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§ 5. *Instability of the representation of functions  
as superpositions of smooth functions*

Let  $A$  be a set of functions of  $n$  variables and  $B$  a set of functions of  $k$  variables ( $k < n$ ). Suppose that a function  $F(x_1, \dots, x_n) \in A$  is in a region  $G_n$  of the space  $x_1, x_2, \dots, x_n$  an  $s$ -fold superposition, generated by a system of functions  $\{f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  of  $B$ .

We say that this superposition is  $(A, B)$ -stable in  $G_n$  if every function  $\tilde{F}(x_1, \dots, x_n) \in A$  can be represented in  $G_n$  as the  $s$ -fold superposition of the same form of functions  $\{\tilde{f}_{\beta_1, \dots, \beta_\alpha}(t_1, t_2, \dots, t_k)\}$  of  $B$  such that

$$\begin{aligned} & \max_{\alpha; \beta_1, \dots, \beta_\alpha} \sup_t |\tilde{f}_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k) - f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)| \\ & \leq \lambda \sup_{x \in G_n} |\tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n)|, \end{aligned}$$

where  $\lambda$  is a constant not depending either on  $F$  or on the  $\{\tilde{f}_{\beta_1, \dots, \beta_\alpha}\}$ .

We denote by  $C_{\omega(\delta)}^{(1)}$  the space of all continuously differentiable functions of  $k$  variables whose partial derivatives have modulus of continuity  $\omega(\delta)$  ( $\omega(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ ).

**THEOREM 5.5.1.** *Suppose that each function  $F(x_1, \dots, x_n) \in A$  is in some region  $D_n$  of the space  $x_1, \dots, x_n$  a superposition of order  $s$  of functions of  $k$  variables  $\{f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  belonging to  $C_{\omega(\delta)}^{(1)}$  ( $k < n$ ). If for any subregion  $G_n \subset D_n$  the functional “dimension” of  $A$  at  $F(x_1, \dots, x_n) \in A$  is greater than  $k$ , then the function  $F(x_1, \dots, x_n)$  cannot be an  $(A, C_{\omega(\delta)}^{(1)})$ -stable superposition in any such region  $G \subset D_n$ .*

*Proof.* Assume the contrary, that is, in a region  $G_n \subset D_n$  the function  $F(x_1, \dots, x_n) \in A$  is an  $(A, C_{\omega(\delta)}^{(1)})$ -stable  $s$ -fold superposition of functions  $\{f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  of  $C_{\omega(\delta)}^{(1)}$ . Then any function  $\tilde{F}(x_1, \dots, x_n) \in A$  can be represented as the superposition of the same form of functions  $\{\tilde{f}_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  of  $C_{\omega(\delta)}^{(1)}$  such that

$$\max_{\alpha; \beta_1, \dots, \beta_\alpha} \sup_t |\varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)| \leq \lambda \sup_{x \in G_n} |\tilde{F} - F|,$$

where  $\varphi_{\beta_1, \dots, \beta_\alpha} = \tilde{f}_{\beta_1, \dots, \beta_\alpha} - f_{\beta_1, \dots, \beta_\alpha}$ . By Lemma 5.4.2 we have (for definiteness,  $k > 1$ )

$$\tilde{F} - F = \sum_{\alpha; \beta_1, \dots, \beta_\alpha} p_{\beta_1, \dots, \beta_\alpha}(x_1, \dots, x_n) \\ \times \varphi_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}(x_1, \dots, x_n), \dots, q_{\beta_1, \dots, \beta_\alpha, k}(x_1, \dots, x_n)) + R(x_1, \dots, x_n),$$

where  $|R(x_1, \dots, x_n)| \leq \gamma(\varepsilon)\varepsilon$ ,  $\gamma(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , and

$$\varepsilon = \max_{\alpha; \beta_1, \dots, \beta_\alpha} \sup_t |\varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)| \\ \leq \lambda \sup_{x \in G_n} |\tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n)|.$$

That  $\gamma(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  follows from the fact that as  $\varepsilon \rightarrow 0$  the quantity

$$\varepsilon' = \max_{\alpha; \beta_1, \dots, \beta_\alpha} \sum_{i=1}^k \sup \left| \frac{\partial \varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)}{\partial t_i} \right| \rightarrow 0,$$

provided only that the modulus of continuity of the partial derivatives of the functions  $\{\varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  is fixed. By 5.1.10 it follows that  $r(A, F) \leq k$  in some subregion  $G_n \subset D_n$ . So we have obtained a contradiction to the assumption that  $r(A, F) > k$  in any subregion  $G_n \subset D_n$  and this proves the theorem.

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