

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 23 (1977)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON REPRESENTATION OF FUNCTIONS BY MEANS OF SUPERPOSITIONS AND RELATED TOPICS  
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**Kapitel:** §3. Functional "dimension" of the space of linear superpositions  
**DOI:** <https://doi.org/10.5169/seals-48931>

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$$|v_i(t_{k+1}, t_l') - v_i(t_k, t_l')| \leq c_8 \frac{1}{\delta^2} \left( \int_{-1}^1 \frac{\delta m d\tau}{\sqrt{1-\tau^2}} \right) \delta \frac{\alpha}{m} = c_9(\gamma) \alpha$$

(here we again use the mean value theorem), to store the numbers  $v_i(t_{k+1}, t_l') - v_i(t_k, t_l')$  to within  $\alpha$ ,  $\log_2 C_9$  binary digits are sufficient. Therefore to write the numbers  $v_i(t_k, t_l')$  ( $i, l$  fixed;  $k$  any admissible number)

$C_{10}(\gamma) \left[ \log_2 \frac{m}{\alpha} + (b_i - a_i) \frac{m}{\delta \alpha} \right] = \mathcal{H}_{i,l}$  binary digits are sufficient. Consequently the total number of digits sufficient to store all the numbers  $v_i(t_k, t_l')$  to within  $\alpha$ , that is, to store the functions  $f_\delta(z)$  to within  $\varepsilon$ , is

$$\mathcal{H} = \sum_{i,l} \mathcal{H}_{i,l} \leq N c_{10}(\gamma) \left[ \log_2 \frac{m}{\alpha} + (b_i - a_i) \frac{m}{\delta \alpha} \right] \frac{1}{\gamma} \frac{m}{\alpha} \leq \frac{B(\gamma, N, D)}{\delta} \left( \frac{m}{\varepsilon} \right)^2.$$

This proves the theorem.

### § 3. Functional “dimension” of the space of linear superpositions

Suppose that continuous functions  $p_i(x, y)$  and continuously differentiable functions  $q_i(x, y)$  ( $i = 1, 2, \dots, N$ ) are fixed. Let  $G$  be a closed region of the  $x, y$  plane. We denote by  $F = F(G, \{p_i\}, \{q_i\})$  the set of superpositions of the form  $f(x, y) = \sum_{i=1}^N p_i(x, y) f_i(q_i(x, y))$ , where  $(x, y) \in G$  and  $\{f_i(t)\}$  are arbitrary continuous functions of one variable. We are interested in the functional dimension of the set  $F$ .

**THEOREM 5.3.1.** *In every region  $D$  of the  $x, y$  plane there exists a closed subregion  $G \subset D$  such that*

$$r(F(G, \{p_i\}, \{q_i\})) \leq 1.$$

*Proof.* By Theorem 4.5.1, in  $D$  there exists a closed subregion  $G^* \subset D$  such that the set of superpositions  $F(G^*, \{p_i\}, \{q_i\})$  is closed (in the uniform metric) in  $C(G^*)$ , and the functions  $\{q_i(x, y)\}$  satisfy the condition: for any  $i$ , either  $\text{grad}[q_i(x, y)] \neq 0$  on  $G^*$  or  $q_i(x, y) \equiv \text{const}$  on  $G^*$ . We show that  $r(F(G^*, \{p_i\}, \{q_i\})) \leq 1$ . By Banach's open mapping theorem, there exists a constant  $K$  such that for any superposition  $\sum_{i=1}^N p_i(x, y) f_i(q_i(x, y)) = f(x, y) \in F(G^*, \{p_i\}, \{q_i\})$  there are con-

tinuous functions  $\{f_i^*(t)\}$ , defined on the sets  $\{q_i(G^*)\}$  and satisfying the conditions

$$8) \quad f(x, y) = \sum_{i=1}^N p_i(x, y) f_i^*(q_i(x, y)) \text{ for all } (x, y) \in G^*;$$

$$9) \quad \max_i \max_{t \in q_i(G^*)} |f_i^*(t)| \geq K \max_{(x, y) \in G^*} |f(x, y)|.$$

Denote by  $F_{\lambda\varepsilon} = F_{\lambda\varepsilon}(G^*, \{p_i\}, \{q_i\})$  the set of superpositions  $f(x, y) \in F(G^*, \{p_i\}, \{q_i\})$  such that  $\max_{(x, y) \in G^*} |f(x, y)| \leq \lambda\varepsilon$ . By Theorem 5.2.1 and (8), (9), there exist constants  $A$  and  $B$  such that if  $\omega(\delta) \leq (\lambda AK)^{-1}$  then  $H_{\varepsilon, \delta}(F_{\lambda\varepsilon}) \leq B(\lambda K)^2/\delta$ . Hence the functional dimension

$$r(F_{\lambda\varepsilon}(G^*, \{p_i\}, \{q_i\})) \leq \lim_{\lambda \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 \log_2 \frac{B(\lambda K)^2}{\delta}}{\log_2 \delta} = 1$$

This proves the theorem.

From Theorem 5.3.1 and the properties of functional dimension (§ 1) we have the following result, which is a stronger form of Theorem 4.6.1.

**COROLLARY 5.3.1.** *For any continuous functions  $\{p_i(x, y)\}$  and continuously differentiable functions  $\{q_i(x, y)\}$  and every region  $D$  the set of linear superpositions  $F(D, \{p_i\}, \{q_i\})$  is nowhere dense in any space of functions that has in every region  $G \subset D$  functional “dimension” greater than 1.*

*Remark 5.3.1.* All the results about linear superpositions of the form  $\sum_{i=1}^N p_i(x, y) f_i(q_i(x, y))$  remain valid if we assume that  $\{f_i(t)\}$  are arbitrary bounded measurable functions.

#### § 4. Variation of superpositions of smooth functions

Let  $G_n$  be a closed region of the space of the variables  $x_1, x_2, \dots, x_n$  ( $n \geq 2$ ). A function  $F(x) = F(x_1, x_2, \dots, x_n)$  is called a superposition of order  $s$  generated by the functions of  $k$  ( $k > 1$ ) variables

$$f_{\beta_1, \beta_2, \dots, \beta_s}(t_1, t_2, \dots, t_k) \quad (\alpha = 0, 1, 2, \dots, s; \beta_i = 1, 2, \dots, k)$$

if it is defined in  $G$  by relations