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§ 5. *Instability of the representation of functions
as superpositions of smooth functions*

Let A be a set of functions of n variables and B a set of functions of k variables ($k < n$). Suppose that a function $F(x_1, \dots, x_n) \in A$ is in a region G_n of the space x_1, x_2, \dots, x_n an s -fold superposition, generated by a system of functions $\{f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$ of B .

We say that this superposition is (A, B) -stable in G_n if every function $\tilde{F}(x_1, \dots, x_n) \in A$ can be represented in G_n as the s -fold superposition of the same form of functions $\{\tilde{f}_{\beta_1, \dots, \beta_\alpha}(t_1, t_2, \dots, t_k)\}$ of B such that

$$\begin{aligned} \max_{\alpha; \beta_1, \dots, \beta_\alpha} \sup_t & \left| \tilde{f}_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k) - f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k) \right| \\ & \leq \lambda \sup_{x \in G_n} \left| \tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n) \right|, \end{aligned}$$

where λ is a constant not depending either on \tilde{F} or on the $\{\tilde{f}_{\beta_1, \dots, \beta_\alpha}\}$.

We denote by $C_{\omega(\delta)}^{(1)}$ the space of all continuously differentiable functions of k variables whose partial derivatives have modulus of continuity $\omega(\delta)$ ($\omega(\delta) \rightarrow 0$ as $\delta \rightarrow 0$).

THEOREM 5.5.1. *Suppose that each function $F(x_1, \dots, x_n) \in A$ is in some region D_n of the space x_1, \dots, x_n a superposition of order s of functions of k variables $\{f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$ belonging to $C_{\omega(\delta)}^{(1)}$ ($k < n$). If for any subregion $G_n \subset D_n$ the functional "dimension" of A at $F(x_1, \dots, x_n) \in A$ is greater than k , then the function $F(x_1, \dots, x_n)$ cannot be an $(A, C_{\omega(\delta)}^{(1)})$ -stable superposition in any such region $G \subset D_n$.*

Proof. Assume the contrary, that is, in a region $G_n \subset D_n$ the function $F(x_1, \dots, x_n) \in A$ is an $(A, C_{\omega(\delta)}^{(1)})$ -stable s -fold superposition of functions $\{f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$ of $C_{\omega(\delta)}^{(1)}$. Then any function $\tilde{F}(x_1, \dots, x_n) \in A$ can be represented as the superposition of the same form of functions $\{\tilde{f}_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$ of $C_{\omega(\delta)}^{(1)}$ such that

$$\max_{\alpha; \beta_1, \dots, \beta_\alpha} \sup_t \left| \varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k) \right| \leq \lambda \sup_{x \in G_n} \left| \tilde{F} - F \right|,$$

where $\varphi_{\beta_1, \dots, \beta_\alpha} = \tilde{f}_{\beta_1, \dots, \beta_\alpha} - f_{\beta_1, \dots, \beta_\alpha}$. By Lemma 5.4.2 we have (for definiteness, $k > 1$)

$$\begin{aligned} \tilde{F} - F &= \sum_{\alpha; \beta_1, \dots, \beta_\alpha} p_{\beta_1, \dots, \beta_\alpha}(x_1, \dots, x_n) \\ &\times \varphi_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}(x_1, \dots, x_n), \dots, q_{\beta_1, \dots, \beta_\alpha, k}(x_1, \dots, x_n)) + R(x_1, \dots, x_n), \end{aligned}$$

where $|R(x_1, \dots, x_n)| \leq \gamma(\varepsilon)\varepsilon$, $\gamma(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, and

$$\begin{aligned} \varepsilon &= \max_{\alpha; \beta_1, \dots, \beta_\alpha} \sup_t |\varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)| \\ &\leq \lambda \sup_{x \in G_n} |\tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n)|. \end{aligned}$$

That $\gamma(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ follows from the fact that as $\varepsilon \rightarrow 0$ the quantity

$$\varepsilon' = \max_{\alpha; \beta_1, \dots, \beta_\alpha} \sum_{i=1}^k \sup \left| \frac{\partial \varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)}{\partial t_i} \right| \rightarrow 0,$$

provided only that the modulus of continuity of the partial derivatives of the functions $\{\varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$ is fixed. By 5.1.10 it follows that $r(A, F) \leq k$ in some subregion $G_n \subset D_n$. So we have obtained a contradiction to the assumption that $r(A, F) > k$ in any subregion $G_n \subset D_n$ and this proves the theorem.

REFERENCES

- [1] HILBERT, D. Mathematische Probleme. *Nachr. Akad. Wiss. Göttingen* (1900), 253-297; *Gesammelte Abhandlungen*, Bd. 3 (1935), 290-329.
- [2] OSTROWSKI, A. Über Dirichletsche Reihen und algebraische Differentialgleichungen. *Math. Z.* 8 (1920), 241-298.
- [3] HILBERT, D. Über die Gleichung neunten Grades. *Math. Ann.* 97 (1927), 243-250; *Gesammelte Abhandlungen*, Bd. 2 (1933), 393-400.
- [4] VITUSHKIN, A. G. On Hilbert's thirteenth problem. *Dokl. Akad. Nauk SSSR* 95 (1954), 701-704.
- [5] BIEBERBACH, L. Bemerkung zum dreizehnten Hilbertschen Problem. *J. Reine Angew. Math.* 165 (1931), 89-92.
- [6] ——— Einfluss von Hilberts Pariser Vortrag über „Mathematische Probleme“. *Naturwissenschaften* 51 (1930), 1101-1111.
- [7] KOLMOGOROV, A. N. On the representation of continuous functions of several variables by superpositions of continuous functions of fewer variables. *Dokl. Akad. Nauk SSSR* 108 (1956), 179-182. *Amer. Math. Soc. Transl.* (2) 17 (1961), 369-373.
- [8] ARNOL'D, V. I. On functions of three variables. *Dokl. Akad. Nauk SSSR* 114 (1957), 679-681.
- [9] KOLMOGOROV, A. N. On the representation of continuous functions of several variables by superpositions of continuous functions of one variable and addition. *Dokl. Akad. Nauk SSSR* 114 (1957), 953-956. *Amer. Math. Soc. Transl.* (2) 28 (1963), 55-59.