

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 23 (1977)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE NUMBER OF ZEROS OF FUNCTIONS
Autor: van der Poorten, A. J.
Kapitel: 1. A BASIC LEMMA
DOI: <https://doi.org/10.5169/seals-48918>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 09.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

1. A BASIC LEMMA

One learns that the essential step in constructing an estimate for the number of zeros of a function in a given disc consists of obtaining an upper bound for a ratio

$$(1) \quad |F|_{S^*} / |F|_S ,$$

where $S^* > S > 0$, and, if the given disc has centre z_0 , then $|F|_R = \max_{|z-z_0|=R} |F(z)|$. We see that this is sufficient by virtue of the following lemma, (see Waldschmidt [36], p. 166, for a slightly weaker statement).

LEMMA 1. *Let S^*, S, R be real numbers satisfying*

$$S^* > S > 0 \quad \text{and} \quad S^* \geq R > 0 .$$

Let F be a function holomorphic in some open set containing the disc $|z - z_0| \leq S^$. If F does not vanish identically in the disc $|z - z_0| \leq S^*$ then the number of zeros $n(F, R, z_0)$ of F in the disc $|z - z_0| \leq R$ satisfies*

$$(2) \quad n(F, R, z_0) \operatorname{Log} \left(\frac{S^{*2} + SR}{S^*(S+R)} \right) \leq \operatorname{Log} \frac{|F|_{S^*}}{|F|_S}$$

Proof. There is no loss of generality in supposing for convenience that $z_0 = 0$. Suppose then that F has zeros at z_1, \dots, z_n in the disc $|z| \leq R$ and write

$$G(z) = F(z) \prod_{h=1}^n \frac{S^{*2} - z\bar{z}_h}{S^*(z - z_h)} .$$

Then G is holomorphic in an open set containing the disc $|z| \leq S^*$, a simple calculation confirms that

$$|G|_{S^*} = |F|_{S^*},$$

and, by the maximum-modulus principle,

$$|G|_S \leq |G|_{S^*}$$

However

$$|G|_S \geq |F|_S \prod_h \min \left| \frac{S^{*2} - Se^{i\theta} \bar{z}_h}{S^*(Se^{i\theta} - z_h)} \right|$$

Each factor in the product on the right is the square root of an expression of the shape

$$(3) \quad (S^{*4} - 2SR_h S^{*2} \cos \psi + S^2 R_h^2)/S^{*2} (S^2 - 2SR_h \cos \psi + R_h^2),$$

where $z_h = R_h e^{i\phi_h}$ and $\psi = \theta - \phi_h$. One sees that the turning points of (3) as a function of ψ occur when $\sin \psi = 0$ and that the minimal value of (3) is

$$(4) \quad ((S^{*2} + SR_h)/S^* (S + R_h))^2$$

One easily confirms that (4) is minimal for $0 \leq R_h \leq R$ when $R_h = R$, whence we obtain

$$|G|_S \geq |F|_S \left(\frac{S^{*2} + SR}{S^* (S + R)} \right)^n,$$

and the assertion of the lemma follows.

The lemma is “best possible”; the function $F(z) = \left(\frac{S^* (R - z)}{S^{*2} - Rz} \right)^n$ being the extreme case. I am indebted to Michel Waldschmidt for mentioning the result of the lemma to me. The lemma improves upon a similar result obtainable via Jensen’s theorem, (see, for example, Tijdeman [26], p. 3).

According to the above observations, our principal attention below is directed towards the finding of upper bounds for ratios of the shape (1). Although the principles of our techniques are not new, many of the details have been little more than folklore and are presented here explicitly for the first time.

2. A USEFUL IDENTITY

The following lemma is presented in somewhat exaggerated generality. Its implications will become clear when below we come to look at specific examples.

LEMMA 2. *Let S^*, S be real numbers satisfying $S^* \geq S > 0$ and let G be a function of the shape*

$$G(z) = \sum_{k=1}^{\sigma} b_k g_k(z),$$

b_1, \dots, b_{σ} complex constants, where g_1, \dots, g_{σ} are functions holomorphic in some open set containing the disc $|z - z_0| \leq S^$. Further let z_1, \dots, z_{σ} be points in the disc $|z - z_0| < S$ and let t_1, \dots, t_{σ} be non-negative integers.*