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ON THE EVALUATION OF GAUSSIAN SUMS FOR NON-PRIMITIVE DIRICHLET CHARACTERS

by Henri JORIS

1. INTRODUCTION

Let χ be a Dirichlet character mod m. We denote the conductor of χ by f and the corresponding primitive character by ψ. For a natural number α, we have the Gaussian sum

$$\mathcal{G}(\alpha, \chi) = \sum_{k \bmod m} \chi(k) \, exp\left(2\pi i \frac{k\alpha}{m}\right).$$

We will write $\tau(\chi)$ for $\mathcal{G}(1, \chi)$. It is well known that

$$\mathcal{G}(\alpha, \chi) = \bar{\chi}(\alpha) \tau(\chi), \tag{1}$$

if χ is primitive, i.e. if $\chi = \psi$. For non primitive characters, (1) does not hold; according to H. Hasse [1], one has the following result:

THEOREM A. Let χ, m, ψ, f be as above. For $\alpha \in \mathbf{N}$ we put $\alpha_0 = \alpha/(\alpha, m)$, $m_0 = m/(\alpha, m)$. Then we have:

$$\left.\begin{aligned} \mathcal{G}(\alpha, \chi) &= 0 \qquad \text{if} \quad f \nmid m_0, \\ \mathcal{G}(\alpha, \chi) &= \frac{\varphi(m)}{\varphi(m_0)} \mu\left(\frac{m_0}{f}\right) \psi\left(\frac{m_0}{f}\right) \bar{\psi}(\alpha_0) \tau(\psi) \\ &\qquad \text{if} \quad f \mid m_0. \end{aligned}\right\} \tag{2}$$

Here, as throughout this note, φ and μ stand for the Euler totient and the Moebius function.

In [1], theorem A is proved in an elementary way, using several steps of reduction. See also [2].

In the present note we give another evaluation of $\mathcal{G}(\alpha, \chi)$, using the functional equation for Dirichlet L-series.

THEOREM B. Let m, χ, f, ψ be as above. We put

$$q = \prod_{\substack{p \mid m \\ p \nmid f}} p , \quad R = \frac{m}{fq}$$

where p denotes rational primes. Let the multiplicative function g be defined by

$$g(n) = \mu((n, q)) \varphi((n, q)) \bar{\psi}(n) .$$

Then we have:

$$\left.\begin{aligned} \mathscr{G}(\alpha, \chi) &= 0 \text{ if } R \nmid \alpha \\ \mathscr{G}(Rn, \chi) &= \mu(q) \psi(q) \tau(\psi) R g(n), \quad n = 1, 2, \ldots \end{aligned}\right\} \qquad (3)$$

2. PROOF OF THEOREM B

For a Dirichlet character χ mod m let the function $L(s, \chi)$ be given by

$$L(s, \chi) = \sum_{1}^{\infty} \chi(n) n^{-s} , \quad Re\, s > 1 .$$

The series defines an analytic function for Re $s > 1$, which can be extended to a meromorphic function on the whole complex plane, with at most one simple pole at $s = 1$. If χ is primitive, then $L(s, \chi)$ satisfies the equation

$$L(1-s, \chi) = m^{s-1} (2\pi)^{-s} \Gamma(s) \left(e^{-\frac{\pi i s}{2}} + \chi(-1) e^{\frac{\pi i s}{2}}\right) \tau(\chi) L(s, \bar{\chi}) . \qquad (4)$$

Because of (1), this can also be written, for Re $s > 1$, as

$$L(1-s, \chi) = m^{s-1} (2\pi)^{-s} \Gamma(s) \left(e^{-\frac{\pi i s}{2}} + \chi(-1) e^{\frac{\pi i s}{2}}\right) \sum_{1}^{\infty} \mathscr{G}(n, \chi) n^{-s} . \qquad (5)$$

Whereas (4) holds only for primitive characters, (5) turns out to be valid in the general case. In fact, a much more general formula is proved in [3], th. 6.1; if we put there $x = \alpha = 0$, $\alpha_n = \chi(n)$ and observe that $\mathscr{G}(-n, \chi) = \chi(-1) \mathscr{G}(n, \chi)$, we get (5) immediately. But also most of the classical (= non-adelic) proofs of (4) will give (5) after very small changes. The only use of the primitivity of χ in these proofs is that they replace $\mathscr{G}(n, \chi)$ by $\bar{\chi}(n) \tau(\chi)$ at some stage (See for instance [7]).

Now let χ, m, ψ, f be as in the theorem. We have, by the Euler-product,

$$\begin{aligned} L(1-s, \chi) &= L(1-s, \psi) \prod_{p \mid q} \left(1 - \frac{\psi(p)}{p^{1-s}}\right) \\ &= L(1-s, \psi) \mu(q) q^{s-1} \psi(q) \prod_{p \mid q} (1 - p \bar{\psi}(p) p^{-s}) \end{aligned} \qquad (6)$$