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# ON THE EVALUATION OF GAUSSIAN SUMS FOR NON-PRIMITIVE DIRICHLET CHARACTERS

by Henri JORIS

## 1. INTRODUCTION

Let  $\chi$  be a Dirichlet character mod  $m$ . We denote the conductor of  $\chi$  by  $f$  and the corresponding primitive character by  $\psi$ . For a natural number  $\alpha$ , we have the Gaussian sum

$$\mathcal{G}(\alpha, \chi) = \sum_{k \bmod m} \chi(k) \exp\left(2\pi i \frac{k\alpha}{m}\right).$$

We will write  $\tau(\chi)$  for  $\mathcal{G}(1, \chi)$ . It is well known that

$$\mathcal{G}(\alpha, \chi) = \bar{\chi}(\alpha) \tau(\chi), \tag{1}$$

if  $\chi$  is primitive, i.e. if  $\chi = \psi$ . For non primitive characters, (1) does not hold; according to H. Hasse [1], one has the following result:

**THEOREM A.** Let  $\chi, m, \psi, f$  be as above. For  $\alpha \in \mathbf{N}$  we put  $\alpha_0 = \alpha/(\alpha, m)$ ,  $m_0 = m/(\alpha, m)$ . Then we have:

$$\left. \begin{aligned} \mathcal{G}(\alpha, \chi) &= 0 && \text{if } f \nmid m_0, \\ \mathcal{G}(\alpha, \chi) &= \frac{\varphi(m)}{\varphi(m_0)} \mu\left(\frac{m_0}{f}\right) \psi\left(\frac{m_0}{f}\right) \bar{\psi}(\alpha_0) \tau(\psi) && \text{if } f \mid m_0. \end{aligned} \right\} \tag{2}$$

Here, as throughout this note,  $\varphi$  and  $\mu$  stand for the Euler totient and the Moebius function.

In [1], theorem A is proved in an elementary way, using several steps of reduction. See also [2].

In the present note we give another evaluation of  $\mathcal{G}(\alpha, \chi)$ , using the functional equation for Dirichlet  $L$ -series.

THEOREM B. Let  $m, \chi, f, \psi$  be as above. We put

$$q = \prod_{\substack{p|m \\ p \nmid f}} p, \quad R = \frac{m}{fq}$$

where  $p$  denotes rational primes. Let the multiplicative function  $g$  be defined by

$$g(n) = \mu((n, q)) \varphi((n, q)) \bar{\psi}(n).$$

Then we have:

$$\left. \begin{aligned} \mathcal{G}(\alpha, \chi) &= 0 \text{ if } R \nmid \alpha \\ \mathcal{G}(Rn, \chi) &= \mu(q) \psi(q) \tau(\psi) R g(n), \quad n = 1, 2, \dots \end{aligned} \right\} \quad (3)$$

## 2. PROOF OF THEOREM B

For a Dirichlet character  $\chi$  mod  $m$  let the function  $L(s, \chi)$  be given by

$$L(s, \chi) = \sum_1^{\infty} \chi(n) n^{-s}, \quad \text{Re } s > 1.$$

The series defines an analytic function for  $\text{Re } s > 1$ , which can be extended to a meromorphic function on the whole complex plane, with at most one simple pole at  $s = 1$ . If  $\chi$  is primitive, then  $L(s, \chi)$  satisfies the equation

$$L(1-s, \chi) = m^{s-1} (2\pi)^{-s} \Gamma(s) \left( e^{-\frac{\pi is}{2}} + \chi(-1) e^{\frac{\pi is}{2}} \right) \tau(\chi) L(s, \bar{\chi}). \quad (4)$$

Because of (1), this can also be written, for  $\text{Re } s > 1$ , as

$$L(1-s, \chi) = m^{s-1} (2\pi)^{-s} \Gamma(s) \left( e^{-\frac{\pi is}{2}} + \chi(-1) e^{\frac{\pi is}{2}} \right) \sum_1^{\infty} \mathcal{G}(n, \chi) n^{-s}. \quad (5)$$

Whereas (4) holds only for primitive characters, (5) turns out to be valid in the general case. In fact, a much more general formula is proved in [3], th. 6.1; if we put there  $x = \alpha = 0$ ,  $\alpha_n = \chi(n)$  and observe that  $\mathcal{G}(-n, \chi) = \chi(-1) \mathcal{G}(n, \chi)$ , we get (5) immediately. But also most of the classical (= non-adelic) proofs of (4) will give (5) after very small changes. The only use of the primitivity of  $\chi$  in these proofs is that they replace  $\mathcal{G}(n, \chi)$  by  $\bar{\chi}(n) \tau(\chi)$  at some stage (See for instance [7]).

Now let  $\chi, m, \psi, f$  be as in the theorem. We have, by the Euler-product,

$$\begin{aligned} L(1-s, \chi) &= L(1-s, \psi) \prod_{p|q} \left( 1 - \frac{\psi(p)}{p^{1-s}} \right) \\ &= L(1-s, \psi) \mu(q) q^{s-1} \psi(q) \prod_{p|q} (1 - p\bar{\psi}(p) p^{-s}) \end{aligned} \quad (6)$$