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Autor: Vitushkin, A. G.
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$$|v_i(t_{k+1}, t'_l) - v_i(t_k, t'_l)| \leq c_8 \frac{1}{\delta^2} \left(\int_{-1}^1 \frac{\delta m d\tau}{\sqrt{1-\tau^2}} \right) \delta \frac{\alpha}{m} = c_9(\gamma) \alpha$$

(here we again use the mean value theorem), to store the numbers $v_i(t_{k+1}, t'_l) - v_i(t_k, t'_l)$ to within α , $\log_2 C_9$ binary digits are sufficient. Therefore to write the numbers $v_i(t_k, t'_l)$ (i, l fixed; k any admissible number)

$C_{10}(\gamma) \left[\log_2 \frac{m}{\alpha} + (b_i - a_i) \frac{m}{\delta \alpha} \right] = \mathcal{H}_{i,l}$ binary digits are sufficient. Consequently the total number of digits sufficient to store all the numbers $v_i(t_k, t'_l)$ to within α , that is, to store the functions $f_\delta(z)$ to within ε , is

$$\mathcal{H} = \sum_{i,l} \mathcal{H}_{i,l} \leq N c_{10}(\gamma) \left[\log_2 \frac{m}{\alpha} + (b_i - a_i) \frac{m}{\delta \alpha} \right] \frac{1}{\gamma} \frac{m}{\alpha} \leq \frac{B(\gamma, N, D)}{\delta} \left(\frac{m}{\varepsilon} \right)^2.$$

This proves the theorem.

§ 3. Functional "dimension" of the space of linear superpositions

Suppose that continuous functions $p_i(x, y)$ and continuously differentiable functions $q_i(x, y)$ ($i=1, 2, \dots, N$) are fixed. Let G be a closed region of the x, y plane. We denote by $F = F(G, \{p_i\}, \{q_i\})$ the set of superpositions of the form $f(x, y) = \sum_{i=1}^N p_i(x, y) f_i(q_i(x, y))$, where $(x, y) \in G$ and $\{f_i(t)\}$ are arbitrary continuous functions of one variable. We are interested in the functional dimension of the set F .

THEOREM 5.3.1. *In every region D of the x, y plane there exists a closed subregion $G \subset D$ such that*

$$r(F(G, \{p_i\}, \{q_i\})) \leq 1.$$

Proof. By Theorem 4.5.1, in D there exists a closed subregion $G^* \subset D$ such that the set of superpositions $F(G^*, \{p_i\}, \{q_i\})$ is closed (in the uniform metric) in $C(G^*)$, and the functions $\{q_i(x, y)\}$ satisfy the condition: for any i , either $\text{grad}[q_i(x, y)] \neq 0$ on G^* or $q_i(x, y) \equiv \text{const}$ on G^* . We show that $r(F(G^*, \{p_i\}, \{q_i\})) \leq 1$. By Banach's open mapping theorem, there exists a constant K such that for any superposition

$\sum_{i=1}^N p_i(x, y) f_i(q_i(x, y)) = f(x, y) \in F(G^*, \{p_i\}, \{q_i\})$ there are con-

tinuous functions $\{f_i^*(t)\}$, defined on the sets $\{q_i(G^*)\}$ and satisfying the conditions

$$8) \quad f(x, y) = \sum_{i=1}^N p_i(x, y) f_i^*(q_i(x, y)) \text{ for all } (x, y) \in G^* ;$$

$$9) \quad \max_i \max_{t \in q_i(G^*)} |f_i^*(t)| \geq K \max_{(x, y) \in G^*} |f(x, y)| .$$

Denote by $F_{\lambda\varepsilon} = F_{\lambda\varepsilon}(G^*, \{p_i\}, \{q_i\})$ the set of superpositions $f(x, y) \in F(G^*, \{p_i\}, \{q_i\})$ such that $\max_{(x, y) \in G^*} |f(x, y)| \leq \lambda\varepsilon$. By Theorem 5.2.1

and (8), (9), there exist constants A and B such that if $\omega(\delta) \leq (\lambda AK)^{-1}$ then $H_{\varepsilon, \delta}(F_{\lambda\varepsilon}) \leq B(\lambda K)^2/\delta$. Hence the functional dimension

$$r(F_i(G^*, \{p_i\}, \{q_i\})) \leq \lim_{\lambda \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 \log_2 \frac{B(\lambda K)^2}{\delta}}{\log_2 \delta} = 1$$

This proves the theorem.

From Theorem 5.3.1 and the properties of functional dimension (§ 1) we have the following result, which is a stronger form of Theorem 4.6.1.

COROLLARY 5.3.1. *For any continuous functions $\{p_i(x, y)\}$ and continuously differentiable functions $\{q_i(x, y)\}$ and every region D the set of linear superpositions $F(D, \{p_i\}, \{q_i\})$ is nowhere dense in any space of functions that has in every region $G \subset D$ functional "dimension" greater than 1.*

Remark 5.3.1. All the results about linear superpositions of the form $\sum_{i=1}^N p_i(x, y) f_i(q_i(x, y))$ remain valid if we assume that $\{f_i(t)\}$ are arbitrary bounded measurable functions.

§ 4. Variation of superpositions of smooth functions

Let G_n be a closed region of the space of the variables x_1, x_2, \dots, x_n ($n \geq 2$). A function $F(x) = F(x_1, x_2, \dots, x_n)$ is called a superposition of order s generated by the functions of k ($k > 1$) variables

$$f_{\beta_1, \beta_2, \dots, \beta_\alpha}(t_1, t_2, \dots, t_k) \quad (\alpha = 0, 1, 2, \dots, s; \beta_i = 1, 2, \dots, k)$$

if it is defined in G by relations