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tinuous functions $\{f_i^*(t)\}$, defined on the sets $\{q_i(G^*)\}$ and satisfying the conditions

$$8) \quad f(x, y) = \sum_{i=1}^N p_i(x, y) f_i^*(q_i(x, y)) \text{ for all } (x, y) \in G^* ;$$

$$9) \quad \max_i \max_{t \in q_i(G^*)} |f_i^*(t)| \geq K \max_{(x, y) \in G^*} |f(x, y)| .$$

Denote by $F_{\lambda\varepsilon} = F_{\lambda\varepsilon}(G^*, \{p_i\}, \{q_i\})$ the set of superpositions $f(x, y) \in F(G^*, \{p_i\}, \{q_i\})$ such that $\max_{(x, y) \in G^*} |f(x, y)| \leq \lambda\varepsilon$. By Theorem 5.2.1

and (8), (9), there exist constants A and B such that if $\omega(\delta) \leq (\lambda AK)^{-1}$ then $H_{\varepsilon, \delta}(F_{\lambda\varepsilon}) \leq B(\lambda K)^2/\delta$. Hence the functional dimension

$$r(F_i(G^*, \{p_i\}, \{q_i\})) \leq \lim_{\lambda \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 \log_2 \frac{B(\lambda K)^2}{\delta}}{\log_2 \delta} = 1$$

This proves the theorem.

From Theorem 5.3.1 and the properties of functional dimension (§ 1) we have the following result, which is a stronger form of Theorem 4.6.1.

COROLLARY 5.3.1. *For any continuous functions $\{p_i(x, y)\}$ and continuously differentiable functions $\{q_i(x, y)\}$ and every region D the set of linear superpositions $F(D, \{p_i\}, \{q_i\})$ is nowhere dense in any space of functions that has in every region $G \subset D$ functional "dimension" greater than 1.*

Remark 5.3.1. All the results about linear superpositions of the form $\sum_{i=1}^N p_i(x, y) f_i(q_i(x, y))$ remain valid if we assume that $\{f_i(t)\}$ are arbitrary bounded measurable functions.

§ 4. Variation of superpositions of smooth functions

Let G_n be a closed region of the space of the variables x_1, x_2, \dots, x_n ($n \geq 2$). A function $F(x) = F(x_1, x_2, \dots, x_n)$ is called a superposition of order s generated by the functions of k ($k > 1$) variables

$$f_{\beta_1, \beta_2, \dots, \beta_\alpha}(t_1, t_2, \dots, t_k) \quad (\alpha = 0, 1, 2, \dots, s; \beta_i = 1, 2, \dots, k)$$

if it is defined in G by relations

LEMMA 5.4.1. *The inequality*

$$\sup_{x \in G} \left| \tilde{F}(x_1, x_2, \dots, x_n) - F(x_1, x_2, \dots, x_n) \right| \leq A(\mu, s) \varepsilon.$$

holds, where the constant $A(\mu, s)$ depends only on μ and s .

Proof. We proceed by induction on s . For definiteness suppose that $k < 1$. Having verified the statement of the lemma for $s = 1$ and having made an appropriate inductive assumption for superpositions of order $s - 1$, we have

$$\begin{aligned} \sup_{x \in G} \left| \tilde{F}(x_1, x_2, \dots, x_n) - F(x_1, x_2, \dots, x_n) \right| \\ \leq \left| f(\tilde{q}_1, \dots, \tilde{q}_k) - f(q_1, \dots, q_k) \right| + \left| \varphi(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_k) \right| \\ \leq \mu \max_{\beta_1} \sup_{x \in G} \left| \tilde{q}_{\beta_1} - q_{\beta_1} \right| + \varepsilon \leq \mu \cdot A(\mu, s-1) \varepsilon + \varepsilon = A(\mu, s) \varepsilon. \end{aligned}$$

(the last by the inductive assumption). This proves the lemma.

Further, let $\omega(\delta)$ be the common modulus of continuity of all the functions $\left\{ \frac{\partial f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)}{\partial t_i} \right\}$ and, in addition, put

$$\varepsilon' = \max_{\alpha, \beta_1, \dots, \beta_\alpha} \sum_{i=1}^k \sup_t \left| \frac{\partial \varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)}{\partial t_i} \right|$$

LEMMA 5.4.2. *We have (for case $k > 1$)*

$$\begin{aligned} \tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n) = \sum_{\alpha, \beta_1, \dots, \beta_\alpha} p_{\beta_1, \dots, \beta_\alpha}(x_1, x_2, \dots, x_n) \\ \times \varphi_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}(x_1, \dots, x_n), \dots, q_{\beta_1, \dots, \beta_\alpha, k}(x_1, \dots, x_n)) \\ + R(x_1, x_2, \dots, x_n), \end{aligned}$$

where

$$\left| R(x_1, x_2, \dots, x_n) \right| \leq B(\mu, s, k) [\varepsilon' + \omega(A(\mu, s) \varepsilon)] \varepsilon,$$

$$p_{\beta_1, \dots, \beta_\alpha}(x_1, x_2, \dots, x_n) = \prod_{i=0}^{\alpha-1} \frac{\partial f_{\beta_1, \dots, \beta_i}}{\partial q_{\beta_1, \dots, \beta_{i+1}}}$$

(for $\alpha=0$ $p(x_1, x_2, \dots, x_n) \equiv 1$),

$B(\mu, s, k)$ is a constant depending only on μ, s, k . For $k = 1$ the corresponding equation is slightly different (see Chapter I, (III)):

$$\begin{aligned} & \tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n) \\ = & \sum_{\alpha, \beta_1, \dots, \beta_\alpha} p_{\beta_1, \dots, \beta_\alpha}(x_1, x_2, \dots, x_n) \varphi_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}(x_1, \dots, x_n) \\ & + q_{\beta_1, \dots, \beta_\alpha, 2}(x_1, \dots, x_n)) + R(x_1, \dots, x_n). \end{aligned}$$

Proof. As in the preceding lemma we proceed by induction on s . Again for definiteness we limit ourselves to the case $k > 1$. For $s = 1$ the assertion of the lemma is easily verified. We assume that it is true for superpositions of order $s - 1$. By Lemma 5.4.1, for superpositions of order s we have

$$\begin{aligned} \tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n) &= f(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_k) - f(q_1, q_2, \dots, q_k) \\ &+ \varphi(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_k) - \varphi(q_1, q_2, \dots, q_k) + \sum_{\beta_1=1}^k \frac{\partial f}{\partial q_{\beta_1}}(\tilde{q}_{\beta_1} - q_{\beta_1}) \\ &+ A(\mu, s) \varepsilon' \cdot \varepsilon + k \cdot A(\mu, s) \omega(A(\mu, s) \varepsilon) \varepsilon. \end{aligned}$$

Since \tilde{q}_{β_1} and q_{β_1} ($\beta_1 = 1, 2, \dots, k$) are superpositions of order $s - 1$, by the inductive hypothesis we have

$$\begin{aligned} \tilde{q}_{\beta_1} - q_{\beta_1} &= \sum_{\substack{\alpha > 0 \\ \beta_2, \beta_3, \dots, \beta_\alpha}} \hat{p}_{\beta_1, \dots, \beta_\alpha}(x_1, x_2, \dots, x_n) \\ &\times \varphi_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}(x_1, x_2, \dots, x_n), \dots, q_{\beta_1, \dots, \beta_\alpha, k}(x_1, x_2, \dots, x_n)) \\ &+ \hat{R}(x_1, x_2, \dots, x_n), \end{aligned}$$

where

$$\begin{aligned} |\hat{R}(x_1, x_2, \dots, x_n)| &\leq B(\mu, s - 1, k) [\varepsilon' + \omega(A(\mu, s - 1) \varepsilon)] \varepsilon, \\ \hat{p}_{\beta_1, \dots, \beta_\alpha}(x_1, \dots, x_n) &= \prod_{i=1}^{\alpha-1} \frac{\partial f_{\beta_1, \beta_2, \dots, \beta_i}}{\partial q_{\beta_1, \dots, \beta_{i+1}}} \end{aligned}$$

(for $\alpha = 1, \hat{p}_{\beta_1}(x_1, \dots, x_n) \equiv 1$).

When we now substitute the expressions for the differences $\tilde{q}_{\beta_1} - q_{\beta_1}$ in the formula for $\tilde{F} - F$ above, we obtain the required representation of the difference of two superpositions $\tilde{F} - F$. This proves the lemma.