$$|v_i(t_{k+1}, t_l') - v_i(t_k, t_l')| \leqslant c_8 \frac{1}{\delta^2}\left(\int_{-1}^{1} \frac{\delta m d\tau}{\sqrt{1-\tau^2}}\right)\delta\frac{\alpha}{m} = c_9(\gamma)\alpha$$

(here we again use the mean value theorem), to store the numbers $v_i(t_{k+1}, t_l') - v_i(t_k, t_l')$ to within α, $\log_2 C_9$ binary digits are sufficient. Therefore to write the numbers $v_i(t_k, t_l')$ (i, l fixed; k any admissible number) $C_{10}(\gamma)\left[\log_2\frac{m}{\alpha} + (b_i - a_i)\frac{m}{\delta\alpha}\right] = \mathscr{H}_{i,l}$ binary digits are sufficient. Consequently the total number of digits sufficient to store all the numbers $v_i(t_k, t_l')$ to within α, that is, to store the functions $f_\delta(z)$ to within ε, is

$$\mathscr{H} = \sum_{i,l}\mathscr{H}_{i,l} \leqslant Nc_{10}(\gamma)\left[\log_2\frac{m}{\alpha} + (b_i - a_i)\frac{m}{\delta\alpha}\right]\frac{1}{\gamma}\frac{m}{\alpha} \leqslant \frac{B(\gamma, N, D)}{\delta}\left(\frac{m}{\varepsilon}\right)^2.$$

This proves the theorem.

§ 3. *Functional "dimension" of the space of linear superpositions*

Suppose that continuous functions $p_i(x, y)$ and continuously differentiable functions $q_i(x, y)$ ($i = 1, 2, \ldots, N$) are fixed. Let G be a closed region of the x, y plane. We denote by $F = F(G, \{p_i\}, \{q_i\})$ the set of superpositions of the form $f(x, y) = \sum_{i=1}^{N} p_i(x, y) f_i(q_i(x, y))$, where $(x, y) \in G$ and $\{f_i(t)\}$ are arbitrary continuous functions of one variable. We are interested in the functional dimension of the set F.

THEOREM 5.3.1. *In every region D of the x, y plane there exists a closed subregion $G \subset D$ such that*

$$r(F(G, \{p_i\}, \{q_i\})) \leqslant 1.$$

Proof. By Theorem 4.5.1, in D there exists a closed subregion $G^* \subset D$ such that the set of superpositions $F(G^*, \{p_i\}, \{q_i\})$ is closed (in the uniform metric) in $C(G^*)$, and the functions $\{q_i(x, y)\}$ satisfy the condition: for any i, either grad $[q_i(x, y)] \neq 0$ on G^* or $q_i(x, y) \equiv$ const on G^*. We show that $r(F(G^*, \{p_i\}, \{q_i\})) \leqslant 1$. By Banach's open mapping theorem, there exists a constant K such that for any superposition $\sum_{i=1}^{N} p_i(x, y) f_i(q_i(x, y)) = f(x, y) \in F(G^*, \{p_i\}, \{q_i\})$ there are continuous functions $\{f_i^*(t)\}$, defined on the sets $\{q_i(G^*)\}$ and satisfying the conditions

8) $$f(x, y) = \sum_{i=1}^{N} p_i(x, y) f_i^*(q_i(x, y)) \text{ for all } (x, y) \in G^*;$$

9) $$\max_i \max_{t \in q_i(G^*)} |f_i^*(t)| \geqslant K \max_{(x,y) \in G^*} |f(x, y)|.$$

Denote by $F_{\lambda\varepsilon} = F_{\lambda\varepsilon}(G^*, \{p_i\}, \{q_i\})$ the set of superpositions $f(x,y) \in F(G^*, \{p_i\}, \{q_i\})$ such that $\max_{(x,y) \in G^*} |f(x, y)| \leqslant \lambda\varepsilon$. By Theorem 5.2.1 and (8), (9), there exist constants A and B such that if $\omega(\delta) \leqslant (\lambda AK)^{-1}$ then $H_{\varepsilon,\delta}(F_{\lambda\varepsilon}) \leqslant B(\lambda K)^2/\delta$. Hence the functional dimension

$$r(F_i(G^*, \{p_i\}, \{q_i\})) \leqslant \lim_{\lambda\to\infty} \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \frac{\log_2 \log_2 \frac{B(\lambda K)^2}{\delta}}{\log_2 \delta} = 1$$

This proves the theorem.

From Theorem 5.3.1 and the properties of functional dimension (§ 1) we have the following result, which is a stronger form of Theorem 4.6.1.

COROLLARY 5.3.1. *For any continuous functions* $\{p_i(x, y)\}$ *and continuously differentiable functions* $\{q_i(x, y)\}$ *and every region* D *the set of linear superpositions* $F(D, \{p_i\}, \{q_i\})$ *is nowhere dense in any space of functions that has in every region* $G \subset D$ *functional "dimension" greater than* 1.

Remark 5.3.1. All the results about linear superpositions of the form $\sum_{i=1}^{N} p_i(x, y) f_i(q_i(x, y))$ remain valid if we assume that $\{f_i(t)\}$ are arbitrary bounded measurable functions.

§ 4. *Variation of superpositions of smooth functions*

Let G_n be a closed region of the space of the variables $x_1, x_2, ..., x_n$ $(n \geqslant 2)$. A function $F(x) = F(x_1, x_2, ..., x_n)$ is called a superposition of order s generated by the functions of k $(k > 1)$ variables

$$f_{\beta_1,\beta_2...,\beta_\alpha}(t_1, t_2, ..., t_k) \ (\alpha = 0, 1, 2, ..., s;\ \beta_i = 1, 2, ..., k)$$

if it is defined in G by relations