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## 1. A BASIC LEMMA

One learns that the essential step in constructing an estimate for the number of zeros of a function in a given disc consists of obtaining an upper bound for a ratio

$$|F|_{S^*} / |F|_S , \tag{1}$$

where $S^* > S > 0$, and, if the given disc has centre $z_0$, then $|F|_R = \max_{|z-z_0|=R} |F(z)|$. We see that this is sufficient by virtue of the following lemma, (see Waldschmidt [36], p. 166, for a slightly weaker statement).

LEMMA 1. *Let* $S^*, S, R$ *be real numbers satisfying*

$$S^* > S > 0 \quad \text{and} \quad S^* \geqslant R > 0 .$$

*Let* $F$ *be a function holomorphic in some open set containing the disc* $|z - z_0| \leqslant S^*$. *If* $F$ *does not vanish identically in the disc* $|z - z_0| \leqslant S^*$ *then the number of zeros* $n(F, R, z_0)$ *of* $F$ *in the disc* $|z - z_0| \leqslant R$ *satisfies*

$$n(F, R, z_0) \operatorname{Log}\left(\frac{S^{*2} + SR}{S^*(S+R)}\right) \leqslant \operatorname{Log} \frac{|F|_{S^*}}{|F|_S} \tag{2}$$

*Proof.* There is no loss of generality in supposing for convenience that $z_0 = 0$. Suppose then that $F$ has zeros at $z_1, \ldots, z_n$ in the disc $|z| \leqslant R$ and write

$$G(z) = F(z) \prod_{h=1}^{n} \frac{S^{*2} - z\bar{z}_h}{S^*(z - z_h)} .$$

Then $G$ is holomorphic in an open set containing the disc $|z| \leqslant S^*$, a simple calculation confirms that

$$|G|_{S^*} = |F|_{S^*},$$

and, by the maximum-modulus principle,

$$|G|_S \leqslant |G|_{S^*}$$

However

$$|G|_S \geqslant |F|_S \prod_h \min \left| \frac{S^{*2} - Se^{i\theta} \bar{z}_h}{S^*(Se^{i\theta} - z_h)} \right|$$

Each factor in the product on the right is the square root of an expression of the shape

$$(3) \qquad (S^{*4} - 2SR_h S^{*2} \cos \psi + S^2 R_h^2)/S^{*2}(S^2 - 2SR_h \cos \psi + R_h^2),$$

where $z_h = R_h e^{i\phi_h}$ and $\psi = \theta - \phi_h$. One sees that the turning points of (3) as a function of $\psi$ occur when $\sin \psi = 0$ and that the minimal value of (3) is

$$(4) \qquad ((S^{*2} + SR_h)/S^*(S + R_h))^2$$

One easily confirms that (4) is minimal for $0 \leqslant R_h \leqslant R$ when $R_h = R$, whence we obtain

$$|G|_S \geqslant |F|_S \left(\frac{S^{*2} + SR}{S^*(S+R)}\right)^n,$$

and the assertion of the lemma follows.

The lemma is " best possible "; the function $F(z) = \left(\frac{S^*(R-z)}{S^{*2} - Rz}\right)^n$ being the extreme case. I am indebted to Michel Waldschmidt for mentioning the result of the lemma to me. The lemma improves upon a similar result obtainable via Jensen's theorem, (see, for example, Tijdeman [26], p. 3).

According to the above observations, our principal attention below is directed towards the finding of upper bounds for ratios of the shape (1). Although the principles of our techniques are not new, many of the details have been little more than folklore and are presented here explicitly for the first time.

## 2. A USEFUL IDENTITY

The following lemma is presented in somewhat exaggerated generality. Its implications will become clear when below we come to look at specific examples.

LEMMA 2. *Let* $S^*, S$ *be real numbers satisfying* $S^* \geqslant S > 0$ *and let* $G$ *be a function of the shape*

$$G(z) = \sum_{k=1}^{\sigma} b_k g_k(z),$$

$b_1, \ldots, b_\sigma$ *complex constants, where* $g_1, \ldots g_\sigma$ *are functions holomorphic in some open set containing the disc* $|z - z_0| \leqslant S^*$. *Further let* $z_1, \ldots, z_\sigma$ *be points in the disc* $|z - z_0| < S$ *and let* $t_1, \ldots t_\sigma$ *be non-negative integers.*