\prod (r_i!)} e_L(\mathcal{I}_1^{[r_1]}, \ldots, \mathcal{I}_l^{[r_l]}) \, m_1^{r_1} \ldots m_l^{r_l}$$

Then

ii) $P(\sum m_i; m_1, \ldots, m_l) = P_r(m_1, \ldots, m_l) +$ lower order terms

iii) $\mid \chi(X, L^{\Sigma m_i} / \prod \mathscr{I}_j^{m_j} \otimes L^{\Sigma m_i}) - P_r(m_1, \ldots, m_l) \mid \; = O((\sum_{j=1}^{l} m_j)^{r-1})$

(i.e. *we retain an estimate assuming only* $\sum m_j$ *is large*).

Proof. Making a suitable compactification of X will not alter the Euler characteristics so we may assume X is compact.

Before proceeding we recall certain facts: If $R = \bigoplus_{n_i \geqq 0} R_{n_1, \ldots, n_l}$ is a multigraded ring we can form a scheme Proj (R) in the obvious way from multi-homogeneous prime ideals. Quasi-coherent sheaves $\mathscr{F}$ on Proj (R) correspond to multigraded R-modules $M = \bigoplus M_{n_1, \ldots, n_l}$. Suppose $R_{0, \ldots, 0} = k$ a field and that R is generated by the homogeneous pieces $R_{0, \ldots, 0, 1, 0, \ldots, 0}$. Then we get invertible sheaves $L_1, \ldots, L_l$ on Proj (R) from the modules M_i, where M_i = (R with i^{th}-grading shifted by 1), and the multigraded variant of the F.A.C. vanishing theorem for higher cohomology says that if $\mathscr{F}$ is a coherent sheaf on Proj (R) then

$$H^i(\mathscr{F} \otimes (\otimes L_j^{n_j})) = \begin{cases} M_{n_1, \ldots, n_l}, & i = 0 \\ (0), & i > 0 \end{cases} \qquad \text{if } n_j \gg 0, \text{ all } j$$

Now if $\mathscr{I}_1, \ldots, \mathscr{I}_k$ are ideal sheaves on X such that supp $(\mathcal{O}_X / \mathscr{I}_j)$ is proper for all i, let $\mathscr{A} = \bigoplus_{m_j \geqq 0} \mathscr{I}_1^{m_1} \ldots \mathscr{I}_l^{m_l}$. Then $\mathscr{A}$ is a multigraded sheaf of $\mathcal{O}_X$-algebras. Let B = Proj $(\mathscr{A})$; the blow up of X along $\prod \mathscr{I}_j$ is just $\pi: B \to X$. If E_j is the exceptional divisor corresponding to $\mathscr{I}_j$, then when $\mathcal{O}_B(-\sum m_j E_j)$ is coherent and when all the m_j are large the relative versions of the vanishing theorems say:

a) $R^i \pi_* (\mathcal{O}(-\sum m_j E_j)) = 0, \; i > 0$

b) $\pi_* \mathcal{O}(-\sum m_j E_j) = \prod_{j=1}^{l} \mathscr{I}_j^{m_j}$

In any case,

c) supp $R^i \pi_* (\mathcal{O}(-\sum m_j E_j))$ has dimension less than r, $i > 0$,

d) $\pi_* (\mathcal{O}(-\sum m_j E_j)) = \prod_i \mathscr{I}_i^{m_i}$ except on a set of dimension less than r.

From a) and b) we deduce that when all the m_j are large, $\chi(\prod \mathscr{I}_j^{m_j}) = \chi(\pi^* \mathcal{O}(-\sum m_j E_j))$. Thus, $\chi(X, \otimes L_i^{n_i} / \prod \mathscr{I}_j^{m_j} L_i^{n_i}) = \chi(X, \otimes L_i^{n_i}) - \chi(B, \otimes L_i^{n_i}(-\sum m_j E_j))$ and both of these last Euler characteristics polynomials of degree $\leqq r$ by Snapper [5, 21]. Now if $\pi^* L = \mathcal{O}_B(D)$, his result also says,

$$r\,!\,.\,\mathrm{n.l.c.}(\chi(X, L^{\Sigma m_j}/\prod \mathscr{I}_j^{m_j} \otimes L^{\Sigma m_j}) = (\sum m_j)^r (D^r) - ((\sum m_j (D - E_j))^r)$$

$$= \sum_{\substack{\Sigma r_j = r \\ r_j \geqq 0}} \frac{r\,!}{\prod (r_j\,!)} \prod (m_j (D - E_j))^{r_j}$$

$$= \sum_{\substack{\Sigma r_j = r \\ r_j \geqq 0}} \frac{r\,!}{\prod (r_j\,!)} e_L (\mathscr{I}_1^{[r_1]}, \ldots, \mathscr{I}_l^{[r_l]}) \,.\, m_1^{r_1} \ldots m_l^{r_l}$$

which is ii). Fix an N such that ii) holds when all $m_j \geqq N$.

Now suppose I is a proper subset of $\{1, \ldots, l\}$, J is its complement and that values $m_i < N$ are fixed for all $i \in I$. Let $\pi_J : B_J \to X$ be the blow up of X along $\prod_{j \in J} \mathscr{I}_j$. As above we deduce that $\exists N'$ depending on I and the m_i, $i \in I$ such that if $m_j > N'$, $\forall j \in J$, then

$$\chi (X, \mathscr{I}_1^{m_1} \ldots \mathscr{I}_k^{m_k}) = \chi (B_J, \prod_{i \in I} \mathscr{I}_i^{m_i} (- \sum_{j \in J} m_j E_j)) \,.$$

Then applying c) and d) we see that for some C, also depending on I and the m_i, $i \in I$,

$$| \chi (B, \mathcal{O} (- \sum m_i E_i)) - \chi (B_J, \prod_{i \in I} \mathscr{I}_i^{m_i} (- \sum_{j \in J} m_j E_j)) | \leqq C (\sum_{j \in J} m_j)^{r-1} \,.$$

Combining this with the argument used in the proof of i) and ii) shows that for some C' (depending on I and the m_i, $i \in I$)

$$| \chi (X, L^{\Sigma m_j}/\prod \mathscr{I}_j^{m_j} L^{\Sigma m_j}) - P_r (m_1 \ldots m_l) | \leqq C' (\sum_{j \in J} m_j)^{r-1} \,.$$

From ii), we get an estimate of this type with a uniform constant C', when all the $m_j \geqq N$. Since there are only finitely many sets I and for each of these only finitely many choices for the m_i, $i \in I$ with $m_i < N$ we can combine all these estimates to show: there exists M and C'' such that if any $m_i > M$, then

$$| \chi (X, L^{\Sigma m_j}/ \prod_j \mathscr{I}_j^{m_j} L^{\Sigma m_j}) - P_r (m_1, \ldots, m_l) | \leqq C'' ((\sum_j m_j)^{r-1})$$

which is iii).

The following analogue of Proposition 2.6 allows us to calculate mixed multiplicities in terms of the dimensions of spaces of sections.

PROPOSITION 4.9. *If* L, $\mathscr{I}_1 L, \ldots, \mathscr{I}_l L$ *are generated by their sections, then*

$$| \chi (X, L^{\Sigma m_j}/(\prod \mathscr{I}_j^{m_j}) L^{\Sigma m_j}) - \dim (\Gamma (X, L^{\Sigma m_j})/\Gamma (X, \prod \mathscr{I}^{m_j} L^{\Sigma m_j})) |$$
$$= O ((\sum m_j)^{r-1})$$

Proof. We give only a sketch of the proof which is very similar to that of Proposition 2.6. One first shows as in the proof of 2.6a), that for $i > 0$, $h^i(L^{\Sigma m_j}/\prod \mathscr{I}^{m_j} L^{\Sigma m_j}) = O((\sum m_j)^{r-1})$, hence that

$$\begin{aligned}&|\chi(X, L^{\Sigma m_j}/\prod \mathscr{I}_j^{m_j} L^{\Sigma m_j}) - \dim \Gamma(X, L^{\Sigma m_j}/\prod \mathscr{I}_j^{m_j} L^{\Sigma m_j})|\\ &= O((\textstyle\sum m_j)^{r-1})\end{aligned}$$

Using the long exact sequence

$$0 \to \Gamma(X, \prod \mathscr{I}_j^{m_j} L^{\Sigma m_j}) \to \Gamma(X, L^{\Sigma m_j}) \to \Gamma(X, L^{\Sigma m_j}/\prod \mathscr{I}_j^{m_j} L^{\Sigma m_j}) \to \ldots$$

this reduces the proposition to showing that

$$\dim\left(\operatorname{coker}\left(\Gamma(X, L^{\Sigma m_j}) \to \Gamma(X, L^{\Sigma m_j}/\prod \mathscr{I}_j^{m_j} L^{\Sigma m_j})\right)\right) = O((\textstyle\sum m_j)^{r-1})$$

and this is done exactly as in the proof of 2.6b). (Note that the extra hypotheses of 2.6b) were not used in this part of the proof.)

The global form of Proposition 4.3 is:

PROPOSITION 4.10. *Given a variety X, a line bundle L on X and an ideal $\mathscr{I} \subset \mathscr{O}_{X \times \mathbf{A}^1}$ with* $\operatorname{supp}(\mathscr{O}_{X \times \mathbf{A}^1}/\mathscr{I})$ *proper in* $X \times (0)$, *let $\mathscr{I}_k = \{ a \in \mathscr{O}_X \mid t^k a \in \mathscr{I} \}$ so that $\mathscr{I}_0 \subseteq \mathscr{I}_1 \subseteq \ldots \subseteq \mathscr{I}_N = \mathscr{O}_X$ and let $L_1 = L \otimes \mathscr{O}_{\mathbf{A}^1}$. Suppose that L, $\mathscr{I}_k L$ and $\mathscr{I} L_1$ are generated by their sections. Then for all sequences $0 = r_0 < r_1 < \ldots < r_l = N$,*

$$e_{L_1}(\mathscr{I}) \leqq \sum_{k=0}^{l} (r_{k+1} - r_k) \sum_{j=0}^{r} e_L(\mathscr{I}_{r_k}^{[j]}, \mathscr{I}_{r_{k+1}}^{[r-j]}).$$

Proof. By Proposition 4.9, $e_{L_1}(\mathscr{I})$ is calculated by the order of growth of

$$\dim\left[H^0(X \times \mathbf{A}^1, L_1^n)/H^0(X \times \mathbf{A}^1, \mathscr{I}^n . L_1^n)\right].$$

Exactly as in Proposition 4.3, for each n, we introduce using the r_i's an approximating ideal sheaf $\mathscr{I}'_n$:

$$\mathscr{I}^n \supset \mathscr{I}'_n = \bigoplus_{k=0}^{\infty} t^k . \mathscr{I}_{n,k}$$

where $\mathscr{I}_{n,0} \subset \mathscr{I}_{n,1} \subset \ldots \subset \mathscr{I}_{n,N} = \mathscr{O}_X$ for $N \gg 0$. Since

$$H^0(X \times \mathbf{A}^1, \mathscr{I}^n . L_1^n) \supset H^0(X \times \mathbf{A}^1, \mathscr{I}'_n . L_1^n) = \bigoplus_{k=0}^{\infty} H^0(X, \mathscr{I}_{n,k} . L^n),$$

it follows that

$$\dim \left(H^0 (X \times \mathbf{A}^1, L_1^n)/H^0 (X \times \mathbf{A}^1, \mathscr{I}^n \, . \, L_1^n\right)$$

$$\leqq \sum_{k=0}^{\infty} \dim \left(H^0 (X, L^n)/H^0 (X, \mathscr{I}_{n,k} \, . \, L^n)\right)$$

The rest of the proof follows Proposition 4.3 exactly, using 4.9 again to get the estimate

$$\dim \left(H^0 (X, L^n)/H^0 (X, \mathscr{I}^i_{r_k} \, . \, \mathscr{I}^{n-i}_{r_{k+1}} \, . \, L^n)\right)$$

for $\chi \, (L^n/\mathscr{I}^i_{r_k} \, . \, \mathscr{I}^{n-i}_{r_{k+1}} \, . \, L^n)$.

COROLLARY 4.11. *If in Proposition* 4.10, X *is a curve*

$$e_{L_1} (\mathscr{I}) \leqq \min_{0 = r_0 < r_1 \ldots < r_l = N} \left[\sum_{k=0}^{e} (r_{k+1} - r_k) \, . \left(e_L (\mathscr{I}_{r_k}) + e_L (\mathscr{I}_{r_{k+1}})\right)\right]$$

If X *is a surface,*

$$e_{L_1} (\mathscr{I})$$

$$\leqq \min_{0 = r_0 < r_1 \ldots < r_l = N} \left[\sum_{k=0}^{l} (r_{k+1} - r_k) \, . \left(e_L (\mathscr{I}_{r_k}) + e_L (\mathscr{I}_{r_k}, \mathscr{I}_{r_{k+1}}) + e_L (\mathscr{I}_{r_k})\right)\right]$$

We now show how this upper bound proves the asymptotic stability of non-singular curves. It turns out that the estimate is, however, *not* sufficiently sharp to prove the asymptotic stability of curves with ordinary double points: more precisely, if $\mathscr{I}$ is the ideal associated to a 1-PS λ with normalized weights ρ_i then the estimate of the corollary may be greater than $\dfrac{2 \deg X}{n+1} \, . \sum \rho_i$ (cf. Theorem 2.9)

THEOREM 4.12. *If* $C^1 \subset \mathbf{P}^N$ *is a linearly stable (resp.: semi-stable) curve, then* C *is Chow stable (resp.: semi-stable).*

Proof. We prove the stable case; the semi-stable case follows by replacing the strict inequalities in the proof by inequalities.

Fix coordinates $X_0, \ldots, X_N$ on $\mathbf{P}^N$ and a 1-PS

$$\lambda (t) = \begin{bmatrix} t^{\rho_0} & & 0 \\ & \ddots & \\ 0 & & t^{\rho_N} \end{bmatrix}, \quad \rho_0 \geqq \rho_1 \geqq \ldots \geqq \rho_N = 0$$

Let $\mathscr{I}$ be the associated ideal on $\mathcal{O}_{C \times \mathbf{A}^1}$ and let $\mathscr{I}_k \subset \mathcal{O}_C$ be the ideal defined by $\mathscr{I}_k . L$ = [sheaf generated by $X_k, \ldots, X_N$]; thus $\mathscr{I} = \sum_{k=0}^{N} t^{\rho_k} \mathscr{I}_k$. The linear stability of X implies (cf. 2.16), $e(\mathscr{I}_k) < \frac{\deg C}{N} . \operatorname{codim} < X_k, \ldots, X_N >$ $= \frac{\deg C . k}{N}$. So using Corollary 4.11,

$$e_L(\mathscr{I}) \leqq \min_{0 = s_0 < \ldots < s_k = N} \left[\sum (\rho_{s_k} - \rho_{s_{k+1}}) (e_L(\mathscr{I}_{s_k}) + e_L(\mathscr{I}_{s_{k+1}}))\right]$$

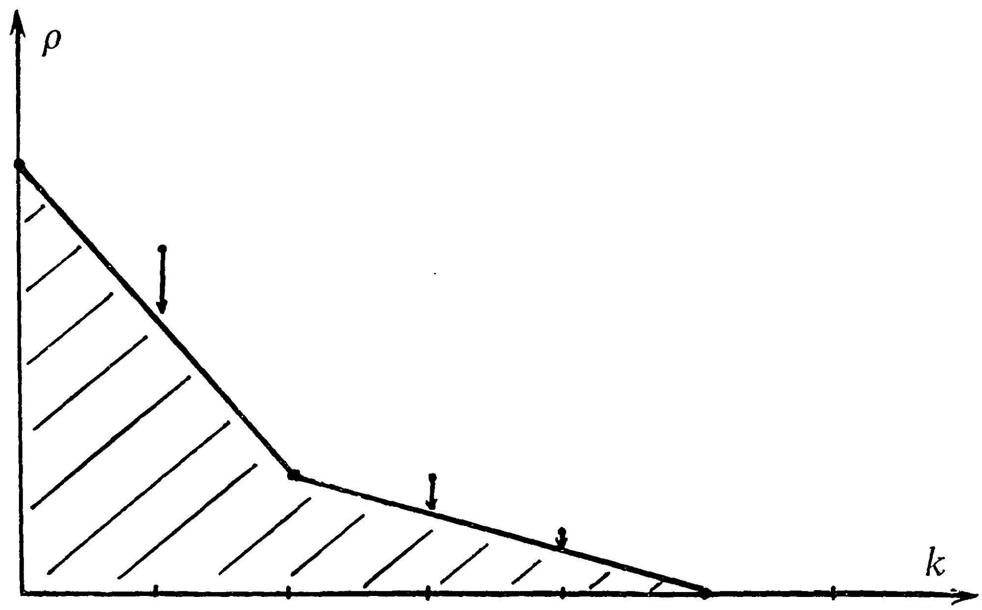
$$< \min_{0 = s_0 < \ldots < s_k = N} \left[\sum (\rho_{s_k} - \rho_{s_{k+1}}) (s_k + s_{k+1}) \frac{\deg C}{N}\right]$$

In view of the Lemma below this implies $e_L(\mathscr{I}) < \frac{2 \deg C}{N+1} \sum_{i=0}^{N} \rho_i$ which in turn implies C is stable by Theorem 2.9.

LEMMA 4.13. *If* $\rho_0 \geqq \ldots \geqq \rho_n = 0$, *then*

$$\min_{0 = s_0 < \ldots < s_l = n} \left[\sum (\rho_{s_k} - \rho_{s_{k+1}}) \cdot \binom{s_k + s_{k+1}}{2}\right] \leqq \frac{n}{n+1} \sum_{k=0}^{n} \rho_k$$

Proof. Draw the Newton polygon of the points (k, ρ_k) as shown below



The left hand side is just the area under this polygon so moving the points above the polygon down onto it as shown, does not affect this expression. Since this can only decrease the right hand side we may assume all the ρ_i are on this polygon. Then the left hand expression can be calculated with $s_k = k$ and it becomes

$$\frac{1}{2}\rho_0 + \rho_1 + \ldots + \rho_{n-1} + \frac{1}{2}\rho_n = \rho_0 + \ldots + \rho_n - \frac{1}{2}(\rho_0 + \rho_n)$$

$$\leqq \rho_0 + \ldots + \rho_n - \frac{1}{n+1}(\rho_0 + \ldots + \rho_n)$$

since the Newton polygon is convex. But the last expression is just $\frac{n}{n+1}(\rho_0 + \ldots + \rho_n)$, hence the lemma.

THEOREM 4.14. *If* $C \subset \mathbf{P}^N$ *is a smooth curve embedded by* $\Gamma(C, L)$ *where* L *is a line bundle of degree* d, *then*

i) $d > 2g > 0 \Rightarrow C$ *linearly stable*,

ii) $d \geqq 2g \geqq 0 \Rightarrow C$ *linearly semi-stable*.

Combining this result with Theorem 4.13 gives the main theorem of this section:

THEOREM 4.15. *If* C *is a smooth curve of genus* $g \geqq 1$ *embedded by a complete linear system of degree* $d > 2g$ *then* C *is Chow-stable.*

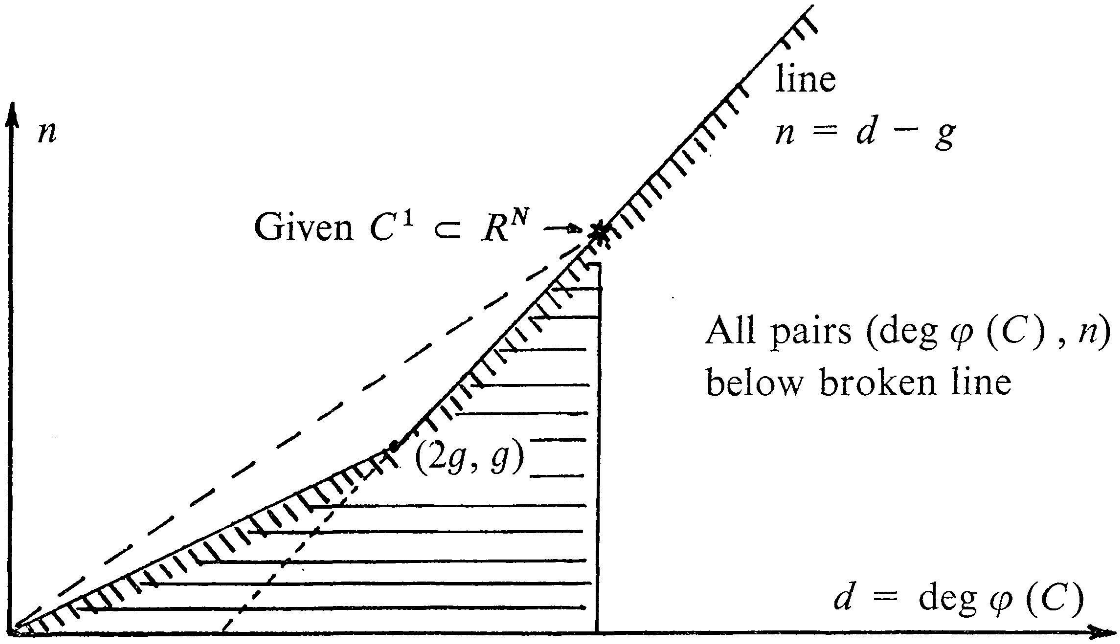
Proof of 4.14. Consider all morphisms $\varphi: C \to \mathbf{P}^n$ for all n, where $\varphi(C) \not\subset$ hyperplane. Let us plot the locus of pairs $(\deg \varphi(C), n)$, where $\varphi(C)$ is counted with multiplicity if φ is not birational. Note that, if $\varphi^*\mathcal{O}(1)$ is non-special, then by Riemann-Roch on C:

$$\begin{aligned} n &= \dim H^0(\mathcal{O}_{\mathbf{P}^n}(1)) - 1 \leqq \dim H^0(\varphi^*\mathcal{O}(1)) - 1 \\ &= \deg \varphi^*\mathcal{O}(1) - g = \deg \varphi(C) - g \end{aligned}$$

while if $\varphi^*\mathcal{O}(1)$ is special, then by Clifford's Theorem on C:

$$\begin{aligned} n &\leqq \dim H^0(\varphi^*\mathcal{O}(1)) - 1 \\ &\leqq \frac{\deg \varphi^*(\mathcal{O}(1))}{2} = \frac{\deg \varphi(C)}{2} \end{aligned}$$

This gives us the diagram



The reduced degree of $\varphi(C)$ is just d/n, the inverse of the slope of the joining (0, 0) to the plotted point (n, d). In case (i), by assumption, the given curve $C^1 \subset \mathbf{P}^N$ corresponds to a point on the upper bounding segment, such as * in our picture. Any projection of C corresponds to a point (n', d') in the shaded area with $d' \leqq d, n' < n$. From the diagram it is clear that the slope decreases, or the reduced degree increases: this is exactly what linear stability means. In case (ii), we allow the given curve C to correspond to the vertex $(2g, g)$ of the boundary, or allow $g = 0$, when the boundary line is just $n = d$. In these cases, the slope at least cannot increase, or the reduced degree cannot decrease under projection.

REMARK. Curves with ordinary double points are *not*, in general, linearly stable since projecting from a double point lowers the degree by 2, but decreases the dimension of the ambient space by only 1. In fact, linear stability is somewhat too strong a condition for most moduli problems: Chow stability for varieties of dimension r apparently allows points of multiplicity up to $(r+1)$! while linear stability allows only points of multiplicity up to r !

§ 5. The Moduli Space of Stable Curves

Our main result is:

THEOREM 5.1. *Fix* $n \geqq 5$, *and for any curve* C *of genus* g *let* $\Phi_n(C) \subset \mathbf{P}^{(2n-1)(g-1)-1}$ *be the image of* C *embedded by a basis of* $\Gamma(C, \omega_c^{\otimes n})$. *Then if* C *is moduli-stable,* $\Phi_n(C)$ *is Chow stable.*

In view of the basic results of § 1, and those of [20], this shows:

COROLLARY 5.2. *(F. Knudsen)* $\overline{\mathcal{M}}_g$ *is a projective variety.*

Recall that C moduli-stable means

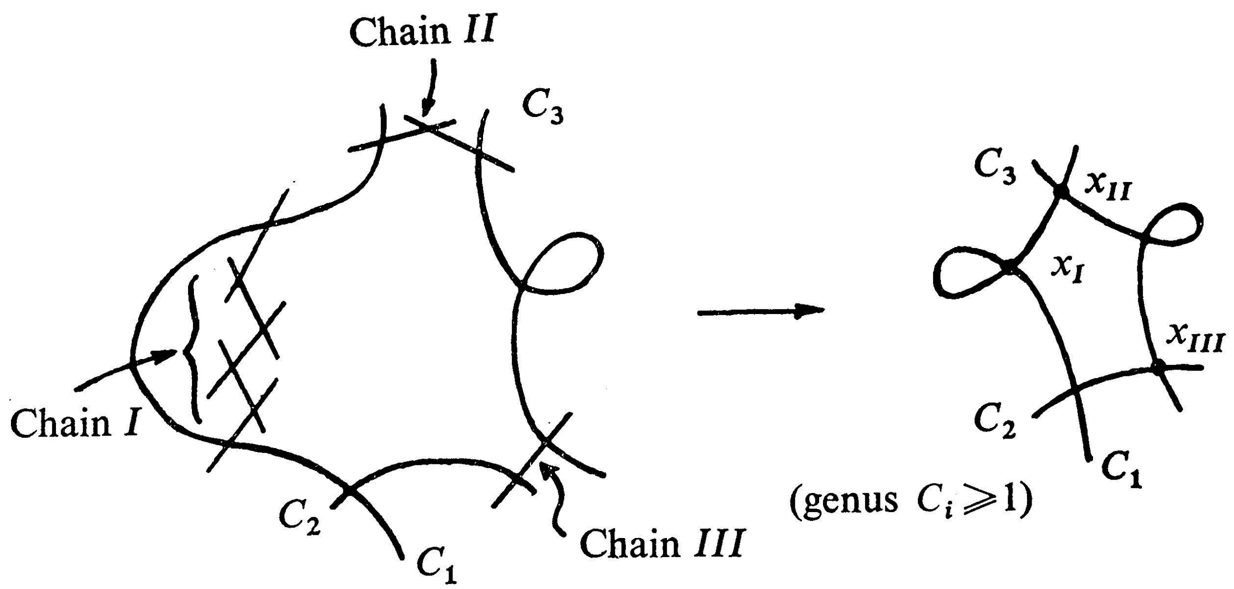
(1) C has at worst ordinary double points (by Proposition 3.12, this is necessary for the asymptotic semi-stability of C) and is connected,

(2) C has no smooth rational components meeting the rest of the curve in fewer than three points:
this condition is necessary to ensure that C has only finitely many automorphisms.

We will call C moduli semi-stable if it satisfies (1) and

(2′) C has no smooth rational components meeting the rest of the curve in only one point.

Note that if C is moduli semi-stable, then the set of its smooth rational components meeting the rest of the curve in exactly 2 points form a finite set of chains and if each of these is replaced by a point, we get a moduli stable curve:



We will case these the rational chains of C.

It would be more satisfactory to have a direct proof of Theorem 5.1 similar to the proof of the stability of smooth curves given in § 4. But curves with double points are not usually linearly stable (cf. the remark following Theorem 4.14) and, in fact, the estimates in Corollary 4.11 do not suffice to prove stability for such curves. We will therefore take an indirect approach.

Proof of 5.1. We begin by recalling the useful valuative criterion:

LEMMA 5.3. *Suppose a reductive group G acts on a k-vector space V. Let $K = k((t))$ and suppose $x \in V_K$ is G-stable. Then there is a finite extension $K' = k'((t')) \supset K$, and elements $g \in G_{K'}$, $\lambda \in (K')^*$ such that the point $\lambda g(x) \in V \otimes_k K'$ lies in $V \otimes_k k'[[t']]$ and specializes as $t \to 0$ to a point $\overline{\lambda g(x)}$ with closed orbit. Thus $\overline{\lambda g(x)}$ is either stable or semi-stable with a positive dimensional stabilizer.*

Proof. The diagram below is defined over k:

$$\begin{array}{l} \mathbf{P}(V) \supset \mathbf{P}(V)_{ss} \\ \qquad\quad \Big\downarrow \pi \\ \qquad\quad X = \text{Proj (graded ring of invariants on } V) \end{array}$$

The point $\pi(x) \in X_K$ specializes to a point $\overline{\pi(x)} \in X_k$. Let $\bar{y}$ be a lifting of this point to V_{ss} with $O^G(\bar{y})$ closed. In the scheme $V \times \operatorname{Spec} k[[t]]$ form the closure Z of $\mathbf{G}_m \,.\, O^G(x)$. The lemma follows if we prove that $\bar{y} \in Z$. If $\bar{y} \notin Z$, then Z and $O^G(\bar{y})$ are closed disjoint G invariant subsets of $V \times \operatorname{Spec} k[[t]]$, hence there exists a homogeneous G-invariant f such that $f(x) = 0$ but $f(\bar{y}) \neq 0$. Then for some n, $f^{\otimes n}$ descends to a section of some line bundle on $X \times \operatorname{Spec} k[[t]]$. But then $f(\pi(x)) = 0$ and $\overline{f(\pi(x))} \neq 0$ are contradictory.

Now suppose that C is a moduli stable curve of genus g over k. Let $\mathscr{C}/k[[t]]$ be a family of curves with fibre C_0 over $t = 0$ equal to C and generic fibre C_η smooth. At the double points of C_0, $\mathscr{C}$ looks formally like $xy = t^n$, that is has only A_{n-1}-type singularities and hence is normal. Embed C_η in $\mathbf{P}^N$ $\big(N = (2n-1)(g-1)-1\big)$ by $\Gamma(C_\eta\, \omega_{C_\eta}{}^{\otimes n})$ and let $\Phi(C_\eta)$ denote its image there. Then Lemma 5.3 says that by replacing $k[[t]]$ with some finite extension and choosing a suitable basis of $\Gamma(C_\eta, \omega_{C_\eta}{}^{\otimes n})$—this corresponds to choosing g, λ—we may assume that the closure $\mathscr{D}$ in $\mathbf{P}^N \times \operatorname{Spec} k[[t]]$ of $\Phi(C_\eta)$ satisfies

i) $D_\eta = C_\eta$

ii) D_0 Chow-stable or Chow semi-stable with positive dimensional stabilizer.

I now claim:

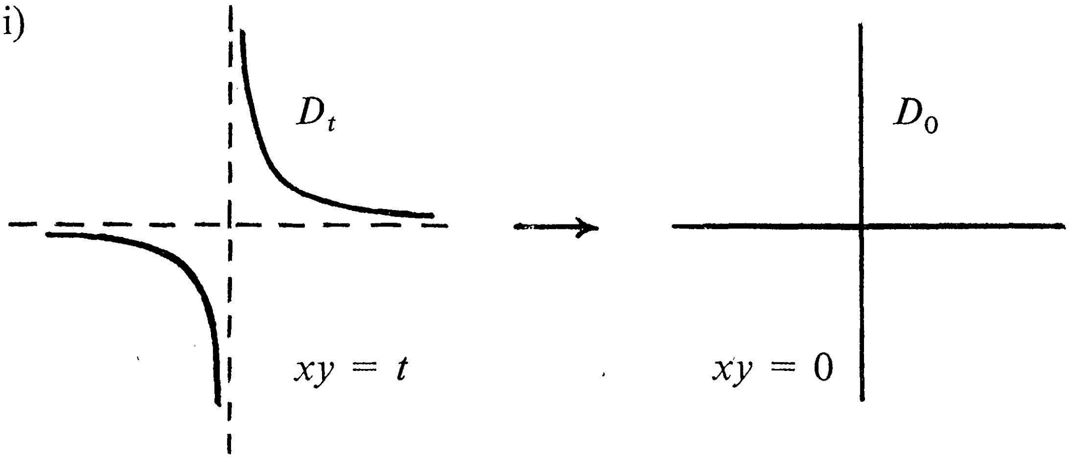
(5.4) $\mathscr{D} = \Phi(\mathscr{C})$, the image of $\mathscr{C}$ under a $k[[t]]$ basis of

$$\Gamma(\mathscr{C}, \omega_{\mathscr{C}}{}^{n}{}_{/k[[t]]})$$

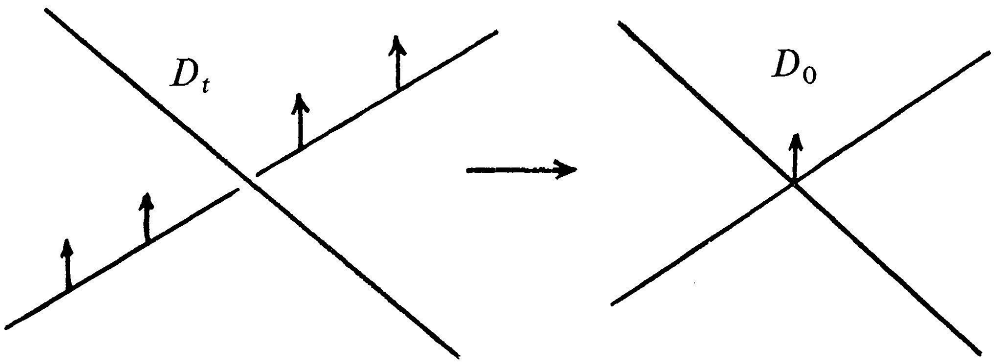
In particular this implies $D_0 = C_0 = C$ and since C has finite stabilizer this means D_0, hence C, is Chow stable.

The main step in the proof of (5.4) is to show that D_0 is moduli semi-stable as a scheme, and the key difficulty in doing this is to show that D_0 has only ordinary double points. At first glance, this seems rather obvious, since from Proposition 3.12 it follows easily that as a cycle D_0 has no multiplicities and has only ordinary double points. But ordinary double points on a limit cycle arise in two ways:

i)



ii)



ideal $= \langle x, z \rangle . \langle y, z-t \rangle$ ideal $= (xy, xz, yz, z^2)$

In the second case the scheme D_0 has an embedded component (the first order normal neighbourhood in the z-direction) at the double point so in the limit scheme the double point is not ordinary. If case (ii) occurred for D_0, then since D_0 is Chow semi-stable, it must span $\mathbf{P}^N$ set-theoretically. But $\Gamma(D_0, \mathcal{O}_{D_0}(1))$ has a torsion section supported at the double point: so D_0 would have to be embedded by a non-complete linear system $\sum \subset \Gamma(D_0, \mathcal{O}_{D_0}(1))$ of torsion-free sections, dim $\sum$ = dim $H^0(D_\eta, \mathcal{O}_{D_\eta}(1))$. Consequently $H^1(D_0, \mathcal{O}_{D_0}(1)) \neq (0)$ too. That this cannot happen in the situation of (5.4) follows from:

PROPOSITION 5.5. *Let* $C \subset \mathbf{P}^n$ *be a* 1*-dimensional scheme such that*

a) $n + 1 = \deg C + \chi(\mathcal{O}_C), \chi(\mathcal{O}_C) < 0$,

b) C *is Chow semi-stable,*

c) $\dfrac{\deg C}{n+1} < \dfrac{8}{7}$.

Then i) C *is embedded by a complete non-special* [1]) *linear system,*

ii) C *is a moduli semi-stable curve with rational chains of length at most one consisting of straight lines.*

Moreover if $v = \dfrac{\deg C}{\deg \omega_C}$ *(where* ω_C *is the Grothendieck dualizing sheaf) and* $C = C_1 \cup C_2$ *is a decomposition of* C *into two sets of components such that* $\mathscr{W} = C_1 \cap C_2$ *and* $w = \# \mathscr{W}$ *then*

iii) $$|\deg C_1 - v \deg_{C_1}(\omega_C)| \leqq \frac{w}{2}$$

REMARKS. 1) It is clear that D_0 satisfies the hypotheses of the lemma. Indeed a) is satisfied by D_η and is preserved under specialization. The key point of the Proposition to replace this by the stronger condition i)

2) Roughly, iii) says that the degrees of the components of C are roughly in proposition to their "natural" degrees. We will see later on that this is enough to force $\mathscr{D} = \mathscr{C}$.

Proof. From b), c) and Proposition 3.1 we know that the cycle of C has no multiplicity and only ordinary double points. Hence C_{red} is a scheme

---

[1]) Non-special means $H^1(C, \mathcal{O}_C(1)) = (0)$.

having only ordinary double points and differing from C only by embedded components.

Suppose we are given a decomposition $C_{\text{red}} = C_1 \cup C_2$; let $\mathscr{W} = C_1 \cap C_2$, $w = \# \mathscr{W}$, L_i be the smallest linear subspace containing C_i and $n_i = \dim L_i$. We can assume $L_1 = V(X_{n_1+1} \dots X_n)$. For the 1-PS λ given by

$$t \mapsto \left[\begin{array}{ccccc|ccccc} t & & & & & & & & & \\ & \cdot & & & & & & 0 & & \\ & & \cdot & & & & & & & \\ & & & \cdot & & & & & & \\ & & & & t & & & & & \\ \hline & & & & & 1 & & & & \\ & & & & & & \cdot & & & \\ & & 0 & & & & & \cdot & & \\ & & & & & & & & \cdot & \\ & & & & & & & & & 1 \end{array}\right] \begin{array}{l} n_1 + 1 \\ \\ \\ n - n_1 \end{array}, \qquad \sum \rho_i = n_1 + 1$$

the associated ideal $\mathscr{I}$ in $\mathcal{O}_{C_{\text{red}} \times \mathbf{A}^1}$ is given by $\mathscr{I} = (t, I(L_1))$. To evaluate $e(\mathscr{I})$ we use an easy lemma whose proof is left to the reader

LEMMA 5.6. *If $X' \xrightarrow{f} X$ is a proper morphism of r-dimensional, possibly reducible "varieties", birational on each component, L is a line bundle on X, and $\mathscr{I}$ is an ideal sheaf on X such that* $\operatorname{supp}(\mathcal{O}_X/\mathscr{I})$ *is proper, then* $e_{f^*(L)}(f^*(\mathscr{I})) = e_L(\mathscr{I})$.

Letting $\mathscr{I}_i$ be the pullback of $\mathscr{I}$ to C_i, the lemma says $e_L(\mathscr{I}) = e_{L_1}(\mathscr{I}_1) + e_{L_2}(\mathscr{I}_2)$. But $\mathscr{I}_1 = t \,.\, \mathcal{O}_{C_1 \times \mathbf{A}^1}$ and support $\mathscr{I}_2$ contains $(0) \times \mathscr{W}$ so this implies[1]) $e_L(\mathscr{I}) \geqq 2 \deg C_1 + w$. Using b) and Theorem 2.8 this gives

$$(5.7) \qquad w + 2 \deg C_1 \leqq \frac{\deg C}{n+1} \,.\, 2 \,.\, (n_1 + 1) \leqq \frac{16}{7} (n_1 + 1)$$

If C_1 as any component of C_{red}, then this implies:

a) $H^1(C_1, \mathcal{O}_{C_1}(1)) = 0$: if not, then by Clifford's theorem

$$h^0(C_1, \mathcal{O}_{C_1}(1)) \leqq \frac{\deg C_1}{2} + 1$$

---

[1]) This argument has a gap: see Appendix, p. 108.

so by (5.7)

$$\deg C_1 \leqq \frac{8}{7} h^0(C_1, \mathcal{O}_{C_1}(1)) \leqq \frac{8}{14} \deg C_1 + \frac{8}{7},$$

which implies $\deg C_1 \leqq 2$, hence C_1 is rational and then $H^1(C_1, \mathcal{O}_{C_1}(1)) = (0)$ anyway.

b) $H^1(C_1, \mathcal{O}_{C_1}(1)(-\mathscr{W})) = (0)$: indeed from (5.7) and Riemann-Roch,

$$\deg C_1 + \frac{1}{2} w \leqq \frac{8}{7} (\deg C_1 - g_1 + 1), \text{ whence}$$

$$\deg \mathcal{O}_{C_1}(1)(-\mathscr{W}) = \deg C_1 - w \geqq 8(g_1 - 1) + \frac{5}{2} w.$$

The last expression is greater than $2g_1 - 2$ unless $w = 0$, when b) reduces to a), or $g_1 = 0$ and $w = 1$ or 2. But in this case $\mathcal{O}_{C_1}(1)(-\mathscr{W}) = \mathcal{O}_{\mathbf{P}^1}(e)$, with $e \geqq 1 - 2 = -1$.

Together a) and b) imply $H^1(C, \mathcal{O}_C(1)) = 0$. In fact, if C_{red} has components C_i, then there is an exact sequence

$$0 \to \oplus \mathcal{O}_{C_i}(1)(-\mathscr{W}_i) \to \mathcal{O}_{C_{\text{red}}}(1) \to \mathscr{M} \to 0$$

where $\mathscr{M}$ has 0-dimensional support, hence $H^1(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}(1)) = 0$, and if $\mathscr{N}$ is the sheaf of nilpotents in $\mathcal{O}_C$, then $\mathscr{N}$ has 0-dimensional support and the conclusion follows from an examination of the exact sequence

$$0 \to \mathscr{N} \to \mathcal{O}_C \to \mathcal{O}_{C\text{red}} \to 0.$$

Therefore hypothesis (a) can be rewritten $n + 1 = h^0(\mathcal{O}_C(1))$. Since C is not contained in a hyperplane, C is embedded by a complete linear system. But now if $\mathscr{N} \neq (0)$, then set-theoretically C will still be contained in a hyperplane, contradicting its Chow semi-stability; so $C = C_{\text{red}}$ and all that we have said about C_{red} above is true of C.

Using the fact that

$$\chi(\mathcal{O}_C) = -\chi(\omega_C) = -(\deg \omega_C + \chi(\mathcal{O}_C))$$

it follows that $\deg C/n + 1 = 2v/2v - 1$ and we can rewrite (5.7) in terms of v as

$$\frac{w}{2} + \deg C_1 \leqq \left(\frac{2v}{2v-1}\right)(\deg C_1 - g_1 + 1)$$

or equivalently

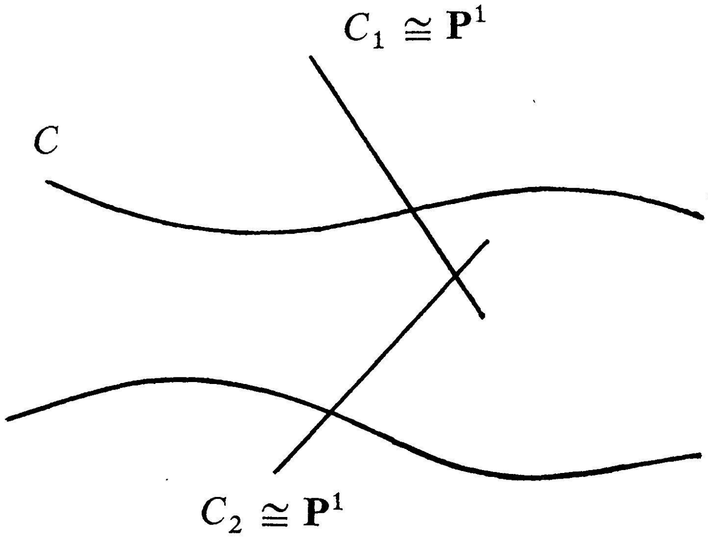
$$\frac{w}{2} \geqq v(2g_1 - 2 + w) - \deg C_1 = v \deg_{C_1}(\omega_C) - \deg C_1 .$$

Then since

$$\begin{aligned} 0 &= v\big(\deg(\omega_C)\big) - \deg C \\ &= v \deg_{C_1}(\omega_C) + v \deg_{C_2}(\omega_C) - \deg C_1 - \deg C_2 , \end{aligned}$$

we obtain iii): $\frac{w}{2} \geqq \left| v \deg_{C_1}(\omega_C) - \deg C_1 \right|$.

Now suppose C has a smooth rational component C_1 meeting the rest of the curve in w points $P_1, \ldots, P_w$. Then $\omega_C \mid C_1$ is just the sheaf of differentials on C_1 with poles at $P_1, \ldots, P_w$, so if $w \leqq 2$, $\deg_{C_1}(\omega_C) \leqq 0$. Using iii) this shows $\deg C_1 \leqq \frac{1}{2}$ if $w = 1$, absurd, and $\deg C_1 \leqq 1$ if $w = 2$. Moreover, if, in this last case one of the P_1 lies on a smooth rational curve C_2 meeting the rest of C in only 1 other point, as in the diagram below



then $\omega_C \mid_{C_1} \cong \mathcal{O}_{C_1}$ and $\omega_C \mid_{C_1} \cong \mathcal{O}_{C_2}$ so $\deg_{C_1 \cup C_2}(\omega_C) = 0$. Using iii) again, we find $\deg(C_1 \cup C_2) \leqq \frac{1}{2} 2 = 1$, and as this is absurd, we have proved all parts of the Proposition.

We are now ready to show that $\mathscr{D} = \mathscr{C}$. Since D_0 is moduli semi-stable, it follows that $\mathscr{D}$ is a normal two-dimensional scheme with only type A_n singularities. Moreover $\omega^{\otimes n}_{\mathscr{D}/k[[t]]}$ is generated by its sections if $n \geqq 3$ and defines a morphism from $\mathscr{D}$ to a scheme $\mathscr{D}'/k\,[[t]]$, where $D'_\eta = D_\eta$, $D'_0 = D_0$ with rational chains blown down to points. Thus $\mathscr{D}'$ is a family of moduli-stable curves over $k\,[[t]]$ with generic fibre $\mathscr{C}_\eta$. Since there is only one such (cf. [6]), it follows that $\mathscr{D}' = \mathscr{C}$. Thus we have a diagram:

$$\begin{array}{ccccc} C_\eta & \xleftarrow{\approx} & D_\eta & \xrightarrow{\Phi\eta} & \mathbf{P}^N \times \operatorname{Spec} k((t)) \\ \cap & & \cap & & \cap \\ \mathscr{C} & \longleftarrow & \mathscr{D} & \xrightarrow[\Phi]{} & \mathbf{P}^N \times \operatorname{Spec} k[[t]] \end{array}$$

$$\Phi_\eta^*(\mathcal{O}_{\mathbf{P}^N}(1)) = \omega_{D_\eta / k((t))}^{\otimes n} \,.$$

Let $L = \mathcal{O}_{\mathscr{D}}(1)$. It follows that $L \cong \omega^{\otimes n}_{\mathscr{D}/k[[t]]}(-\sum r_i D_i)$, where D_i are the components of D_0. Multiplying the isomorphism by $t^{\min(r_i)}$, we can assume $r_i \geqq 0$, $\min r_i = 0$. Let $D_1 = \bigcup_{r_i = 0} D_i$, $D_2 = \bigcup_{r_i > 0} D_i$. If f is a local equation of $\sum r_i D_i$, then $f \not\equiv 0$ in any component of D_1 since $r_i = 0$ on all these while $f(x) = 0$, all $x \in D_1 \cap D_2$, so

$$\#(D_1 \cap D_2) \leqq \deg_{D_1}(\mathcal{O}_{\mathscr{D}_0}(\textstyle\sum r_i D_i)) \,.$$

But this last degree equals $(\deg D_1 - n \deg_{D_1}(\omega_{D_0}))$ which contradicts iii) of Proposition 5.5 unless all r_i are zero. Hence $L = \omega_{\mathscr{C}}^{\otimes n}$ which shows $\mathscr{D} = \mathscr{C}$.

Line bundles on the moduli space

For the remainder of this section we examine $\operatorname{Pic}(\overline{\mathscr{M}}_g)$. We fix a genus $g \geqq 2$ and an $e \geqq 3$. Then for all stable C, $\omega_C^{\otimes e}$ is very ample and in this embedding C has degree $d = 2e(g-1)$, the ambient space has dimension $\nu - 1$ where $\nu = (2e-1)(g-1)$ and C has Hilbert polynomial $P(X) = dX - (g-1)$. Let $H \subset \operatorname{Hilb}^P_{\mathbf{P}^{\nu-1}}$ be the locally closed smooth subscheme of e-canonical stable curves C, let $C \subset H \times \mathbf{P}^{\nu-1}$ be the universal curve and let

$$\mathrm{ch} : H \to \mathrm{Div} = \mathrm{Div}^{d,d} = \left\{ \begin{array}{l} \text{projective space of bihomogeneous forms} \\ \text{of bidegree } (d, d) \text{ in dual coordinates} \\ u, v \text{ (cf. § 1).} \end{array} \right\}$$

be the Chow map. These are related by the diagram

$$\begin{array}{ccccc} & & C & & \\ & & \downarrow{\scriptstyle \pi} & & \\ \mathrm{Div} & \xleftarrow{\mathrm{ch}} & H & \xrightarrow{\rho} & \overline{\mathscr{M}}_g = H/PGL(\nu) \end{array}$$

If $\operatorname{Pic}(H, PGL(\nu))$ is the Picard group of invertible sheaves on H with $PGL(\nu)$-action, we have a diagram

$$\operatorname{Pic}(\bar{\mathcal{M}}_g) \xrightarrow{\rho^*} \operatorname{Pic}(H, PGL(v)) \xrightarrow{\alpha} \operatorname{Pic}(H)^{PGL(v)} \subset \operatorname{Pic}(H).$$

In this situation, we have:

LEMMA 5.8. *In the sequence above, ρ^* is injective with torsion cokernel and α is an isomorphism.*

Proof. α is an isomorphism by Prop. 1.4 [14]; ρ^* injective is easy; coker ρ^* torsion can be proved, for instance, using Seshadri's construction, Th. 6.1 [19].

This lemma allows us to examine Pic $(\bar{\mathcal{M}}_g)$ by looking inside Pic $(H)^{PGL(v)}$ which is a much easier group to come to grips with.

DEFINITION 5.9. *Let $\Delta \subset H$ be the divisor of singular curves, $\delta = \mathcal{O}_H(\Delta)$ and $\lambda_n = \Lambda^{\max}(\pi_*(\omega_{C/H}{}^{\otimes n}))$, $(n \geq 1)$. We write λ for λ_1.*

The sheaves λ_n and δ are the most obviously interesting invertible sheaves on H from a moduli point of view. The next theorem expresses all of these in terms just involving λ and δ.

THEOREM 5.10. $\lambda_n = \mu^{\binom{n}{2}} \otimes \lambda$ *where* $\mu = \lambda^{12} \otimes \delta^{-1}$.

Proof. The proof is based on Grothendieck's relative Riemann-Roch theorem (see Borel-Serre [4]), which we will briefly recall.

Let X and Y be complete smooth varieties over k, $A(X)$ be the Chow ring of X and $\mathcal{F}$ be a coherent sheaf on X. Let $c_i(\mathcal{F}) \in A(X)$ denote the i^{th} Chern class of $\mathcal{F}$, Chern $(\mathcal{F}) \in A(X) \otimes \mathbf{Q}$ its Chern character and $\mathcal{T}(\mathcal{F}) \in A(X) \otimes \mathbf{Q}$ its Todd genus. These are related by:

$$(5.11) \quad \text{Chern}(\mathcal{F}) = rk\,\mathcal{F} + c_1(\mathcal{F}) + \frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F})$$

$$+ \text{ terms of higher codimension,}$$

$$\mathcal{T}(\mathcal{F}) = 1 - \frac{c_1(\mathcal{F})}{2} + \frac{c_1(\mathcal{F})^2 + c_2(\mathcal{F})}{12}$$

$$+ \text{ terms of higher codimension.}$$

Let $K(Y)$ be the Grothendieck group of $Y, f: X \to Y$ be a proper map, and $f_!(\mathcal{F}) = \sum (-1)^i [\mathbf{R}^i f_* \mathcal{F}] \in K(Y)$. The relative Riemann-Roch theorem expresses the Chern character of $f_!(\mathcal{F})$, *modulo torsion* as

$$\text{Chern}(f_!\,\mathcal{F}) = f_*(\text{Chern}\,\mathcal{F} \cdot \mathcal{T}(\Omega^1_{X/Y}))$$

which using (5.11) gives:

(5.12) $rkf_! \mathscr{F} + c_1(f_! \mathscr{F}) + \dots\dots\dots$

$$= f_* \left[\left(rk(\mathscr{F}) + c_1(\mathscr{F}) + \frac{c_1(\mathscr{F})^2}{2} - c_2(\mathscr{F}) \right) \cdot \left(1 - \frac{c_1(\Omega^1_{X/Y})}{2} + \frac{c_1(\Omega^1_{X/Y})^2 + c_2(\Omega^1_{X/Y})}{12} \right) \right]$$

For the time being, we work implicitly modulo torsion.

Now suppose $\mathscr{F}$ is a line bundle such that $R^i f_*(\mathscr{F}) = 0$, $i > 0$ and suppose dim X = dim $Y + 1$. Then the codimension 1 term on the left of (5.12) (i.e. on Y) corresponds to the codimension two term on the right (i.e. on X). Since $c_2(\mathscr{F}) = 0$, this gives

(5.13) $c_1(f_* \mathscr{F}) = c_1(f_! \mathscr{F})$

$$= f_* \left[\frac{c_1(\Omega^1_{X/Y})^2 + c_2(\Omega^1_{X/Y})}{12} - \frac{c_1(\mathscr{F}) c_1(\Omega^1_{X/Y})}{2} + \frac{c_1(\mathscr{F})^2}{2} \right]$$

In case $f: C \to S$ is a moduli-stable curve over S, $X = C$ and $Y = S$, we can simplify this. Indeed I claim that if Sing C is the singular set on C and I_{sing} is its ideal, then

i) codim Sing $C = 2$

ii) the canonical homomorphism $\Omega^1_{C/S} \to \omega_{C/S}$ induces an isomorphism $\Omega^1_{C/S} = I_{\text{sing}} \cdot \omega_{C/S}$.

We certainly have the isomorphism of ii) off Sing C. At a singular point C has a local equation of the form $xy = t^n$, where t is a parameter on S, x and y are affine coordinates on the fibre. Moreover locally C is singular only at the points (0, 0) in the fibres where $t = 0$, so Sing C has codimension 2. Near the singular point

$$\Omega^1_{C/S} = (\mathcal{O}_C dx + \mathcal{O}_C dy)/(xdy + ydx)\, \mathcal{O}_C$$

while $\omega_{C/S}$ is the invertible sheaf generated by the differential ζ which is given by dx/x outside $x = 0$ and by $-dy/y$ outside $y = 0$. Thus

$$\Omega^1_{C/S} = \mathscr{M}_{(0,0),C} \cdot \zeta = \mathscr{M}_{(0,0),C} \cdot \omega_{C/S} \cdot$$

Recall the following corollary to Riemann-Roch: if X is a smooth variety, $Y \subset X$ a subvariety of codim r and $\mathscr{F}$ is coherent on Y, then considering $\mathscr{F}$ as a sheaf on X

$$c_i(\mathscr{F}) = \begin{cases} 0, 1 \leqq i \leqq r-1 \\ \left((-1)^{r-1}(r-1)!\, rk\, \mathscr{F}\right) Y, i = r \end{cases}$$

Set $X = C$, $Y = \operatorname{Sing} C$ and $\mathscr{F} = \Omega^1_{C/S}$. The Whitney product formula applied to the chern classes of the exact sequence

$$0 \to \Omega^1_{C/S} \to \omega_{C/S} \to \omega_{C/S} \otimes \mathcal{O}_{\operatorname{Sing} C} \to 0$$

gives, taking account of the corollary

$$\begin{aligned} &1 + c_1(\omega_{C/S}) \\ &= \left(1 + c_1(\Omega^1_{C/S}) + c_2(\Omega^1_{C/S}) + \ldots..\right) . \left(1 + 0 - [\operatorname{Sing} C] + \ldots..\right) \end{aligned}$$

Equating terms of equal codimension, we see that $c_1(\Omega^1_{C/S}) = c_1(\omega)$ and $c_2(\Omega^1_{C/S}) = [\operatorname{Sing} C]$ so that (5.13) becomes

$$c_1(f_* \mathscr{F}) = f_* \left[\frac{c_1(\omega_{C/S})^2 + [\operatorname{Sing} C]}{12} - \frac{c_1(\mathscr{F})\, c_1(\omega_{C/S})}{2} + \frac{c_1(\mathscr{F})^2}{2}\right]$$

Applying this to the map $\pi\colon C \to H$, when $\mathscr{F} = \omega^{\otimes n}_{C/H}$ gives

$$\begin{aligned} \lambda_n &= \Lambda^{\max}(\pi_* \omega^{\otimes n}_{C/H}) = c_1(\pi_* \omega^{\otimes n}_{C/H}) \\ &= \pi_* \left[\frac{c_1(\omega_{C/H})^2 + [\operatorname{Sing} C]}{12} - \frac{c_1(\omega^{\otimes n}_{C/H})\, c_1(\omega_{C/H})}{2} + \frac{c_1(\omega^{\otimes n}_{C/H})^2}{2}\right] \\ &= \binom{n}{2} \pi_* \left(c_1(\omega_{C/H})^2\right) + \frac{\pi_*\left(c_1(\omega_{C/H})^2\right) + [\Delta]}{12} \end{aligned}$$

Setting[1]) $n = 1$, we see that $\lambda = \left[\dfrac{\pi_*\left(c_1(\omega_{C/H})^2\right) + [\Delta]}{12}\right]$ and $\pi_*\left(c_1(\omega_{C/H})^2\right) = 12\,\lambda - [\Delta]$. Plugging these values back in gives us the theorem up to torsion. But in fact:

LEMMA 5.14. *Over* $\mathbf{C}$, $\operatorname{Pic}(H, PGL(v))$ *is torsion free.*

Note that this will prove what we want because the invertible sheaves that we are trying to show are isomorphic all "live" on the full scheme $H_{\mathbf{Z}}$ over Spec $\mathbf{Z}$ of stable $\mathscr{C}$-canonical curves. If they are isomorphic on $H_{\mathbf{Z}}$, they are isomorphic after any base change. But on the other hand, I claim that $\operatorname{Pic}(H, PGL(v))$ injects into $\operatorname{Pic}(H_{\mathbf{C}}, PGL_{\mathbf{C}}(v))$:

---

[1]) For $n = 1$, $R^1 \pi_*(\omega_{C/H})$ is not zero, but it is the trivial line bundle, hence doesn't affect $\pi_!$.

If L is a line bundle on H with $PGL(v)$ action such that $L \otimes \mathbf{C}$ is trivial over $H_{\mathbf{C}}$, then

$$\begin{array}{ccc} H^0(H, L)^{PGL(v)} \otimes \mathbf{C} = & H^0(H_{\mathbf{C}}, L \otimes \mathbf{C})^{PGL(v)} & \\ & \wr \Big\downarrow \alpha & \\ & H^0(H_{\mathbf{C}}, \mathcal{O}_{H\mathbf{C}})^{PGL(v)} & = \mathbf{C} \end{array}$$

since $H_{\mathbf{C}}/_{PGL(v)}$ is compact. Thus we can find a non-zero section $s \in H^0(H, L)^{PGL(v)}$, which over $\mathbf{C}$ can be used to give the trivialization α. Over $\mathbf{C}$, s has no zeros so the divisor $(s)_0$ of the zeros of s on H, has support only over the closed fibres of Spec $(\mathbf{Z})$. Mumford and Deligne [6] have shown that $H \to$ Spec $\mathbf{Z}$ is smooth with irreducible fibres, hence $(s)_0 = \sum r_i \pi^{-1}(p)$, $r_i \geqq 0$ i.e. $(s)_0 = (n)$ for some integer n. Then $\left(\frac{s}{n}\right)$ is a global section of L with no zeros so L is trivial.

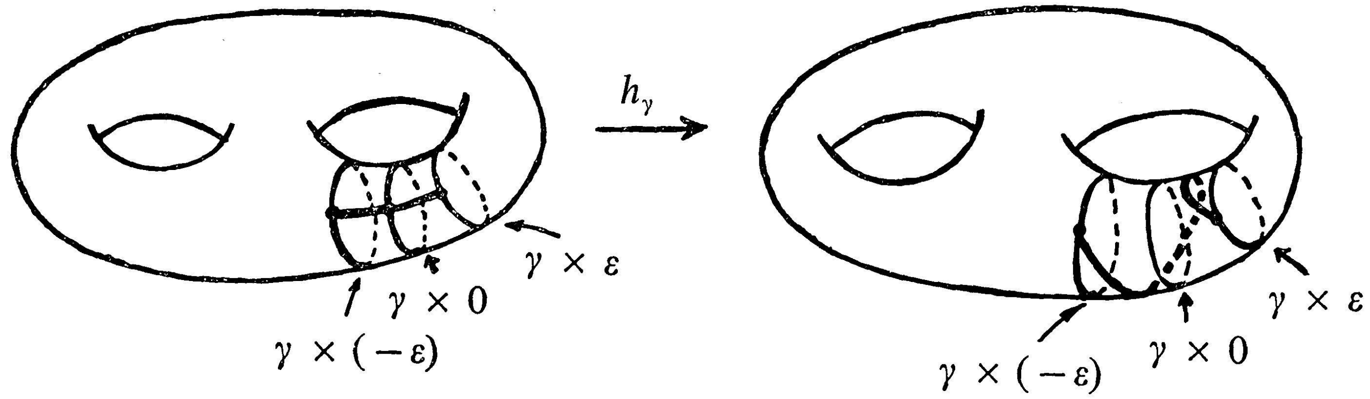
Proof of Lemma. Over $\mathbf{C}$, we have Teichmüller theory at our disposal. Let Π be a standard model of a group with generators $\{a_i, b_i \mid 1 \leqq i \leqq g\}$ mod the relation $\prod_{i=1}^{g} (a_i b_i a_i^{-1} b_i^{-1}) = 1$. Then the Teichmüller modular group Γ is

$$\Gamma = \{\alpha \mid \alpha : \Pi \to \Pi \text{ is an orientation preserving isomorphism}\} / \text{inner automorphisms}$$

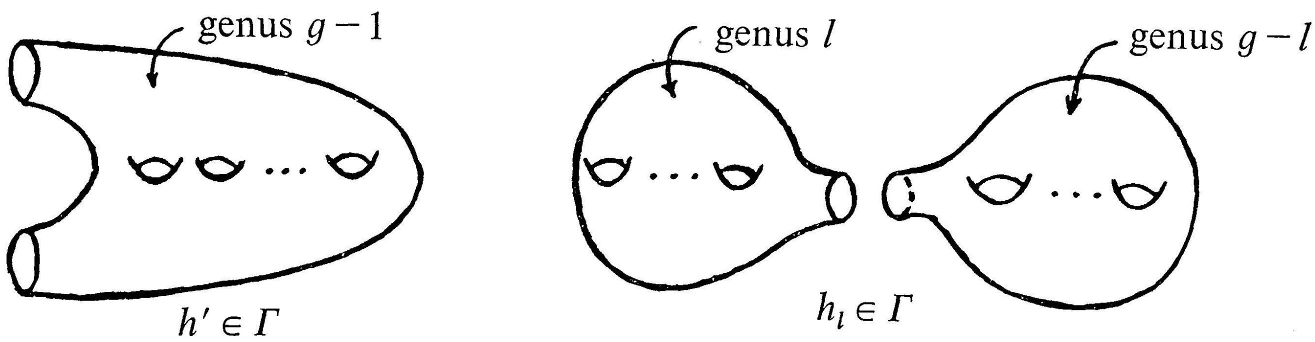
The Teichmüller space $\mathcal{T}_g$ is given by

$$\mathcal{T}_g = \left\{ (C, \alpha) \,\middle|\, \begin{array}{l} C \text{ a smooth curve of genus } g \text{ and } \alpha: \pi_1(C) \to \Pi \text{ an} \\ \text{orientation preserving isomorphism given up to inner} \\ \text{automorphism} \end{array} \right\}$$

Fix a model M_g of the real surface of genus g, and identify $\pi_1(M_g)$ and Π. Then Γ is generated by the maps which are induced by certain automorphisms of M_g, called Dehn twists. The Dehn twist h_γ corresponding to a loop $\gamma: [0, 1] \to M_g$ on M_g is given by taking an ε-collar $\gamma \times [-\varepsilon, \varepsilon]$ about γ, letting h = identify off the collar and letting $h(\gamma(t), \eta - \varepsilon) = \left(\gamma\left(t + \frac{\eta}{2\varepsilon}\right), \eta - \varepsilon\right)$ as shown below.

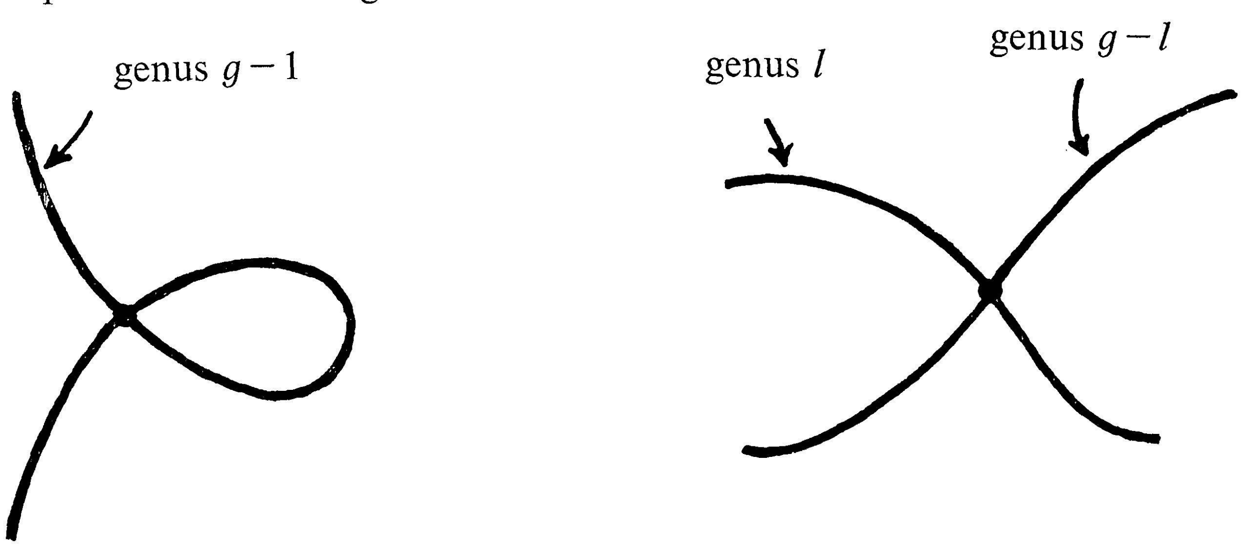


Up to inner automorphism h_γ is determined by which of the pictures below results from cutting open M_g along γ. We have name these elements of Γ in the diagrams:



The Dehn twist h_γ can also be described as the monodromy map obtained by going around a curve C_0 with one double point for which γ is the vanishing cycle.

The components of $\Delta \subset H$ correspond to the different ways of putting a stable double point on a smooth moduli stable curve C. They are the closures of the sets of curves of the forms shown below: again, we name these components in the diagram:



$\Delta' = \begin{pmatrix}\text{closure of set}\\ \text{formed by curves}\\ \text{like this}\end{pmatrix}$ $\qquad$ $\Delta_l = \begin{pmatrix}\text{closure of set}\\ \text{formed by curves}\\ \text{like this}\end{pmatrix}$

Let $\tilde{H} = \left\{ (C, \alpha, B) \;\middle|\; \begin{array}{l} (C, \alpha) \in \mathcal{T}_g, B \text{ a basis of the } e\text{-tuple dif-} \\ \text{ferentials on } C \text{ given up to a scalar} \end{array} \right\}$

Suppose we are given a line bundle L on H with $PGL(\nu)$-action such that $L^n \cong \mathcal{O}_H$. L induces a cyclic covering H' of H plus a lifting of the $PGL(\nu)$-action to H'. If we choose n minimal this covering is not split: we denote its structure group by Γ_L. Let $\tilde{H}'$ be the pullback of covering over $\tilde{H}$, and let $\mathcal{T}'_g$ denote the quotient of $\tilde{H}'$ by $PGL(\nu)$—this is a covering of $\mathcal{T}_g$. These coverings are related by

$$\begin{tikzcd}
 & & & \tilde{H}^1 \arrow[dl] \arrow[dd, no head] \arrow[dr] & \\
H' \arrow[dd, "\Gamma_L"'] & \supset & H' - \Delta' \arrow[dd, "\Gamma_L"'] & & \mathcal{T}'_g \arrow[dd] \\
 & & & \tilde{H} \arrow[ul, dashed] \arrow[dl, "\Gamma"] \arrow[dr] & \\
H & \supset & H - \Delta & & \mathcal{T}_g
\end{tikzcd}$$

$\mathcal{T}_g$ is simply connected so the cover $\mathcal{T}'_g \to \mathcal{T}_g$ splits, hence so does $\tilde{H}' \to \tilde{H}$. A section of this last cover gives a map from $\tilde{H}$ to $H' - \Delta'$ (shown dashed in the diagram), so Γ_L is a quotient of Γ, of finite order.

Let γ' [resp. γ_e] be a loop at a fixed base point $P_0 \in H - \Delta$ going around Δ' [resp.: Δ_e] but homotopic to 0 in H. Fix a point $\tilde{P}_0 \in \tilde{H}$ over P_0. The monodromy characterization of the Dehn twists implies that γ' [resp.: γ_e] lifted to $\tilde{H}$ goes from $\tilde{P}_0$ to $h'(\tilde{P}_0)$ [resp.: to $h_e(\tilde{P}_0)$]. Since γ' [resp.: γ_e] are homotopic to 0 in H, and the covering $H' - \Delta'$ extends over H, this implies that the image of h' [resp.: h_l] in Γ_L is 0. But these elements and their conjugates generate Γ_L, so $\Gamma_L = \{1\}$, hence $L \cong \mathcal{O}_H$, proving the lemma and the theorem.

In order to describe the ample cone on Pic $(\overline{\mathcal{M}_g})$ we prove:

THEOREM 5.15. $\mathrm{Ch}^*(\mathcal{O}_{\mathrm{Div}}(\nu)) = (\mu^e \otimes \lambda^{-4})^{e(g-1)}$

Proof. The proof depends on a result which we simply quote from Fogarty [8] or Knudsen [12]:

PROPOSITION 5.16. *Let* S *be a locally closed subscheme of a Hilbert scheme* $\mathrm{Hilb}^P_{\mathbf{P}^{\nu-1}}$, Ch *be the associated Chow map* $\mathrm{Ch}: S \to \mathrm{Div}$ *and* $Z \subset \mathbf{P}^\nu \times S$ *have relative dimension r over S. Then if* $n \gg 0$, $\Lambda^{\max} p_{2,*}(\mathcal{O}_Z(n)) = \bigotimes_{i=0}^{r+1} \mu_i^{\binom{n}{i}}$ *and* $\mathrm{Ch}^*(\mathcal{O}_{\mathrm{Div}}(1)) = \mu_{r+1}$, *where* μ_i *are suitable invertible sheaves on* S.

In the situation of our theorem, with $S = H$ and $Z = C$, $\mathcal{O}_C(1) = \omega_{C/H}^{\otimes e} \otimes \pi^* Q$ where Q is the invertible sheaf determined by $(\pi_* \omega_{C/H}^{\otimes e}) \otimes Q = \pi_* \mathcal{O}_C(1) = \pi_* \mathcal{O}_{\mathbf{P}^{\nu-1}}(1) = \mathcal{O}_H^\nu$, hence

$$\mathcal{O}_H = [\Lambda^{\max} \pi_*(\omega_{C/H}^{\otimes e})] \otimes Q^\nu = \mu^{\binom{e}{2}} \otimes \lambda \otimes Q^\nu . \tag{5.17}$$

On the other hand,

$$\Lambda^{\max}(\pi_* \mathcal{O}_C(n)) = \Lambda^{\max}[\pi_*(\omega_{C/H}^{\otimes ne}) \otimes Q^n] = \mu^{\binom{ne}{2}} \otimes \lambda \otimes Q^{P(n).n} .$$

This has leading term in n of $\mu^{n^2e^2/2} \otimes Q^{2e(g-1)n^2}$ so

$$\begin{aligned} Ch^*(\mathcal{O}_{\mathrm{Div}}(\nu)) &= \mu^{\nu e^2} \otimes Q^{4e(g-1)\nu} \\ &= \mu^{\nu e^2 - \binom{e}{2}.4e(g-1)} \otimes \lambda^{-4e(g-1)} \qquad \text{using (5.17)}. \end{aligned}$$

Finally, therefore, $\mathrm{Ch}^*(\mathcal{O}_{\mathrm{Div}}(\nu)) = \mu^{e^2(g-1)} \otimes \lambda^{-4e(g-1)}$ as required.

COROLLARY 5.18. *If* $e \geqq 5$, $\mu^e \otimes \lambda^{-4} (= \lambda^{12e-4} \otimes \delta^{-e})$ *is "ample on* $\overline{\mathcal{M}}_g$", *i.e. those positive powers of this bundle which are pull-backs of bundles on* $\overline{\mathcal{M}}_g$ *are ample on* $\overline{\mathcal{M}}_g$.

Proof. This is an immediate consequence of the Theorem and our main result: that $PGL(\nu)$-invariant sections of $\mathrm{Ch}^*(\mathcal{O}_{\mathrm{Div}}(1))$ define a projective embedding of $\overline{\mathcal{M}}_g$.

REMARK 5.19. A similar argument using the facts that

(1) $\omega^{\otimes e}$ is base point free for all canonical curves when $e \geqq 2$,

(2) smooth curves are stable if $d > 2g$,

shows that if $e \geqq 2$, the sections of $\lambda^{12e-4} \otimes \delta^{-e}$ on $\overline{\mathcal{M}}_g$ separate points on $\mathcal{M}_g$.

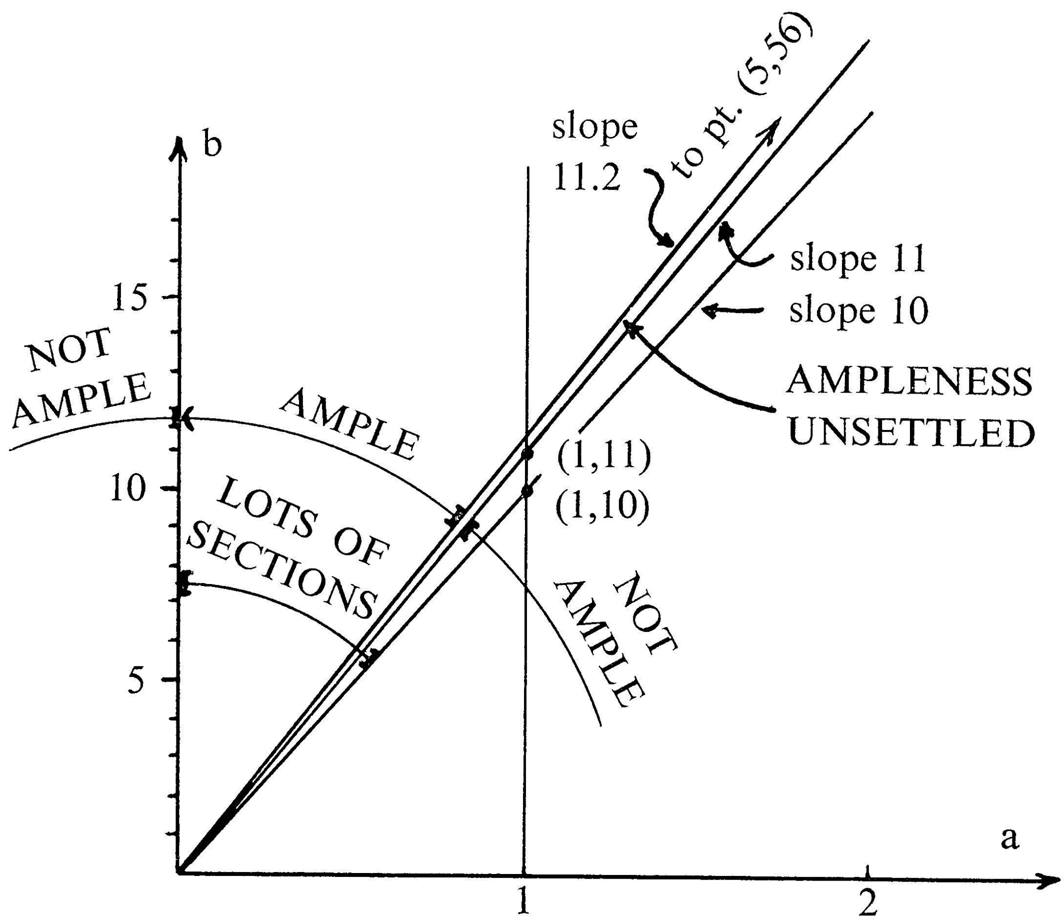
To get a good picture of the ample cone on $\overline{\mathcal{M}}_g$ we need to use the realization via Θ functions $\mathcal{A}_{g,1} \xrightarrow{\Theta} \mathbf{P}^N$ of the moduli scheme $\mathcal{A}_{g,1}$ of principally polarized abelian varieties. More precisely, let $J : \mathcal{M}_g \to \mathcal{A}_{g,1}$ be the map taking a curve C to its Jacobian. Then we have:

THEOREM 5.20. *In characteristic* 0, *the morphism* $\mathcal{M}_g \xrightarrow{J} \mathcal{A}_{g,1} \xrightarrow{\Theta} \mathbf{P}^N$ *extends to a morphism* $\overline{\mathcal{M}_g} \xrightarrow{\theta} \mathbf{P}^N$ *so that for some* m, $\theta^*(\mathcal{O}_{\mathbf{P}^N}(1)) = \lambda^m$.

Proof. See Arakelov [1] or Knudsen [12].

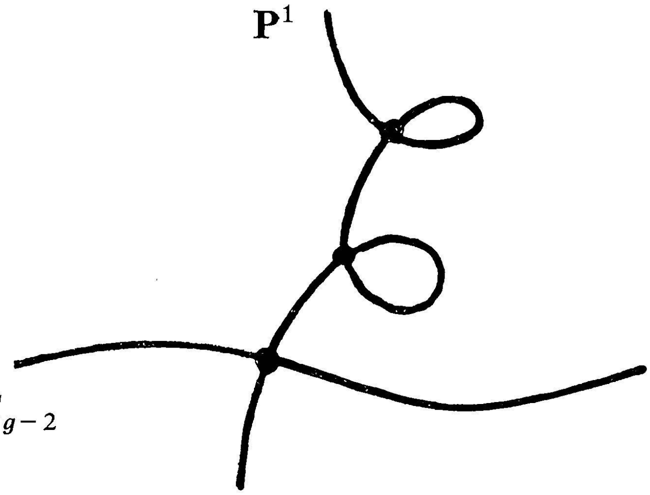
REMARK. This should also hold in characteristic p, but it seems to be a rather messy problem there.

Putting together 5.18 and 5.20, we get a whole sector in the (a, b)-plane such that $\lambda^b \otimes \delta^{-a}$ is ample for (a, b) in this sector. This is depicted in the diagram below:



The fact that λ and $\lambda^{11} \otimes \delta^{-1}$ are not ample can be seen by examining the following 2 curves in $\overline{\mathcal{M}_g}$:

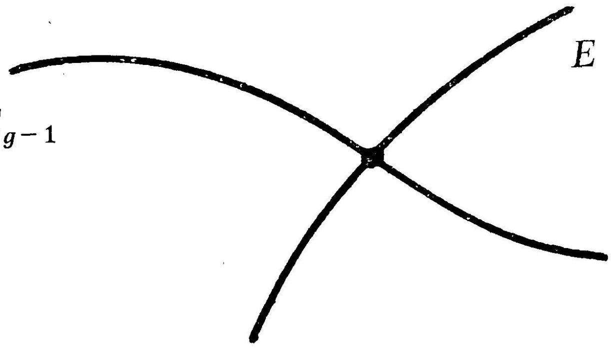
(1) If S_1 is a curve in $\overline{\mathcal{M}_g}$ composed of curves of the form:



$\begin{cases} \text{vary cross ratio of the 4 points} \\ \text{in } \mathbf{P}^1 \text{ making up the 2 double points} \end{cases}$

where C_{g-2} is a fixed genus $(g-2)$ component, then $\lambda \mid_{S_1} = \mathcal{O}_{S_1}$, hence sections of λ always collapse such families.

(2) If S_2 is a curve in $\overline{\mathcal{M}}_g$ composed of curves of the form:



$\begin{cases} E \text{ an elliptic curve: vary its} \\ j\text{-invariant} \end{cases}$

where C_{g-1} is a fixed genus $(g-1)$ component, then $\lambda^{11} \otimes \delta^{-1} \mid_{S_2} = \mathcal{O}_{S_2}$ i.e. $\lambda^{11} \otimes \delta^{-1}$ collapses these families.

We omit the details.

APPENDIX

We wish to fill in the gap in the proof of Proposition 5.5 on page 95. The difficulty occurs if the support of $\mathcal{I}$, i.e. $(0) \times L_1$, contains some of the components of C_2 meeting C_1. In this case, the inequality

$$e_L(\mathcal{I}_2) \geqslant w$$

is not clear. Indeed, if $D_1, \ldots, D_k$ are the components of C_2 meeting C_1, $w_i = \#(D_i \cap C_1)$, and $\mathcal{K}_i$ is the pull-back of $\mathcal{I}_2$ to D_i, then

$$e_L(\mathscr{I}_2) = \sum e_L(\mathscr{K}_i)$$
$$e_L(\mathscr{K}_i) \geqslant w_i \text{ if } L_1 \not\supseteq D_i$$
$$= 2 \deg D_i \text{ if } L_1 \supseteq D_i \,.$$

Now suppose C_1 is *irreducible* and $D_i \subseteq L_1$. Then (5.7) is modified to:

$$2 \deg D_i + 2 \deg C_1 \leqslant \frac{16}{7}(n_1+1)\,.$$

Since C_1 spans L_1, $n_1 \leqslant \deg C_1$. Substituting this, we find

$$\deg D_i \leqslant \frac{\deg C_1}{7} + \frac{8}{7}$$

hence $\deg D_i \leqslant \deg C_1$ (except in the lowest case $\deg C_1 = 1$; in this case, C_1 is a line, so $C_1 = L_1$ and $\operatorname{Supp} \mathscr{K}_i = D_i \cap \mathrm{L}_1 \subsetneqq D_i$). Now the reverse of this inequality cannot be true too. This means that if we apply the same arguement to

$$C_{\text{red}} = D_i \cup \overline{(C-D_i)}$$

then the linear span M of D_i cannot contain C_1. Therefore

$$w_i + 2 \deg D_i \leqslant \frac{16}{7}(\dim M_i+1) \leqslant \frac{16}{7}(\deg D_i+1)$$

$$\therefore\ w_i \leqslant 2 \deg D_i$$
$$\therefore\ e_L(\mathscr{K}_i) \geqslant w_i \quad \text{in all cases}$$
$$\therefore\ e_L(\mathscr{I}_2) \geqslant w \quad \text{as required.}$$

This proves (5.7) if C_1 is irreducible, hence (a) and (b) that follow are correct. In particular, (b) shows that $\mathcal{O}_{C_1}(1)(-\mathscr{W})$ always has sections, unless C_1 is a line and $\# \mathscr{W}=2$. The next paragraph shows that C is embedded by a complete linear system. So when $\Gamma(\mathcal{O}_{C_1}(1)(-\mathscr{W})) \neq (0)$, there is a hyperplane containing all components of C except C_1. Returning to the general case of (5.7) where C_1 is any subset of the components of C, it follows that the linear span L_1 of C_1 contains only C_1 and the *line*s D_i which meet C_1 in 2 points. For these, $\# (D_i \cap C_1) = 2 \deg D_i$, so in all cases it is true that $e_L(\mathscr{I}_2) \geqslant w$ as required.

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