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Artikel: ON REPRESENTATION OF FUNCTIONS BY MEANS OF SUPERPOSITIONS AND RELATED TOPICS

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§ 5. *Instability of the representation of functions as superpositions of smooth functions*

Let A be a set of functions of n variables and B a set of functions of k variables ($k<n$). Suppose that a function $F(x_1, ..., x_n) \in A$ is in a region G_n of the space $x_1, x_2, ..., x_n$ an s-fold superposition, generated by a system of functions $\{f_{\beta_1,...,\beta_\alpha}(t_1, ..., t_k)\}$ of B.

We say that this superposition is (A, B)-stable in G_n if every function $\tilde{F}(x_1, ..., x_n) \in A$ can be represented in G_n as the s-fold superposition of the same form of functions $\{\tilde{f}_{\beta_1,...,\beta_\alpha}(t_1, t_2, ..., t_k)\}$ of B such that

$$\max_{\alpha;\,\beta_1,\dots,\beta_\alpha} \sup_t \left|\tilde{f}_{\beta_1,...,\beta_\alpha}(t_1, \dots, t_k) - f_{\beta_1,...,\beta_\alpha}(t_1, \dots, t_k)\right|$$
$$\leqslant \lambda \sup_{x \in G_n} \left|\tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n)\right|,$$

where λ is a constant not depending either on $\tilde{F}$ or on the $\{\tilde{f}_{\beta_1,...,\beta_\alpha}\}$.

We denote by $C^{(1)}_{\omega(\delta)}$ the space of all continuously differentiable functions of k variables whose partial derivatives have modulus of continuity $\omega(\delta)$ ($\omega(\delta) \to 0$ as $\delta \to 0$).

THEOREM 5.5.1. *Suppose that each function $F(x_1, ..., x_n) \in A$ is in some region D_n of the space $x_1, ..., x_n$ a superposition of order s of functions of k variables $\{f_{\beta_1,...,\beta_\alpha}(t_1, ..., t_k)\}$ belonging to $C^{(1)}_{\omega(\delta)}$ ($k<n$). If for any subregion $G_n \subset D_n$ the functional "dimension" of A at $F(x_1, ..., x_n) \in A$ is greater than k, then the function $F(x_1, ..., x_n)$ cannot be an $(A, C^{(1)}_{\omega(\delta)})$-stable superposition in any such region $G \subset D_n$.*

Proof. Assume the contrary, that is, in a region $G_n \subset D_n$ the function $F(x_1, ..., x_n) \in A$ is an $(A, C^{(1)}_{\omega(\delta)})$-stable s-fold superposition of functions $\{f_{\beta_1,...,\beta_\alpha}(t_1, ..., t_k)\}$ of $C^{(1)}_{\omega(\delta)}$. Then any function $\tilde{F}(x_1, ..., x_n) \in A$ can be represented as the superposition of the same form of functions $\{\tilde{f}_{\beta_1,...,\beta_\alpha}(t_1, ..., t_k)\}$ of $C^{(1)}_{\omega(\delta)}$ such that

$$\max_{\alpha;\,\beta_1,\dots,\beta_\alpha} \sup_t \left|\varphi_{\beta_1,...,\beta_\alpha}(t_1, \dots, t_k)\right| \leqslant \lambda \sup_{x \in G_n} \left|\tilde{F} - F\right|,$$

where $\varphi_{\beta_1,...,\beta_\alpha} = \tilde{f}_{\beta_1,...,\beta_\alpha} - f_{\beta_1,...,\beta_\alpha}$. By Lemma 5.4.2 we have (for definiteness, $k > 1$)

$$\tilde{F} - F = \sum_{\alpha;\,\beta_1,\dots,\beta_\alpha} p_{\beta_1,\dots,\beta_\alpha}(x_1, \dots, x_n)$$

$$\times\ \varphi_{\beta_1,\dots,\beta_\alpha}(q_{\beta_1,\dots,\beta_\alpha,1}(x_1, \dots, x_n), \dots, q_{\beta_1,\dots,\beta_\alpha,k}(x_1, \dots, x_n)) + R(x_1, \dots, x_n),$$

where $|R(x_1, \dots, x_n)| \leqslant \gamma(\varepsilon)\,\varepsilon$, $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$, and

$$\varepsilon = \max_{\alpha;\,\beta_1,\dots,\beta_\alpha} \sup_t |\varphi_{\beta_1,\dots,\beta_\alpha}(t_1, \dots, t_k)|$$

$$\leqslant \lambda \sup_{x \in G_n} |\tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n)|.$$

That $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$ follows from the fact that as $\varepsilon \to 0$ the quantity

$$\varepsilon' = \max_{\alpha;\,\beta_1,\dots,\beta_\alpha} \sum_{i=1}^{k} \sup \left| \frac{\partial \varphi_{\beta_1,\dots,\beta_\alpha}(t_1, \dots, t_k)}{\partial t_i} \right| \to 0,$$

provided only that the modulus of continuity of the partial derivatives of the functions $\{\varphi_{\beta_1,\dots,\beta_\alpha}(t_1, \dots, t_k)\}$ is fixed. By 5.1.10 it follows that $r(A, F) \leqslant k$ in some subregion $G_n \subset D_n$. So we have obtained a contradiction to the assumption that $r(A, F) > k$ in any subregion $G_n \subset D_n$ and this proves the theorem.

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