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Artikel: ON REPRESENTATION OF FUNCTIONS BY MEANS OF SUPERPOSITIONS AND RELATED TOPICS

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tinuous functions $\{f_i^*(t)\}$, defined on the sets $\{q_i(G^*)\}$ and satisfying the conditions

8) $$f(x, y) = \sum_{i=1}^{N} p_i(x, y) f_i^*(q_i(x, y)) \text{ for all } (x, y) \in G^*;$$

9) $$\max_i \max_{t \in q_i(G^*)} |f_i^*(t)| \geqslant K \max_{(x,y) \in G^*} |f(x, y)|.$$

Denote by $F_{\lambda\varepsilon} = F_{\lambda\varepsilon}(G^*, \{p_i\}, \{q_i\})$ the set of superpositions $f(x,y) \in F(G^*, \{p_i\}, \{q_i\})$ such that $\max_{(x,y) \in G^*} |f(x, y)| \leqslant \lambda\varepsilon$. By Theorem 5.2.1 and (8), (9), there exist constants A and B such that if $\omega(\delta) \leqslant (\lambda A K)^{-1}$ then $H_{\varepsilon,\delta}(F_{\lambda\varepsilon}) \leqslant B(\lambda K)^2/\delta$. Hence the functional dimension

$$r(F_i(G^*, \{p_i\}, \{q_i\})) \leqslant \lim_{\lambda\to\infty} \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \frac{\log_2 \log_2 \frac{B(\lambda K)^2}{\delta}}{\log_2 \delta} = 1$$

This proves the theorem.

From Theorem 5.3.1 and the properties of functional dimension (§ 1) we have the following result, which is a stronger form of Theorem 4.6.1.

COROLLARY 5.3.1. *For any continuous functions* $\{p_i(x, y)\}$ *and continuously differentiable functions* $\{q_i(x, y)\}$ *and every region* D *the set of linear superpositions* $F(D, \{p_i\}, \{q_i\})$ *is nowhere dense in any space of functions that has in every region* $G \subset D$ *functional "dimension" greater than* 1.

Remark 5.3.1. All the results about linear superpositions of the form $\sum_{i=1}^{N} p_i(x, y) f_i(q_i(x, y))$ remain valid if we assume that $\{f_i(t)\}$ are arbitrary bounded measurable functions.

§ 4. *Variation of superpositions of smooth functions*

Let G_n be a closed region of the space of the variables $x_1, x_2, \ldots, x_n$ $(n \geqslant 2)$. A function $F(x) = F(x_1, x_2, \ldots, x_n)$ is called a superposition of order s generated by the functions of k $(k > 1)$ variables

$$f_{\beta_1, \beta_2 \ldots, \beta_\alpha}(t_1, t_2, \ldots, t_k) \ (\alpha = 0, 1, 2, \ldots, s;\ \beta_i = 1, 2, \ldots, k)$$

if it is defined in G by relations

$$\begin{cases} F = f(q_1, q_2, \dots, q_k), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ q_{\beta_1,\beta_2,\dots,\beta_\alpha} = f_{\beta_1,\dots,\beta_\alpha}(q_{\beta_1,\dots,\beta_\alpha,1} q_{\beta_1,\dots,\beta_\alpha,2}, \dots, q_{\beta_1,\dots,\beta_\alpha,k}), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ q_{\beta_1,\beta_2,\dots,\beta_{s+1}} = x_{\gamma(\beta_1,\beta_2,\dots,\beta_{s+1})}, \end{cases} \tag{VI}$$

where $\gamma(\beta_1, \beta_2, \dots, \beta_{s+1})$ is a function of the indices $\beta_1, \beta_2, \dots, \beta_{s+1}$ and takes one of the values $1, 2, \dots, n$. As before, we assume that the functions $\{\varphi_{\beta_1,\beta_2,\dots,\beta_\alpha}(t_1, t_2, \dots, t_k)\}$ are defined for all values of the arguments.

A superposition of any order, generated by functions of one variable, is again a function of one variable. Therefore in this case ($k=1$) we consider superpositions of functions of one variable and the operation of addition, that is, superpositions definable in the following way.

A function $F(x) = F(x_1, x_2, \dots, x_n)$ $(n>1)$ is called a superposition of order s of the functions $f_{\beta_1,\dots,\beta_\alpha}(t)$ $(\alpha=0, 1, 2, \dots, s;\ \beta_i=1, 2)$ if the following relations are satisfied:

$$\left.\begin{aligned} F &= f(q_1+q_2), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ q_{\beta_1,\beta_2,\dots,\beta_\alpha} &= f_{\beta_1,\beta_2,\dots,\beta_\alpha}(q_{\beta_1,\beta_2,\dots,\beta_\alpha,1} + q_{\beta_1,\beta_2,\dots,\beta_\alpha,2}) \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ q_{\beta_1,\beta_2,\dots,\beta_{s+1}} &= x_{\gamma(\beta_1,\beta_2,\dots,\beta_{s+1})}, \end{aligned}\right\} \tag{VII}$$

where $\gamma(\beta_1, \beta_2, \dots, \beta_{s+1})$ takes one of the values $1, 2, \dots, n$.

Note that we can represent as superpositions of the form (VII), for example, all rational functions of $x_1, x_2, \dots, x_n$ since we can write any arithmetic operation by such superpositions, for example, $u \cdot v = e^{\ln u + \ln v} = f(f_1(u) + f_2(v))$.

Let $F(x_1, x_2, \dots, x_n)$ be a superposition of order s of the continuously differentiable functions $\{f_{\beta_1,\beta_2,\dots,\beta_\alpha}(t_1, t_2, \dots, t_k)\}$ and $\tilde{F}(x_1, x_2, \dots, x_n)$ the superposition of the same form of the continuously differentiable functions $\{\tilde{f}_{\beta_1,\beta_2,\dots,\beta_\alpha}(t_1, t_2, \dots, t_k)\}$. We put

$$\varphi_{\beta_1,\beta_2,\dots,\beta_\alpha} = \tilde{f}_{\beta_1,\dots,\beta_\alpha} - f_{\beta_1,\dots,\beta_\alpha} \quad (\alpha=0, 1, 2, \dots, s;\ \beta_i=1, 2, \dots, k)$$

$$\mu = \max_{\alpha, \beta_1, \dots, \beta_\alpha} \sum_{i=1}^{k} \sup_t \left| \frac{\partial f_{\beta_1,\dots,\beta_\alpha}(t_1, \dots, t_k)}{\partial t_i} \right|,$$

$$\varepsilon = \max_{\alpha, \beta_1, \dots, \beta_\alpha} \sup_t \left| \varphi_{\beta_1,\dots,\beta_\alpha}(t_1, t_2, \dots, t_k) \right|$$

LEMMA 5.4.1. *The inequality*

$$\sup_{x \in G} \left| \tilde{F}(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n) \right| \leqslant A(\mu, s)\, \varepsilon .$$

holds, where the constant $A(\mu, s)$ *depends only on* μ *and* s.

Proof. We proceed by induction on s. For definiteness suppose that $k < 1$. Having verified the statement of the lemma for $s = 1$ and having made an appropriate inductive assumption for superpositions of order $s - 1$, we have

$$\sup_{x \in G} \left| \tilde{F}(x_1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n) \right|$$

$$\leqslant \left| f(\tilde{q}_1, \ldots, \tilde{q}_k) - f(q_1, \ldots, q_k) \right| + \left| \varphi(\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_k) \right|$$

$$\leqslant \mu \max_{\beta_1} \sup_{x \in G} \left| \tilde{q}_{\beta_1} - q_{\beta_1} \right| + \varepsilon \leqslant \mu \cdot A(\mu, s-1)\, \varepsilon + \varepsilon = A(\mu, s)\, \varepsilon .$$

(the last by the indictive assumption). This proves the lemma.

Further, let $\omega(\delta)$ be the common modulus of continuity of all the functions $\left\{ \dfrac{\partial f_{\beta_1, \ldots, \beta_\alpha}(t_1, \ldots, t_k)}{\partial t_i} \right\}$ and, in addition, put

$$\varepsilon' = \max_{\alpha, \beta_1, \ldots, \beta_\alpha} \sum_{i=1}^{k} \sup_{t} \left| \frac{\partial \varphi_{\beta_1, \ldots, \beta_\alpha}(t_1, \ldots, t_k)}{\partial t_i} \right|$$

LEMMA 5.4.2. *We have (for case* $k > 1$*)*

$$\tilde{F}(x_1, \ldots, x_n) - F(x_1, \ldots, x_n) = \sum_{\alpha, \beta_1, \ldots, \beta_\alpha} p_{\beta_1, \ldots, \beta_\alpha}(x_1, x_2, \ldots, x_n)$$

$$\times \varphi_{\beta_1, \ldots \beta_\alpha,}(q_{\beta_1, \ldots, \beta_\alpha, 1}(x_1, \ldots, x_n), \ldots, q_{\beta_1, \ldots, \beta_\alpha, k}(x_1, \ldots, x_n))$$

$$+ R(x_1, x_2, \ldots, x_n),$$

where

$$\left| R(x_1, x_2, \ldots, x_n) \right| \leqslant B(\mu, s, k) \left[\varepsilon' + \omega\left(A(\mu, s)\, \varepsilon\right) \right] \varepsilon ,$$

$$p_{\beta_1, \ldots, \beta_\alpha}(x_1, x_2, \ldots, x_n) = \prod_{i=0}^{\alpha-1} \frac{\partial f_{\beta_1, \ldots, \beta_i}}{\partial q_{\beta_1, \ldots, \beta_{i+1}}}$$

(for $\alpha = 0$ $p(x_1, x_2, \ldots, x_n) \equiv 1$),

$B(\mu, s, k)$ is a constant depending only on μ, s, k. For $k = 1$ the corresponding equation is slightly different (see Chapter I, (III)) :

$$\tilde{F}(x_1, \ldots, x_n) - F(x_1, \ldots, x_n)$$
$$= \sum_{\alpha, \beta_1, \ldots, \beta_\alpha} p_{\beta_1,\ldots,\beta_\alpha}(x_1, x_2, \ldots, x_n)\, \varphi_{\beta_1,\ldots,\beta_\alpha}(q_{\beta_1,\ldots,\beta_\alpha,1}(x_1, \ldots, x_n)$$
$$+ q_{\beta_1,\ldots,\beta_\alpha,2}(x_1, \ldots, x_n)) + R(x_1, \ldots, x_n)\,.$$

Proof. As in the preceding lemma we proceed by induction on s. Again for definiteness we limit ourselves to the case $k > 1$. For $s = 1$ the assertion of the lemma is easily verified. We assume that it is true for superpositions of order $s - 1$. By Lemma 5.4.1, for superpositions of order s we have

$$\tilde{F}(x_1, \ldots, x_n) - F(x_1, \ldots, x_n) = f(\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_k) - f(q_1, q_2, \ldots, q_k)$$
$$+ \varphi(\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_k) = \varphi(q_1, q_2, \ldots, q_k) + \sum_{\beta_1=1}^{k} \frac{\partial f}{\partial q_{\beta_1}}(\tilde{q}_{\beta_1} - q_{\beta_1})$$
$$+ A(\mu, s)\,\varepsilon' \cdot \varepsilon + k \cdot A(\mu, s)\,\omega\bigl(A(\mu, s)\,\varepsilon\bigr)\,\varepsilon\,.$$

Since $\tilde{q}_{\beta_1}$ and q_{β_1} $(\beta_1 = 1, 2, \ldots, k)$ are superpositions of order $s - 1$, by the inductive hypothesis we have

$$\tilde{q}_{\beta_1} - q_{\beta_1} = \sum_{\substack{\alpha > 0 \\ \beta_2, \beta_3, \ldots, \beta_\alpha}} \hat{p}_{\beta_1,\ldots,\beta_\alpha}(x_1, x_2, \ldots, x_n)$$
$$\times\, \varphi_{\beta_1,\ldots\beta_\alpha,}\bigl(q_{\beta_1,\ldots,\beta_\alpha,1}(x_1, x_2, \ldots, x_n), \ldots, q_{\beta_1,\ldots,\beta_\alpha,k}(x_1, x_2, \ldots, x_n)\bigr)$$
$$+ \acute{R}(x_1, x_2, \ldots, x_n)\,,$$

where

$$\bigl|\hat{R}(x_1, x_2, \ldots, x_n)\bigr| \leqslant B(\mu, s-1, k)\bigl[\varepsilon' + \omega\bigl(A(\mu, s-1)\,\varepsilon\bigr)\bigr]\,\varepsilon\,,$$
$$\hat{p}_{\beta_1,\ldots,\beta_\alpha}(x_1, \ldots, x_n) = \prod_{i=1}^{\alpha-1} \frac{\partial f_{\beta_1,\beta_2,\ldots,\beta_i}}{\partial q_{\beta_1,\ldots,\beta_{i+1}}}$$

(for $\alpha = 1$, $\hat{p}_{\beta_1}(x_1, \ldots, x_n) \equiv 1$).
When we now substative the expressions for the differences $\tilde{q}_{\beta_1} - q_{\beta_1}$ in the formula for $\tilde{F} - F$ above, we obtain the required representation of the difference of two superpositions $\tilde{F} - F$. This proves the lemma.