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Each factor in the product on the right is the square root of an expression of the shape

$$(3)\qquad (S^{*4} - 2SR_h S^{*2} \cos\psi + S^2 R_h^2)/S^{*2}(S^2 - 2SR_h \cos\psi + R_h^2),$$

where $z_h = R_h e^{i\phi_h}$ and $\psi = \theta - \phi_h$. One sees that the turning points of (3) as a function of ψ occur when $\sin\psi = 0$ and that the minimal value of (3) is

$$(4)\qquad ((S^{*2} + SR_h)/S^*(S + R_h))^2$$

One easily confirms that (4) is minimal for $0 \leqslant R_h \leqslant R$ when $R_h = R$, whence we obtain

$$|G|_S \geqslant |F|_S \left(\frac{S^{*2} + SR}{S^*(S+R)}\right)^n,$$

and the assertion of the lemma follows.

The lemma is " best possible "; the function $F(z) = \left(\dfrac{S^*(R-z)}{S^{*2} - Rz}\right)^n$ being the extreme case. I am indebted to Michel Waldschmidt for mentioning the result of the lemma to me. The lemma improves upon a similar result obtainable via Jensen's theorem, (see, for example, Tijdeman [26], p. 3).

According to the above observations, our principal attention below is directed towards the finding of upper bounds for ratios of the shape (1). Although the principles of our techniques are not new, many of the details have been little more than folklore and are presented here explicitly for the first time.

2. A USEFUL IDENTITY

The following lemma is presented in somewhat exaggerated generality. Its implications will become clear when below we come to look at specific examples.

LEMMA 2. *Let* S^*, S *be real numbers satisfying* $S^* \geqslant S > 0$ *and let* G *be a function of the shape*

$$G(z) = \textstyle\sum_{k=1}^{\sigma} b_k g_k(z),$$

$b_1, \ldots, b_\sigma$ *complex constants, where* $g_1, \ldots g_\sigma$ *are functions holomorphic in some open set containing the disc* $|z - z_0| \leqslant S^*$. *Further let* $z_1, \ldots, z_\sigma$ *be points in the disc* $|z - z_0| < S$ *and let* $t_1, \ldots t_\sigma$ *be non-negative integers.*

Finally denote by Δ_{ji} *the cofactor of the typical element in the* $\sigma \times \sigma$ *determinant*

$$\Delta = | g_i^{(t_j)}(z_j) |_{1 \leq i,j \leq \sigma},$$

suppose that $\Delta \neq 0$, *and assume the notational conventions of the introduction above. Then for* w *such that* $| w - z_0 | = S^*$ *we have*

$$(5) \quad G(w) = \sum_{\lambda=1}^{\sigma} \sum_{k=1}^{\sigma} \left\{ \frac{1}{2\pi i} \int_{|\gamma - z_0| = S} \frac{\Delta_{\lambda,k}}{\Delta} g_k(w) \left(\frac{d}{dz}\right)^{t_\lambda} G(\gamma) \frac{d\gamma}{(\gamma - z)} \right\}_{z = z_\lambda}$$

and it follows that if G *does not vanish identically*

$$(6) \quad | G |_{S^*} / | G |_S \leqslant \sum_{\lambda=1}^{\sigma} \max \left| \sum_{k=1}^{\sigma} \frac{\Delta_{\lambda,k}}{\Delta} g_k(w) \right| \cdot S \frac{t_\lambda!}{(S - |z_\lambda - z_0|)^{t_\lambda + 1}}$$

Proof. By the residue theorem the right-hand side of (5) is

$$\sum_{\lambda=1}^{\sigma} \sum_{k=1}^{\sigma} \frac{\Delta_{\lambda,k}}{\Delta} g_k(w) G^{(t_\lambda)}(z_\lambda)$$

$$= \sum_{k=1}^{\sigma} \sum_{h=1}^{\sigma} b_h g_k(w) \sum_{\lambda=1}^{\sigma} g_h^{(t_\lambda)}(z_\lambda) \frac{\Delta_{\lambda,k}}{\Delta}$$

$$= \sum_{k=1}^{\sigma} \sum_{h=1}^{\sigma} b_h g_k(w) \delta_{hk}, \quad (\delta_{hk}, \text{ the Kronecker delta})$$

$$= G(w),$$

as was asserted. Having thus established the identity (5), we conclude that

$$| G |_{S^*} \leqslant \sum_{\lambda=1}^{\sigma} \max \left| \sum_{k=1}^{\sigma} \frac{\Delta_{\lambda,k}}{\Delta} g_k(w) \right| \left| \frac{1}{2\pi} \oint G(\gamma) \frac{t_\lambda!}{(\gamma - z_\lambda)^{t_\lambda + 1}} \right| |$$

and estimating the integral on the circle $| \gamma - z_0 | = S$, the assertion (6) is immediate.

We have stated the lemma in such generality as might be appropriate for the purposes of this note. The reader should observe that, moreover, the same idea can be used to obtain any combination

$$\sum_{k=1}^{\sigma} b_k h_k$$

on the left-hand side of an identity similar to (5); this is useful in isolating the coefficients b_k which is necessary when one is investigating the number of points in a disc at which the given function $G(z)$ may be small; see theorem 2 below for details. We remark that the identity (5) should be viewed as a (degenerate) case of the integral form of the Hermite interpolation formula.