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Autor: Verner, Lawrence
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$e_i = 1 / \# E(S_i)$, where $\#$ denotes cardinality. We now define the “number of representations of T by the genus of S ” as

$$A(\text{genus}(S), T) = \frac{e_1 A(S_1, T) + \dots + e_h A(S_h, T)}{e_1 + \dots + e_h}.$$

Now S is a real symmetric matrix, and so we may view it as a point in \mathbf{R}^{n_1} , where $n_1 = n(n+1)/2$. Similarly, T is a point in \mathbf{R}^{m_1} . Let dt be the usual measure in \mathbf{R}^{m_1} , and let dx be the usual measure in the real vector space of $m \times n$ matrices. Given $\varepsilon > 0$, let B_ε denote the ε -neighborhood of T in \mathbf{R}^{m_1} , and let C_ε denote the set of $x \in M_{m \times n}(\mathbf{R})$ satisfying $S[x] \in B_\varepsilon$. Then B_ε and C_ε are open sets with compact closure, and the following limit is known to exist:

$$A_\infty(S, T) = \lim_{\varepsilon \rightarrow 0} \int_{C_\varepsilon} dx / \int_{B_\varepsilon} dt.$$

THEOREM (Siegel [4]). For $m - n \geq 3$,

$$(S) \quad A(\text{genus}(S), T) = A_\infty(S, T) \lim_q \frac{A_q(S, T)}{q^{mn - (n+1)n/2}}.$$

4. DERIVATION OF SIEGEL'S THEOREM

Let $G = \{g \in SL(m) : S[g] = S\}$, and let $X = \{x \in M_{m \times n} : S[x] = T\}$. If $m \geq 4$, both $G_{\mathbf{C}}$ and $G_{\mathbf{R}}$ have fundamental groups of order 2. Condition (c) of § 2 is the classical Witt theorem for (G, X) . We assume that $X_{\mathbf{Q}}$ is nonempty.

We will show that (A) implies (S). This reduces Siegel's theorem to the computation of the Tamagawa number $\tau(G)$.

Let $\Phi_\infty =$ the constant function 1 on $X_{\mathbf{R}}$, and let $\Phi_p =$ the characteristic function of $X_{\mathbf{Z}_p}$ in $X_{\mathbf{Q}_p}$. Then $\Phi = \Phi_\infty \cdot \prod \Phi_p$ is the characteristic function of $X_{S_\infty} = X_{\mathbf{R}} \cdot \prod X_{\mathbf{Z}_p}$ in X_A . Because of the positive definiteness of S , Φ has compact support.

Consider the right-hand side of formula (S). Siegel has shown that there exists an algebraic gauge form dx on X such that $A_\infty(S, T) = \int_{X_{\mathbf{R}}} dx_\infty$, and

$$\lim_q \frac{A_q(S, T)}{q^{mn - (n+1)n/2}} = \prod_p \int_{X_{\mathbf{Z}_p}} dx_p,$$

where dx and dx_p are the positive measures induced on $X_{\mathbf{R}}$ and $X_{\mathbf{Q}_p}$ by dx .

It remains to identify the left-hand sides of (A) and (S). First we analyze the denominator of the left-hand side of (A). Since $\tau(G) = \tau(G_\xi) = 2$ ([5]), this denominator is $\int_{G_A/G_Q} dg$. Now G_A admits a double coset decomposition

$$G_A = G_{S_\infty} \sigma_1 G_Q \dots G_{S_\infty} \sigma_h G_Q.$$

Then, following Tamagawa [5].

$$\begin{aligned} \int_{G_A/G_Q} dg &= \sum_1^h \int_{(G_{S_\infty} \sigma_i G_Q)/G_Q} dg = \\ &= \sum_1^h \int_{(\sigma_i^{-1} G_{S_\infty} \sigma_i G_Q)/G_Q} dg \\ &= \sum_1^h \int_{(\sigma_i^{-1} G_{S_\infty} \sigma_i)/G(\sigma_i)} dg, \end{aligned}$$

where $G(\sigma_i) = \sigma_i^{-1} G_{S_\infty} \sigma_i \cap G_Q$. This reduces to

$$\frac{\sum_1^h \int_{G_{S_\infty}} dg}{\# G(\sigma_i)} = \int_{G_{S_\infty}} dg \cdot \sum_1^h e_i.$$

A similar reduction applies to the numerator. First observe that for our choice of Φ ,

$$\begin{aligned} \sum_{x \in X_Q} \Phi(gx) &= \#(X_Q \cap g^{-1} X_{S_\infty}) \\ &= \#(gX_Q \cap X_{S_\infty}). \end{aligned}$$

Then

$$\begin{aligned} \int_{G_A/G_Q} \sum_{x \in X_Q} \Phi(gx) dg &= \int_{G_A/G_Q} \#(gX_Q \cap X_{S_\infty}) dg \\ &= \sum_1^h \int_{(G_{S_\infty} \sigma_i G_Q)/G_Q} \#(gX_Q \cap X_{S_\infty}) dg = \\ &= \sum_1^h \int_{(\sigma_i^{-1} G_{S_\infty} \sigma_i G_Q)/G_Q} \#(\sigma_i gX_Q \cap X_{S_\infty}) dg \\ &= \sum_1^h \int_{(\sigma_i^{-1} G_{S_\infty} \sigma_i)/G(\sigma_i)} \#(\sigma_i gX_Q \cap X_{S_\infty}) dg \\ &= \sum_1^h e_i \int_{G_{S_\infty}} \#(g\sigma_i X_Q \cap X_{S_\infty}) dg \\ &= \sum_1^h e_i \int_{G_{S_\infty}} \#(\sigma_i X_Q \cap g^{-1} X_{S_\infty}) dg \end{aligned}$$

$$\begin{aligned}
 &= \sum_1^h e_i \int_{G_{S_\infty}} \#(\sigma_i X_{\mathbf{Q}} \cap X_{S_\infty}) dg \\
 &= \int_{G_{S_\infty}} dg \cdot \sum_1^h e_i \#(\sigma_i X_{\mathbf{Q}} \cap X_{S_\infty}).
 \end{aligned}$$

The left-hand side of (A) therefore becomes

$$\frac{\int_{G_{S_\infty}} dg \cdot \sum_1^h e_i \#(\sigma_i X_{\mathbf{Q}} \cap X_{S_\infty})}{\int_{G_{S_\infty}} dg \cdot \sum_1^h e_i} = \frac{\sum_1^h e_i \#(\sigma_i X_{\mathbf{Q}} \cap X_{S_\infty})}{\sum_1^h e_i}.$$

The following result completes the identification of the left-hand side of (A) with A (genus $(S), T$).

PROPOSITION. $A(S_i, T) = \#(\sigma_i X_{\mathbf{Q}} \cap X_{S_\infty})$.

Before giving the proof, we reinterpret the matrices S_1, \dots, S_h . G_A acts on the set of \mathbf{Z} -lattices in \mathbf{Q}^n as follows: for $\sigma \in G_A$ and a lattice L , $\sigma * L$ is the unique lattice satisfying

$$(\sigma * L) \otimes \mathbf{Z}_p = \sigma_p(L \otimes \mathbf{Z}_p),$$

for all p .

The matrix S defines a quadratic form on \mathbf{Q}^n by $q(x) = S[x]$. Consider the lattices $\sigma_1 * \mathbf{Z}^n, \dots, \sigma_h * \mathbf{Z}^n$. In each lattice $\sigma_i * \mathbf{Z}^n$ choose a \mathbf{Z} -basis, and let S_i be the matrix of q with respect to this basis. Then S_1, \dots, S_h form a complete set of representatives of the h classes in genus (S) (see [7]).

Decomposing $(SL_m)_A = (SL_m)_{S_\infty} (SL_m)_{\mathbf{Q}}$ we see that each $\sigma_i \in G_A$ can be written $\sigma_i = u_i a_i$, where $u_i \in (SL_m)_{S_\infty}$, $a_i \in (SL_m)_{\mathbf{Q}}$. Then

$$\begin{aligned}
 \sigma_i^{-1} * \mathbf{Z}^m &= a_i^{-1} u_i^{-1} * \mathbf{Z}^m = a_i^{-1} * (u_i^{-1} * \mathbf{Z}^m) \\
 &= a_i^{-1} * \mathbf{Z}^m = a_i^{-1} \mathbf{Z}^m.
 \end{aligned}$$

Let w_1, \dots, w_m be the standard \mathbf{Z} -basis of \mathbf{Z}^m ; then $a_i^{-1} w_1, \dots, a_i^{-1} w_m$ is a \mathbf{Z} -basis of $\sigma_i^{-1} * \mathbf{Z}^m$. The matrix of q with respect to this basis is

$$S_i = S[a_i^{-1}] = S[\sigma_i][a_i^{-1}] = S[\sigma_i a_i^{-1}] = S[u_i].$$

LEMMA. Let $X_i = \{x \in M_{m \times n} : S_i[x] = T\}$. Then

- (1) $(X_i)_{\mathbf{Q}} = a_i X_{\mathbf{Q}}$,
- (2) $(X_i)_{S_\infty} = u_i^{-1} X_{S_\infty}$.

Proof of (1): Let $x \in X_A$. $a_i x \in a_i X_{\mathbf{Q}} \Leftrightarrow x$ is \mathbf{Q} -rational and $S[x] = T \Leftrightarrow a_i x$ is \mathbf{Q} -rational and $S_i[a_i x] = T \Leftrightarrow a_i x \in (X_i)_{\mathbf{Q}}$.

The proof of (2) is similar.

Now we prove the proposition.

$$\begin{aligned} A(S_i, T) &= \#(X_i)_{\mathbf{Z}} = \#((X_i)_{\mathbf{Q}} \cap (X_i)_{S_{\infty}}) = \#(a_i X_{\mathbf{Q}} \cap u_i^{-1} X_{S_{\infty}}) \\ &= \#(u_i a_i X_{\mathbf{Q}} \cap X_{S_{\infty}}) = \#(\sigma_i X_{\mathbf{Q}} \cap X_{S_{\infty}}). \end{aligned}$$

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Lawrence Verner

Department of Mathematics
Baruch College, CUNY
New York, New York 10010