

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 23 (1977)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON REPRESENTATION OF FUNCTIONS BY MEANS OF SUPERPOSITIONS AND RELATED TOPICS  
**Autor:** Vitushkin, A. G.  
**Kapitel:** §5. Instability of the representation of functions as superpositions of smooth functions  
**DOI:** <https://doi.org/10.5169/seals-48931>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

§ 5. *Instability of the representation of functions  
as superpositions of smooth functions*

Let  $A$  be a set of functions of  $n$  variables and  $B$  a set of functions of  $k$  variables ( $k < n$ ). Suppose that a function  $F(x_1, \dots, x_n) \in A$  is in a region  $G_n$  of the space  $x_1, x_2, \dots, x_n$  an  $s$ -fold superposition, generated by a system of functions  $\{f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  of  $B$ .

We say that this superposition is  $(A, B)$ -stable in  $G_n$  if every function  $\tilde{F}(x_1, \dots, x_n) \in A$  can be represented in  $G_n$  as the  $s$ -fold superposition of the same form of functions  $\{\tilde{f}_{\beta_1, \dots, \beta_\alpha}(t_1, t_2, \dots, t_k)\}$  of  $B$  such that

$$\begin{aligned} \max_{\alpha; \beta_1, \dots, \beta_\alpha} \sup_t & \left| \tilde{f}_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k) - f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k) \right| \\ & \leq \lambda \sup_{x \in G_n} \left| \tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n) \right|, \end{aligned}$$

where  $\lambda$  is a constant not depending either on  $\tilde{F}$  or on the  $\{\tilde{f}_{\beta_1, \dots, \beta_\alpha}\}$ .

We denote by  $C_{\omega(\delta)}^{(1)}$  the space of all continuously differentiable functions of  $k$  variables whose partial derivatives have modulus of continuity  $\omega(\delta)$  ( $\omega(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ ).

**THEOREM 5.5.1.** *Suppose that each function  $F(x_1, \dots, x_n) \in A$  is in some region  $D_n$  of the space  $x_1, \dots, x_n$  a superposition of order  $s$  of functions of  $k$  variables  $\{f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  belonging to  $C_{\omega(\delta)}^{(1)}$  ( $k < n$ ). If for any subregion  $G_n \subset D_n$  the functional "dimension" of  $A$  at  $F(x_1, \dots, x_n) \in A$  is greater than  $k$ , then the function  $F(x_1, \dots, x_n)$  cannot be an  $(A, C_{\omega(\delta)}^{(1)})$ -stable superposition in any such region  $G \subset D_n$ .*

*Proof.* Assume the contrary, that is, in a region  $G_n \subset D_n$  the function  $F(x_1, \dots, x_n) \in A$  is an  $(A, C_{\omega(\delta)}^{(1)})$ -stable  $s$ -fold superposition of functions  $\{f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  of  $C_{\omega(\delta)}^{(1)}$ . Then any function  $\tilde{F}(x_1, \dots, x_n) \in A$  can be represented as the superposition of the same form of functions  $\{\tilde{f}_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  of  $C_{\omega(\delta)}^{(1)}$  such that

$$\max_{\alpha; \beta_1, \dots, \beta_\alpha} \sup_t \left| \varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k) \right| \leq \lambda \sup_{x \in G_n} \left| \tilde{F} - F \right|,$$

where  $\varphi_{\beta_1, \dots, \beta_\alpha} = \tilde{f}_{\beta_1, \dots, \beta_\alpha} - f_{\beta_1, \dots, \beta_\alpha}$ . By Lemma 5.4.2 we have (for definiteness,  $k > 1$ )

$$\begin{aligned} \tilde{F} - F &= \sum_{\alpha; \beta_1, \dots, \beta_\alpha} p_{\beta_1, \dots, \beta_\alpha}(x_1, \dots, x_n) \\ &\times \varphi_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}(x_1, \dots, x_n), \dots, q_{\beta_1, \dots, \beta_\alpha, k}(x_1, \dots, x_n)) + R(x_1, \dots, x_n), \end{aligned}$$

where  $|R(x_1, \dots, x_n)| \leq \gamma(\varepsilon)\varepsilon$ ,  $\gamma(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , and

$$\begin{aligned} \varepsilon &= \max_{\alpha; \beta_1, \dots, \beta_\alpha} \sup_t |\varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)| \\ &\leq \lambda \sup_{x \in G_n} |\tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n)|. \end{aligned}$$

That  $\gamma(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  follows from the fact that as  $\varepsilon \rightarrow 0$  the quantity

$$\varepsilon' = \max_{\alpha; \beta_1, \dots, \beta_\alpha} \sum_{i=1}^k \sup \left| \frac{\partial \varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)}{\partial t_i} \right| \rightarrow 0,$$

provided only that the modulus of continuity of the partial derivatives of the functions  $\{\varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  is fixed. By 5.1.10 it follows that  $r(A, F) \leq k$  in some subregion  $G_n \subset D_n$ . So we have obtained a contradiction to the assumption that  $r(A, F) > k$  in any subregion  $G_n \subset D_n$  and this proves the theorem.

#### REFERENCES

- [1] HILBERT, D. Mathematische Probleme. *Nachr. Akad. Wiss. Göttingen* (1900), 253-297; *Gesammelte Abhandlungen*, Bd. 3 (1935), 290-329.
- [2] OSTROWSKI, A. Über Dirichletsche Reihen und algebraische Differentialgleichungen. *Math. Z.* 8 (1920), 241-298.
- [3] HILBERT, D. Über die Gleichung neunten Grades. *Math. Ann.* 97 (1927), 243-250; *Gesammelte Abhandlungen*, Bd. 2 (1933), 393-400.
- [4] VITUSHKIN, A. G. On Hilbert's thirteenth problem. *Dokl. Akad. Nauk SSSR* 95 (1954), 701-704.
- [5] BIEBERBACH, L. Bemerkung zum dreizehnten Hilbertschen Problem. *J. Reine Angew. Math.* 165 (1931), 89-92.
- [6] ——— Einfluss von Hilberts Pariser Vortrag über „Mathematische Probleme“. *Naturwissenschaften* 51 (1930), 1101-1111.
- [7] KOLMOGOROV, A. N. On the representation of continuous functions of several variables by superpositions of continuous functions of fewer variables. *Dokl. Akad. Nauk SSSR* 108 (1956), 179-182. *Amer. Math. Soc. Transl.* (2) 17 (1961), 369-373.
- [8] ARNOL'D, V. I. On functions of three variables. *Dokl. Akad. Nauk SSSR* 114 (1957), 679-681.
- [9] KOLMOGOROV, A. N. On the representation of continuous functions of several variables by superpositions of continuous functions of one variable and addition. *Dokl. Akad. Nauk SSSR* 114 (1957), 953-956. *Amer. Math. Soc. Transl.* (2) 28 (1963), 55-59.