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COROLLARY 5.1.3. For any continuous functions  $p_i$  and continuously differentiable functions  $q_{1,i}, q_{2,i}, \dots, q_{k,i}, k < n$  ( $i = 1, 2, \dots, N$ ) and every region  $G_n$  there exists a continuous function that is not equal in  $G_n$  to any superposition of the form (V).

§ 2.  $(\varepsilon, \delta)$ -entropy of the set of linear superpositions

We denote by  $S(\delta, z)$  the disc of radius  $\delta$  with centre at  $z$ . Let  $p(z) = p(x, y)$  and  $q(z) = q(x, y)$  be functions defined in a closed region  $G$  of the  $x, y$ -plane and having the properties:

a)  $p(x, y), \frac{\partial q(x, y)}{\partial x}, \frac{\partial q(x, y)}{\partial y}$  are continuous in  $G$  and have modulus of continuity  $\omega(\delta)$ ,

b) the inequalities  $0 < \gamma \leq |\text{grad}[q(r)]| \leq \frac{1}{\gamma}$  and  $|p(z)| \leq \frac{1}{\gamma}$ , where  $\gamma$  is some constant, are satisfied everywhere in  $G$ .

LEMMA 5.2.1. Let  $S(\delta, z) \subset G$  and let  $\mu_q(t)$  be the function equal to  $2 \sqrt{\delta^2 - (t - q(z))^2} |\text{grad}[q(z)]|^{-2}$  on

$$q(z) - \delta |\text{grad}[q(z)]| \leq t \leq q(z) + \delta |\text{grad}[q(z)]|$$

and equal to zero elsewhere. Then

$$\int_{-\infty}^{\infty} |\mu_q(t) - h_1(e(q, t) \cap S(\delta, z))| dt \leq c_1(\gamma) \omega(\delta) \delta^2,$$

where  $c_1(\gamma)$  is a constant depending only on  $\gamma$ .

*Proof.* Let  $[a, b] \subset e(q, t) \cap S(\delta, z)$  be the segment of the level curve  $e(q, t)$ , endpoints  $a$  and  $b$ , lying on the boundary of  $S(\delta, z)$ ;  $[z, a]$  and  $[z, b]$  the vectors with origin at  $z$  and endpoints at  $a$  and  $b$ , respectively;

$$\alpha_1 = \gamma(\overrightarrow{[z, a]}, \text{grad}[q(z)]), \alpha_2 = \gamma(\overrightarrow{[z, b]}, \text{grad}[q(z)]).$$

We have

$$\begin{aligned} |t - q(z)| &= |q(a) - q(z)| = \left| \int_{s \in [z, a]} \frac{\partial q}{\partial s} ds \right| \\ &= \delta \cos \alpha_1 |\text{grad}[q(z)]| (1 + o(1) \omega(\delta)) \end{aligned}$$

Hence

$$\delta \sin \alpha_1 = \sqrt{\delta^2 - (t - q(z) + o(\gamma) \delta \omega(\delta))^2} \left| \text{grad } [q(z)] \right|^{-2}$$

and similarly

$$\delta \sin \alpha_2 = \sqrt{\delta^2 - (t - q(z) + o(\gamma) \delta \omega(\delta))^2} \left| \text{grad } [q(z)] \right|^{-2}$$

By b) the size of the angle swept out by the tangent vector to the level curve  $e(q, t)$  on moving along  $[a, b]$  does not exceed  $C_2(\gamma) \omega(\delta)$ . Therefore

$$\begin{aligned} h_1([a, b]) &= \delta (\sin \alpha_1 + \sin \alpha_2) (1 + o(\gamma) \omega(\delta)) \\ &= 2 \sqrt{\delta^2 - (t - q(z) + o(\gamma) \delta \omega(\delta))^2} \left| \text{grad } [q(z)] \right|^{-2} + o(\gamma) \delta \omega(\delta). \end{aligned}$$

If  $\alpha_1 \geq C_3(\gamma) \omega(\delta)$  ( $C_3$  is a sufficiently large constant), then  $[a, b] = e(q, t) \cap S(\delta, z)$ . Consequently, for

$$\left| t - q(z) \right| \leq \theta = \delta \cos [C_3 \omega(\delta)] \left| \text{grad } [q(z)] \right| \times (1 + o(1) \omega(\delta))$$

we have  $h_1(e(q, t) \cap S(\delta, z)) = h_1([a, b])$ . Since for every  $t$  (by b))

$$h_1(e(q, t) \cap S(\delta, z)) \leq C_4(\gamma) \delta (1 + \omega(\delta)),$$

we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \left| h_1(e(q, t) \cap S(\delta, z)) - \mu_q(t) \right| dt = \\ &= \int_{q(z) - \theta}^{q(z) + \theta} \left| h_1(e(q, t) \cap S(\delta, z)) - \mu_q(t) \right| dt + o(\gamma) \delta^2 \omega(\delta). \end{aligned}$$

We now estimate

$$\begin{aligned} &\int_{q(z) - \theta}^{q(z) + \theta} \left| h_1(e(q, t) \cap S(\delta, z)) - \mu_q(t) \right| dt = \\ &= \int_{q(z) - \theta}^{q(z) + \theta} \left| h_1([a, b]) - \mu_q(t) \right| dt \leq \\ &\leq 2 \int_{q(z) - \theta}^{q(z) + \theta} \left( \sqrt{\delta^2 - (t - q(z) + o(\gamma) \delta \omega(\delta))^2} \left| \text{grad } [q(z)] \right|^{-2} \right. \\ &\quad \left. - \sqrt{\delta^2 - (t - q(z))^2} \left| \text{grad } [q(z)] \right|^{-2} \right) dt + o(\gamma) \delta^2 \omega(\delta) \\ &= o(\gamma) \delta^2 \omega(\delta) \int_{-1}^1 \frac{d\tau}{\sqrt{1 - \tau^2}} + o(\gamma) \delta^2 \omega(\delta) = o(\gamma) \delta^2 \omega(\delta). \end{aligned}$$

Here we have the mean value theorem. This proves the lemma.

LEMMA 5.2.2. Let  $p(z), q(z)$  satisfy conditions a) and b);  $S(\delta, z) \subset G$ ; let  $f(t)$  be an arbitrary continuous function, uniformly bounded in modulus by the constant  $m$ . Then

$$\begin{aligned} & \iint_{(u, v) \in S(\delta, z)} p(u, v) f(q(u, v)) \, dudv \\ &= p(z) \left| \operatorname{grad} [q(z)] \right|^{-1} \int_{-\infty}^{\infty} f(t) \mu_q(t) \, dt + \lambda(z) m \delta^2 \omega(\delta), \end{aligned}$$

where  $|\lambda(z)| \leq C_5(\gamma)$ .

*Proof.* Using a) and b) and Lemma 5.2.1 we have

$$\begin{aligned} & \int_{S(\delta, z)} p(u, v) f(q(u, v)) \, dudv \\ &= p(z) \iint_{(u, v) \in S(\delta, z)} f(q(u, v)) \, dudv + O(1) m \delta^2 \omega(\delta) \\ &= p(z) \int_{-\infty}^{\infty} \left\{ f(t) \int_{s \in e(q, t) \cap S(\delta, z)} \left| \operatorname{grad} [q(s)] \right|^{-2} ds \right\} dt + O(1) m \delta^2 \omega(\delta) \\ &= p(z) \left| \operatorname{grad} [q(z)] \right|^{-1} \int_{-\infty}^{\infty} \left\{ f(t) \int_{s \in e(q, t) \cap S(\delta, z)} ds \right\} dt + O(\gamma) m \delta^2 \omega(\delta) \\ &= p(z) \left| \operatorname{grad} [q(z)] \right|^{-2} \int_{-\infty}^{\infty} f(t) h_1(e(q, t) \cap S(\delta, z)) \, dt + O(\gamma) m \delta^2 \omega(\delta) \\ &= p(z) \left| \operatorname{grad} [q(z)] \right|^{-1} \int_{-\infty}^{\infty} f(t) \mu_q(t) \, dt + O(\gamma) m \delta^2 \omega(\delta). \end{aligned}$$

This proves the lemma.

LEMMA 5.2.3. Suppose that a number  $\alpha > 0$  and functions  $p(z), q(z), f(t)$  satisfying the conditions of Lemma 5.2.2. are given. If for every integer  $k$  such that

$$\min_{z \in G} q(z) \leq t_k = k \delta \frac{\alpha}{m} \leq \max_{z \in G} q(z)$$

and any integer  $l$  such that

$$\min_{z \in G} \left| \operatorname{grad} [q(z)] \right| \leq t'_l = l \frac{\alpha}{m} \leq \max_{z \in G} \left| \operatorname{grad} [q(z)] \right|,$$

the inequality

$$\left| \int_{t_k - t'_l \delta}^{t_k + t'_l \delta} f(t) \sqrt{\delta^2 - \left( \frac{t - t_k}{t'_l} \right)^2} dt \right| \leq \alpha \delta^2$$

is satisfied, then for every disc  $S(\delta, z) \subset G$

$$\left| \iint_{(u, v) \in S(\delta, z)} p(u, v) f(q(u, v)) \, dudv \right| \leq c_6(\gamma) (\alpha\delta^2 + m\delta^2\omega(\delta)).$$

*Proof.* Suppose that a disc  $S(\delta, z) \subset G$  is given. By the condition of the lemma there are integers  $k$  and  $l$  such that  $|q(z) - t_k| \leq \delta\alpha/m$  and  $|| \text{grad}[q(z)] | - t'_l | \leq \alpha/m$ . From Lemma 5.2.2 we obtain

$$\begin{aligned} \left| \iint_{(u, v) \in S(\delta, z)} p(u, v) f(q(u, v)) \, dudv \right| &\leq \frac{|p(z)|}{|\text{grad}[q(z)]|} \left| \int_{-\infty}^{\infty} f(t) \mu_q(t) \, dt \right| \\ &+ c_5(\gamma) m\delta^2\omega(\delta) \leq \frac{2}{\gamma^2} \left| \int_{\substack{q(z) + \\ -\delta|\text{grad}[q(z)]|}}^{\substack{q(z) + \\ +\delta|\text{grad}[q(z)]|}} f(t) \sqrt{\delta^2 - \frac{(t - q(z))^2}{|\text{grad}[q(z)]|^2}} \, dt \right. \\ &\left. - \int_{t_k - t'_l\delta}^{t_k + t'_l\delta} f(t) \sqrt{\delta^2 - \left(\frac{t - t_k}{t'_l}\right)^2} \, dt \right| + \frac{2}{\gamma^2} \alpha\delta^2 + c_5(\gamma) m\delta^2\omega(\delta) \leq \end{aligned}$$

(by the mean value theorem)

$$\begin{aligned} &\leq \frac{2}{\gamma^2} \alpha\delta^2 + c_5(\gamma) m\delta^2\omega(\delta) + \frac{2}{\gamma^2} \left( \int_{-1}^1 \frac{\delta m d\tau}{\sqrt{1 - \tau^2}} \right) \delta \frac{\alpha}{m} \\ &+ \frac{2}{\gamma^2} \left( \int_{-1}^1 \frac{\delta^2 m d\tau}{\sqrt{1 - \tau^2}} \right) \frac{\alpha}{m} \leq c_6(\gamma) (\alpha\delta^2 + m\delta^2\omega(\delta)). \end{aligned}$$

This proves the lemma.

We denote by  $F_m = F_m(D; p_1, p_2, \dots, p_N; q_1, q_2, \dots, q_N)$  the set of superpositions of the form

$$f(x, y) = \sum_{i=1}^N p_i(x, y) f_i(q_i(x, y)), \text{ where } \{p_i(x, y)\}$$

and  $\{q_i(x, y)\}$  are fixed functions, defined in the closed region  $D$  of the  $x, y$  plane and satisfying conditions a) and b) with a constant  $\gamma$  not depending on  $i$  and  $\{f_i(t)\}$  are arbitrary continuous functions, defined on  $\{[a_i, b_i]\} = \{[\min_{z \in D} q_i(z); \max_{z \in D} q_i(z)]\}$  and uniformly bounded in modulus by the constant  $m$ .

THEOREM 5.2.1. *There exist constants  $A$  and  $B$  such that if  $\varepsilon > Am\omega(\delta)$  then for the  $(\varepsilon, \delta)$ -entropy of the set of functions  $F_m$ ,  $H_{\varepsilon, \delta}(F_m) \leq \frac{B}{\delta} \left(\frac{m}{\varepsilon}\right)^2$ , where  $A$  and  $B$  depend only on  $\gamma, N$  and  $D$ .*

*Proof.* We put

$$R(f(z), \delta) = \max_{S(\delta, z) \subset D} \left| \frac{1}{\pi\delta^2} \iint_{(u, v) \in S(\delta, z)} f(u, v) \, dudv \right|.$$

We denote by  $\mathcal{H}_{\varepsilon, \delta}(F_m)$  the  $\varepsilon$ -entropy of the space  $F_m$ , taking as the distance between the functions  $f_1(z), f_2(z) \in F_m$  the number  $R(f_1(z) - f_2(z), \delta)$ . The inequality  $H_{2\varepsilon, \delta}(F_m) \leq \mathcal{H}_{\varepsilon, \delta}(F_m)$  holds owing to the fact that if two functions  $f_1(z)$  and  $f_2(z)$  are  $(\varepsilon, \delta)$ -distinguishable, then they are  $\varepsilon$ -distinguishable also in the sense of the metric  $R(f_1(z) - f_2(z), \delta)$ . We now estimate the value of  $\mathcal{H}_{\varepsilon, \delta}(F_m)$ . Let  $k$  and  $l$  be integers such that

$$\min_{z \in D} q_i(z) \leq t_k = k\delta \frac{\alpha}{m} \leq \max_{z \in D} q_i(z)$$

and

$$\min_{z \in D} |\text{grad } [q_i(z)]| \leq t'_l = l \frac{\alpha}{m} \leq \max_{z \in D} |\text{grad } [q_i(z)]|.$$

To compute the function

$$f_\delta(z) = \frac{1}{\pi\delta^2} \iint_{(u, v) \in S(\delta, z)} f(u, v) \, dudv,$$

where  $f(x, y) \in F_m$ ,  $S(\delta, z) \subset D$  to within  $\varepsilon$ , it is sufficient by Lemma 5.2.3 to give the values of

$$v_i(t_k, t'_l) = \frac{1}{\pi\delta^2} \int_{t_k - t'_l\delta}^{t_k + t'_l\delta} f_i(t) \sqrt{\delta^2 - \left(\frac{t - t_k}{t'_l}\right)^2} \, dt$$

to within  $\alpha = \pi\varepsilon / (2NC_B(\gamma))$  and to assume that  $\delta$  is small enough so that

$$\varepsilon > \frac{2NC_B(\gamma) m\omega(\delta)}{\pi} = A(\gamma, N) m\omega(\delta).$$

Since  $|v_i(t_k, t'_l)| \leq C_1 m$ , to write the numbers  $v_i(t_k, t'_l)$  ( $i, k, l$  fixed)  $\log_2(C_1 m/\alpha)$  binary digits are sufficient. Since

$$|v_i(t_{k+1}, t'_l) - v_i(t_k, t'_l)| \leq c_8 \frac{1}{\delta^2} \left( \int_{-1}^1 \frac{\delta m d\tau}{\sqrt{1-\tau^2}} \right) \delta \frac{\alpha}{m} = c_9(\gamma) \alpha$$

(here we again use the mean value theorem), to store the numbers  $v_i(t_{k+1}, t'_l) - v_i(t_k, t'_l)$  to within  $\alpha$ ,  $\log_2 C_9$  binary digits are sufficient. Therefore to write the numbers  $v_i(t_k, t'_l)$  ( $i, l$  fixed;  $k$  any admissible number)

$C_{10}(\gamma) \left[ \log_2 \frac{m}{\alpha} + (b_i - a_i) \frac{m}{\delta \alpha} \right] = \mathcal{H}_{i,l}$  binary digits are sufficient. Consequently the total number of digits sufficient to store all the numbers  $v_i(t_k, t'_l)$  to within  $\alpha$ , that is, to store the functions  $f_\delta(z)$  to within  $\varepsilon$ , is

$$\mathcal{H} = \sum_{i,l} \mathcal{H}_{i,l} \leq N c_{10}(\gamma) \left[ \log_2 \frac{m}{\alpha} + (b_i - a_i) \frac{m}{\delta \alpha} \right] \frac{1}{\gamma} \frac{m}{\alpha} \leq \frac{B(\gamma, N, D)}{\delta} \left( \frac{m}{\varepsilon} \right)^2.$$

This proves the theorem.

### § 3. Functional “dimension” of the space of linear superpositions

Suppose that continuous functions  $p_i(x, y)$  and continuously differentiable functions  $q_i(x, y)$  ( $i=1, 2, \dots, N$ ) are fixed. Let  $G$  be a closed region of the  $x, y$  plane. We denote by  $F = F(G, \{p_i\}, \{q_i\})$  the set of superpositions of the form  $f(x, y) = \sum_{i=1}^N p_i(x, y) f_i(q_i(x, y))$ , where  $(x, y) \in G$  and  $\{f_i(t)\}$  are arbitrary continuous functions of one variable. We are interested in the functional dimension of the set  $F$ .

**THEOREM 5.3.1.** *In every region  $D$  of the  $x, y$  plane there exists a closed subregion  $G \subset D$  such that*

$$r(F(G, \{p_i\}, \{q_i\})) \leq 1.$$

*Proof.* By Theorem 4.5.1, in  $D$  there exists a closed subregion  $G^* \subset D$  such that the set of superpositions  $F(G^*, \{p_i\}, \{q_i\})$  is closed (in the uniform metric) in  $C(G^*)$ , and the functions  $\{q_i(x, y)\}$  satisfy the condition: for any  $i$ , either  $\text{grad}[q_i(x, y)] \neq 0$  on  $G^*$  or  $q_i(x, y) \equiv \text{const}$  on  $G^*$ . We show that  $r(F(G^*, \{p_i\}, \{q_i\})) \leq 1$ . By Banach’s open mapping theorem, there exists a constant  $K$  such that for any superposition

$\sum_{i=1}^N p_i(x, y) f_i(q_i(x, y)) = f(x, y) \in F(G^*, \{p_i\}, \{q_i\})$  there are con-