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of a compact set K . The right-hand side of the above formula reduces to the volume of K , while the left-hand side gives the mean value of

$$\text{card } (L - \{0\} \cap K),$$

as L varies over all \mathbf{Z} -lattices in \mathbf{R}^n with volume 1.

We turn now to the adelic mean value formula. Let G be a linear algebraic group defined over \mathbf{Q} , and let X be an algebraic homogeneous space for G , defined over \mathbf{Q} . For $\xi \in X$, let $G_\xi = \{g \in G: g\xi = \xi\}$. We assume that

- a) X has at least one rational point
- b) for any $\xi \in X_{\mathbf{C}}$, both $G_{\mathbf{C}}$ and $(G_\xi)_{\mathbf{C}}$ have finite fundamental groups
- c) for any extension field K of \mathbf{Q} , G_K acts transitively on X_K .

We then have the following result.

THEOREM (Ono [2]). There are canonical measures on the adèle spaces G_A and X_A such that, given any continuous function Φ on X_A with compact support,

$$(A) \quad \frac{\int_{G_A/G_{\mathbf{Q}}} \sum_{x \in X_{\mathbf{Q}}} \Phi(gx) dg}{\tau(G_\xi)} = \int_{X_A} \Phi(x) dx,$$

where ξ is any element of $X_{\mathbf{Q}}$, and $\tau(G_\xi)$ = the invariant measure of $(G_\xi)_A / (G_\xi)_{\mathbf{Q}}$. The analogy to the previous mean value theorem is clear in the cases when $\tau(G) = \tau(G_\xi)$.

3. FORMULATION OF SIEGEL'S THEOREM

Let S and T be square matrices with integral entries of size m and n , respectively. We assume that both are positive definite. For any matrix x , denote $S[x] = {}^t x S x$ (when defined). Let $A(S, T)$ = the number of integral $m \times n$ matrices x such that $S[x] = T$. For each positive integer q , let $A_q(S, T)$ = the number of integral $m \times n$ matrices x , mod q , such that $S[x] \equiv T \pmod{q}$.

A positive definite integral matrix S' is said to be in the same class as S if $S' = S[U]$, for some $U \in SL(m, \mathbf{Z})$. S' is in the same genus as S if for each q , there exists $U \in SL(m, \mathbf{Z})$ such that $S' \equiv S[U] \pmod{q}$. Let S_1, \dots, S_h be the representatives of the classes in genus (S) . Let $E(S_i)$ = the finite group consisting of all $U \in SL(m, \mathbf{Z})$ such that $S_i[U] = S_i$, and put

$e_i = 1 / \# E(S_i)$, where $\#$ denotes cardinality. We now define the "number of representations of T by the genus of S " as

$$A(\text{genus}(S), T) = \frac{e_1 A(S_1, T) + \dots + e_h A(S_h, T)}{e_1 + \dots + e_h}.$$

Now S is a real symmetric matrix, and so we may view it as a point in \mathbf{R}^{n_1} , where $n_1 = n(n+1)/2$. Similarly, T is a point in \mathbf{R}^{m_1} . Let dt be the usual measure in \mathbf{R}^{m_1} , and let dx be the usual measure in the real vector space of $m \times n$ matrices. Given $\varepsilon > 0$, let B_ε denote the ε -neighborhood of T in \mathbf{R}^{m_1} , and let C_ε denote the set of $x \in M_{m \times n}(\mathbf{R})$ satisfying $S[x] \in B_\varepsilon$. Then B_ε and C_ε are open sets with compact closure, and the following limit is known to exist:

$$A_\infty(S, T) = \lim_{\varepsilon \rightarrow 0} \int_{C_\varepsilon} dx / \int_{B_\varepsilon} dt.$$

THEOREM (Siegel [4]). For $m - n \geq 3$,

$$(S) \quad A(\text{genus}(S), T) = A_\infty(S, T) \lim_q \frac{A_q(S, T)}{q^{mn - (n+1)n/2}}.$$

4. DERIVATION OF SIEGEL'S THEOREM

Let $G = \{g \in SL(m): S[g] = S\}$, and let $X = \{x \in M_{m \times n}: S[x] = T\}$. If $m \geq 4$, both $G_{\mathbf{C}}$ and $G_{\mathbf{R}}$ have fundamental groups of order 2. Condition (c) of § 2 is the classical Witt theorem for (G, X) . We assume that $X_{\mathbf{Q}}$ is nonempty.

We will show that (A) implies (S). This reduces Siegel's theorem to the computation of the Tamagawa number $\tau(G)$.

Let $\Phi_\infty =$ the constant function 1 on $X_{\mathbf{R}}$, and let $\Phi_p =$ the characteristic function of $X_{\mathbf{Z}_p}$ in $X_{\mathbf{Q}_p}$. Then $\Phi = \Phi_\infty \cdot \prod \Phi_p$ is the characteristic function of $X_{S_\infty} = X_{\mathbf{R}} \cdot \prod X_{\mathbf{Z}_p}$ in X_A . Because of the positive definiteness of S , Φ has compact support.

Consider the right-hand side of formula (S). Siegel has shown that there exists an algebraic gauge form dx on X such that $A_\infty(S, T) = \int_{X_{\mathbf{R}}} dx_\infty$, and

$$\lim_q \frac{A_q(S, T)}{q^{mn - (n+1)n/2}} = \prod_p \int_{X_{\mathbf{Z}_p}} dx_p,$$

where dx and dx_p are the positive measures induced on $X_{\mathbf{R}}$ and $X_{\mathbf{Q}_p}$ by dx .