

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 23 (1977)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE NUMBER OF ZEROS OF FUNCTIONS
Autor: van der Poorten, A. J.
Kapitel: 5. Further results
DOI: <https://doi.org/10.5169/seals-48918>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 15.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

b_{ks} , or even the $b_{ks}/(s-1)!$ rational integers; thus as soon as $|b_{k\rho(k)}| < 1$ one has $b_{k\rho(k)} = 0$ and then *a fortiori* one has sequentially $b_{k\rho(k)} \downarrow_1 = \dots = b_{k1} = 0$ which eventually shows that F vanishes identically. Thus in this circumstance it suffices to have an estimate only for the leading coefficients of the polynomial coefficients.

In applications it is of course necessary to have good lower bounds for the D_k and the Δ_h . For some such estimates see Cijssouw and Tijdeman [5], lemmas 5 and 6.

One case is of sufficient interest to mention specifically:

COROLLARY. If $\alpha_1 = 1, \alpha_2 = 2, \dots, \alpha_n = R$ (so $n = R$) and $\tau(h) = T$ ($h=1, \dots, n$), so $N = RT$, then

$$\left| \frac{F^{(t-1)}(h)}{(t-1)!} \right| \leq \chi_{ht} \leq \chi_h \leq \chi \quad (h = 1, \dots, R; t = 1, \dots, T)$$

and

$$N = RT \geq 2(\sigma - 1) + 5\Omega R,$$

implies that for $k = 1, \dots, m$

$$\begin{aligned} |b_{k\rho(k)}| &< D_k^{-1} (N/eR)^{\sigma-1} \sum_{h=1}^R \sum_{t=1}^T \chi_{ht} N^{-1} (5R)^N / ((h-1)!(R-h)!)^T \\ &< D_k^{-1} (N/eR)^{\sigma-1} 30^N \chi \end{aligned}$$

Proof. Note only that $(h-1)!(R-h)! > 2^{-(R-1)}(R-1)! > (R/6)^R$; (by sharpening lemma 7 for this case one can improve the 30 to about 15).

5. FURTHER RESULTS

We consider some further applications of the method of this note. It is instructive to observe that the success of these applications depends, in effect, on forcing an analogy with the simplest case, that of exponential polynomials. The methods of Hayman [8] applies to a different class of functions, which does however intersect with the class considered here. For an example of this different method at work, see Voorhoeve, van der Poorten and Tijdeman [33]. In this context see also Voorhoeve and van der Poorten [32]; the ideas here however relate to the new method of Voorhoeve [31].

Continuing to use the notation of the previous sections, we observe that if in lemma 2 we take $t_\lambda = \lambda - 1, z_\lambda = 0$ and $g_i(z) = g(\omega_i z)$ where g is given by (14) then the ratio $\Delta_{\lambda, k}/\Delta$ of lemma 2 is given by

$$\Delta_{\lambda,k} / \Delta = D_{\lambda,k} / Dc_{\lambda-1}.$$

where D is the Vandermonde determinant of lemma 3. Then lemma 4 and lemma 5 allow us to estimate the number of zeros of functions F of the shape

$$(44) \quad F(z) = \sum_{h=1}^m \sum_{t=1}^{\rho(h)} a_{ht} z^{t-1} g^{(t-1)}(\omega_h z)$$

in discs with centre the origin. Indeed, the analogue of (21) becomes

$$|F|_{S^*} / |F|_S < \sum_{\lambda=1}^{\sigma} \left(\frac{S^*}{S} \right)^{\lambda-1} |c_{\lambda-1}|^{-1} \sum_{h=1}^{\sigma} \frac{(\Omega S^*)^{h-\lambda}}{(h-\lambda)!} |g|^{(h-1)}(\Omega S^*),$$

and the only important new addition is that one requires, if $g(z) = \sum \frac{c_n}{n!} z^n$, that $c_0 c_1 \dots c_{\sigma-1} \neq 0$.

An easy example is given by the class of functions

$$(45) \quad g(z) = f_{\mu}(z) = \sum_{n=0}^{\infty} z^n / (\mu+1) \dots (\mu+n)$$

for μ in \mathbf{C} , μ not a negative integer. Here it is amusing to observe that one has

$$\begin{aligned} z f_{\mu}'(z) &= \mu + (z-\mu) f_{\mu}(z) \text{ and hence } z^{t-1} f_{\mu}^{(t-1)}(z) \\ &= r_t(z; \mu) + q_t(z; \mu) f_{\mu}(z) \end{aligned}$$

for $t = 1, 2, \dots$, where the polynomials r_t, q_t have degree respectively at most $t-2$ and $t-1$ in z . It follows that, with a slight change of notation, the function (44) can be taken to be of the shape

$$F(z) = \sum_{h=0}^m P_h(z) f_{\mu}(\omega_h z),$$

where the P_h are polynomials of degrees respectively at most $\rho(0), \rho(1) - 1, \dots, \rho(m) - 1$ and $\rho(0) \geq \max_k \rho(k), \omega_0 = 0$ (so $f_{\mu}(\omega_0 z) \equiv 1$), and we take $\sum_{h=0}^m \rho(h) = \sigma + 1$.

However one need not be as explicit as regards the Taylor coefficients of the given function g . For example consider a Weierstrass elliptic function p with given fixed algebraic invariants. Then one easily shows that there is a point v such that

$$|p(v)| \leq c \text{ and } |p^{(\lambda-1)}(v)| \geq \sigma^{-c\sigma}, \lambda = 1, \dots, \sigma$$

for some c depending only on p . It is then easy to conclude by the method we have described that if $\max_k |\omega_k| = \Omega \leq 1$ then a function $F \neq 0$ of the shape

$$F(z) = \sum_{h=1}^m \sum_{t=1}^{\rho(h)} a_{ht} z^{t-1} p^{(t-1)}(\omega_h z + v)$$

cannot have more than $c' \sigma \log \sigma$ zeros in a disc $|z| \leq c''$, where c', c'' depend only on p . We are indebted for the above details to D. W. Masser (for a problem involving zeros of polynomials in several variables see his [13]).

To extend our method to a class of functions wider than that given by (44) is practical provided only that one can usefully estimate the determinants arising in lemma 2. This can certainly be done in the case

$$F(z) = \sum_{h=1}^m \sum_{t=1}^{\rho(h)} a_{ht} (\log z)^{t-1} z^{\alpha_h},$$

for details see van der Poorten [22]. A similar argument should allow one to deal with functions

$$\sum_{h=1}^{\sigma} b_h f_{\mu_h}(z),$$

where f_{μ} is given by (45); now lemma 5 allows one to consider rather surprising functions. There are further, rather isolated cases where one can deal with the determinants; for some examples, and further references see [21].

REFERENCES

- [1] BALKEMA A. A. and R. TIJDEMAN. Some estimates in the theory of exponential sums. *Acta Math. Acad. Sci. Hungar.* 24 (1973), pp. 115-133.
- [2] BROWNAWELL, D. Gel'fond's method for algebraic independence. *Trans. Amer. Math. Soc.* 205 (1975), pp. 1-26.
- [3] CUDNOVSKII, G. V. Algebraic independence of some values of the exponential function. *Mat. Zametki* 15 (1974), pp. 661-672 = *Math. Notes* 15 (1974), pp. 391-398.
- [4] CIJSOUW, P. L. On the simultaneous approximation of certain numbers. *Duke Math. J.* 42 (1975), pp. 249-257.
- [5] CIJSOUW P. and R. TIJDEMAN. An auxiliary result in the theory of transcendental numbers II. *Duke Math. J.* 42 (1975), pp. 239-247.
- [6] DICKSON, D. G. Asymptotic distribution of zeros of exponential sums. *Publ. Math. Debrecen* 11 (1964), pp. 295-300.
- [7] GEL'FOND, A. O. *Transcendental and algebraic numbers*. Dover New York, 1960.
- [8] HAYMAN, W. K. Differential inequalities and local valency. *Pacific J. of Maths.* 44 (1973), pp. 117-137.
- [9] KNUTH, D. E. *Surreal Numbers* (how two ex-students turned onto pure mathematics and found total happiness) Ad-Wess., 1975.
- [10] MAHLER, K. On a class of entire functions. *Acta Math. Acad. Sci. Hungar* 18 (1967) pp. 83-86.
- [11] MAKAI, E. The first main theorem of P. Turán. *Acta Math. Acad. Sci. Hungar* 10 (1959), pp. 405-411.
- [12] ——— On a minimum problem. *Ann. Univ. Sci. Budapes. Eötvös Sect. Math.* 3-4 (1960-61), pp. 177-182.
- [13] MASSER, D. W. *Elliptic functions and transcendence*. Lecture Notes in Math. 437 Springer Verlag.
- [14] MUIR, T. *A treatise on the theory of determinants*. Dover, New York, 1960.