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COROLLARY 5.1.3. For any continuous functions p_i and continuously differentiable functions $q_{1,i}, q_{2,i}, \dots, q_{k,i}, k < n$ ($i = 1, 2, \dots, N$) and every region G_n there exists a continuous function that is not equal in G_n to any superposition of the form (V).

§ 2. (ε, δ) -entropy of the set of linear superpositions

We denote by $S(\delta, z)$ the disc of radius δ with centre at z . Let $p(z) = p(x, y)$ and $q(z) = q(x, y)$ be functions defined in a closed region G of the x, y -plane and having the properties:

a) $p(x, y), \frac{\partial q(x, y)}{\partial x}, \frac{\partial q(x, y)}{\partial y}$ are continuous in G and have modulus of continuity $\omega(\delta)$,

b) the inequalities $0 < \gamma \leq |\text{grad}[q(r)]| \leq \frac{1}{\gamma}$ and $|p(z)| \leq \frac{1}{\gamma}$, where γ is some constant, are satisfied everywhere in G .

LEMMA 5.2.1. Let $S(\delta, z) \subset G$ and let $\mu_q(t)$ be the function equal to $2 \sqrt{\delta^2 - (t - q(z))^2} |\text{grad}[q(z)]|^{-2}$ on

$$q(z) - \delta |\text{grad}[q(z)]| \leq t \leq q(z) + \delta |\text{grad}[q(z)]|$$

and equal to zero elsewhere. Then

$$\int_{-\infty}^{\infty} |\mu_q(t) - h_1(e(q, t) \cap S(\delta, z))| dt \leq c_1(\gamma) \omega(\delta) \delta^2,$$

where $c_1(\gamma)$ is a constant depending only on γ .

Proof. Let $[a, b] \subset e(q, t) \cap S(\delta, z)$ be the segment of the level curve $e(q, t)$, endpoints a and b , lying on the boundary of $S(\delta, z)$; $[z, a]$ and $[z, b]$ the vectors with origin at z and endpoints at a and b , respectively;

$$\alpha_1 = \gamma(\overrightarrow{[z, a]}, \text{grad}[q(z)]), \alpha_2 = \gamma(\overrightarrow{[z, b]}, \text{grad}[q(z)]).$$

We have

$$\begin{aligned} |t - q(z)| &= |q(a) - q(z)| = \left| \int_{s \in [z, a]} \frac{\partial q}{\partial s} ds \right| \\ &= \delta \cos \alpha_1 |\text{grad}[q(z)]| (1 + o(1) \omega(\delta)) \end{aligned}$$

Hence

$$\delta \sin \alpha_1 = \sqrt{\delta^2 - (t - q(z) + o(\gamma) \delta \omega(\delta))^2} \left| \text{grad } [q(z)] \right|^{-2}$$

and similarly

$$\delta \sin \alpha_2 = \sqrt{\delta^2 - (t - q(z) + o(\gamma) \delta \omega(\delta))^2} \left| \text{grad } [q(z)] \right|^{-2}$$

By b) the size of the angle swept out by the tangent vector to the level curve $e(q, t)$ on moving along $[a, b]$ does not exceed $C_2(\gamma) \omega(\delta)$. Therefore

$$\begin{aligned} h_1([a, b]) &= \delta (\sin \alpha_1 + \sin \alpha_2) (1 + o(\gamma) \omega(\delta)) \\ &= 2\sqrt{\delta^2 - (t - q(z) + o(\gamma) \delta \omega(\delta))^2} \left| \text{grad } [q(z)] \right|^{-2} + o(\gamma) \delta \omega(\delta). \end{aligned}$$

If $\alpha_1 \geq C_3(\gamma) \omega(\delta)$ (C_3 is a sufficiently large constant), then $[a, b] = e(q, t) \cap S(\delta, z)$. Consequently, for

$$\left| t - q(z) \right| \leq \theta = \delta \cos [C_3 \omega(\delta)] \left| \text{grad } [q(z)] \right| \times (1 + o(1) \omega(\delta))$$

we have $h_1(e(q, t) \cap S(\delta, z)) = h_1([a, b])$. Since for every t (by b))

$$h_1(e(q, t) \cap S(\delta, z)) \leq C_4(\gamma) \delta (1 + \omega(\delta)),$$

we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \left| h_1(e(q, t) \cap S(\delta, z)) - \mu_q(t) \right| dt = \\ &= \int_{q(z) - \theta}^{q(z) + \theta} \left| h_1(e(q, t) \cap S(\delta, z)) - \mu_q(t) \right| dt + o(\gamma) \delta^2 \omega(\delta). \end{aligned}$$

We now estimate

$$\begin{aligned} &\int_{q(z) - \theta}^{q(z) + \theta} \left| h_1(e(q, t) \cap S(\delta, z)) - \mu_q(t) \right| dt = \\ &= \int_{q(z) - \theta}^{q(z) + \theta} \left| h_1([a, b]) - \mu_q(t) \right| dt \leq \\ &\leq 2 \int_{q(z) - \theta}^{q(z) + \theta} \left(\sqrt{\delta^2 - (t - q(z) + o(\gamma) \delta \omega(\delta))^2} \left| \text{grad } [q(z)] \right|^{-2} \right. \\ &\quad \left. - \sqrt{\delta^2 - (t - q(z))^2} \left| \text{grad } [q(z)] \right|^{-2} \right) dt + o(\gamma) \delta^2 \omega(\delta) \\ &= o(\gamma) \delta^2 \omega(\delta) \int_{-1}^1 \frac{d\tau}{\sqrt{1 - \tau^2}} + o(\gamma) \delta^2 \omega(\delta) = o(\gamma) \delta^2 \omega(\delta). \end{aligned}$$

Here we have the mean value theorem. This proves the lemma.

LEMMA 5.2.2. Let $p(z), q(z)$ satisfy conditions a) and b); $S(\delta, z) \subset G$; let $f(t)$ be an arbitrary continuous function, uniformly bounded in modulus by the constant m . Then

$$\begin{aligned} & \iint_{(u, v) \in S(\delta, z)} p(u, v) f(q(u, v)) \, dudv \\ &= p(z) \left| \operatorname{grad} [q(z)] \right|^{-1} \int_{-\infty}^{\infty} f(t) \mu_q(t) \, dt + \lambda(z) m \delta^2 \omega(\delta), \end{aligned}$$

where $|\lambda(z)| \leq C_5(\gamma)$.

Proof. Using a) and b) and Lemma 5.2.1 we have

$$\begin{aligned} & \iint_{S(\delta, z)} p(u, v) f(q(u, v)) \, dudv \\ &= p(z) \iint_{(u, v) \in S(\delta, z)} f(q(u, v)) \, dudv + O(1) m \delta^2 \omega(\delta) \\ &= p(z) \int_{-\infty}^{\infty} \left\{ f(t) \int_{s \in e(q, t) \cap S(\delta, z)} \left| \operatorname{grad} [q(s)] \right|^{-2} ds \right\} dt + O(1) m \delta^2 \omega(\delta) \\ &= p(z) \left| \operatorname{grad} [q(z)] \right|^{-1} \int_{-\infty}^{\infty} \left\{ f(t) \int_{s \in e(q, t) \cap S(\delta, z)} ds \right\} dt + O(\gamma) m \delta^2 \omega(\delta) \\ &= p(z) \left| \operatorname{grad} [q(z)] \right|^{-2} \int_{-\infty}^{\infty} f(t) h_1(e(q, t) \cap S(\delta, z)) \, dt + O(\gamma) m \delta^2 \omega(\delta) \\ &= p(z) \left| \operatorname{grad} [q(z)] \right|^{-1} \int_{-\infty}^{\infty} f(t) \mu_q(t) \, dt + O(\gamma) m \delta^2 \omega(\delta). \end{aligned}$$

This proves the lemma.

LEMMA 5.2.3. Suppose that a number $\alpha > 0$ and functions $p(z), q(z), f(t)$ satisfying the conditions of Lemma 5.2.2. are given. If for every integer k such that

$$\min_{z \in G} q(z) \leq t_k = k \delta \frac{\alpha}{m} \leq \max_{z \in G} q(z)$$

and any integer l such that

$$\min_{z \in G} \left| \operatorname{grad} [q(z)] \right| \leq t'_l = l \frac{\alpha}{m} \leq \max_{z \in G} \left| \operatorname{grad} [q(z)] \right|,$$

the inequality

$$\left| \int_{t_k - t'_l \delta}^{t_k + t'_l \delta} f(t) \sqrt{\delta^2 - \left(\frac{t - t_k}{t'_l} \right)^2} dt \right| \leq \alpha \delta^2$$

is satisfied, then for every disc $S(\delta, z) \subset G$

$$\left| \int \int_{(u, v) \in S(\delta, z)} p(u, v) f(q(u, v)) \, dudv \right| \leq c_6(\gamma) (\alpha\delta^2 + m\delta^2\omega(\delta)).$$

Proof. Suppose that a disc $S(\delta, z) \subset G$ is given. By the condition of the lemma there are integers k and l such that $|q(z) - t_k| \leq \delta\alpha/m$ and $|| \text{grad}[q(z)] | - t'_l | \leq \alpha/m$. From Lemma 5.2.2 we obtain

$$\begin{aligned} \left| \int \int_{(u, v) \in S(\delta, z)} p(u, v) f(q(u, v)) \, dudv \right| &\leq \frac{|p(z)|}{|\text{grad}[q(z)]|} \left| \int_{-\infty}^{\infty} f(t) \mu_q(t) \, dt \right| \\ &+ c_5(\gamma) m\delta^2\omega(\delta) \leq \frac{2}{\gamma^2} \left| \int_{\substack{q(z) + \\ -\delta|\text{grad}[q(z)]|}}^{\substack{q(z) + \\ +\delta|\text{grad}[q(z)]|}} f(t) \sqrt{\delta^2 - \frac{(t - q(z))^2}{|\text{grad}[q(z)]|^2}} \, dt \right. \\ &\left. - \int_{t_k - t'_l\delta}^{t_k + t'_l\delta} f(t) \sqrt{\delta^2 - \left(\frac{t - t_k}{t'_l}\right)^2} \, dt \right| + \frac{2}{\gamma^2} \alpha\delta^2 + c_5(\gamma) m\delta^2\omega(\delta) \leq \end{aligned}$$

(by the mean value theorem)

$$\begin{aligned} &\leq \frac{2}{\gamma^2} \alpha\delta^2 + c_5(\gamma) m\delta^2\omega(\delta) + \frac{2}{\gamma^2} \left(\int_{-1}^1 \frac{\delta m d\tau}{\sqrt{1 - \tau^2}} \right) \delta \frac{\alpha}{m} \\ &+ \frac{2}{\gamma^2} \left(\int_{-1}^1 \frac{\delta^2 m d\tau}{\sqrt{1 - \tau^2}} \right) \frac{\alpha}{m} \leq c_6(\gamma) (\alpha\delta^2 + m\delta^2\omega(\delta)). \end{aligned}$$

This proves the lemma.

We denote by $F_m = F_m(D; p_1, p_2, \dots, p_N; q_1, q_2, \dots, q_N)$ the set of superpositions of the form

$$f(x, y) = \sum_{i=1}^N p_i(x, y) f_i(q_i(x, y)), \text{ where } \{p_i(x, y)\}$$

and $\{q_i(x, y)\}$ are fixed functions, defined in the closed region D of the x, y plane and satisfying conditions a) and b) with a constant γ not depending on i and $\{f_i(t)\}$ are arbitrary continuous functions, defined on $\{[a_i, b_i]\} = \{[\min_{z \in D} q_i(z); \max_{z \in D} q_i(z)]\}$ and uniformly bounded in modulus by the constant m .

THEOREM 5.2.1. *There exist constants A and B such that if $\varepsilon > Am\omega(\delta)$ then for the (ε, δ) -entropy of the set of functions F_m , $H_{\varepsilon, \delta}(F_m) \leq \frac{B}{\delta} \left(\frac{m}{\varepsilon}\right)^2$, where A and B depend only on γ, N and D .*

Proof. We put

$$R(f(z), \delta) = \max_{S(\delta, z) \subset D} \left| \frac{1}{\pi\delta^2} \iint_{(u, v) \in S(\delta, z)} f(u, v) \, dudv \right|.$$

We denote by $\mathcal{H}_{\varepsilon, \delta}(F_m)$ the ε -entropy of the space F_m , taking as the distance between the functions $f_1(z), f_2(z) \in F_m$ the number $R(f_1(z) - f_2(z), \delta)$. The inequality $H_{2\varepsilon, \delta}(F_m) \leq \mathcal{H}_{\varepsilon, \delta}(F_m)$ holds owing to the fact that if two functions $f_1(z)$ and $f_2(z)$ are (ε, δ) -distinguishable, then they are ε -distinguishable also in the sense of the metric $R(f_1(z) - f_2(z), \delta)$. We now estimate the value of $\mathcal{H}_{\varepsilon, \delta}(F_m)$. Let k and l be integers such that

$$\min_{z \in D} q_i(z) \leq t_k = k\delta \frac{\alpha}{m} \leq \max_{z \in D} q_i(z)$$

and

$$\min_{z \in D} |\text{grad } [q_i(z)]| \leq t'_l = l \frac{\alpha}{m} \leq \max_{z \in D} |\text{grad } [q_i(z)]|.$$

To compute the function

$$f_\delta(z) = \frac{1}{\pi\delta^2} \iint_{(u, v) \in S(\delta, z)} f(u, v) \, dudv,$$

where $f(x, y) \in F_m$, $S(\delta, z) \subset D$ to within ε , it is sufficient by Lemma 5.2.3 to give the values of

$$v_i(t_k, t'_l) = \frac{1}{\pi\delta^2} \int_{t_k - t'_l\delta}^{t_k + t'_l\delta} f_i(t) \sqrt{\delta^2 - \left(\frac{t - t_k}{t'_l}\right)^2} \, dt$$

to within $\alpha = \pi\varepsilon / (2NC_B(\gamma))$ and to assume that δ is small enough so that

$$\varepsilon > \frac{2NC_B(\gamma) m\omega(\delta)}{\pi} = A(\gamma, N) m\omega(\delta).$$

Since $|v_i(t_k, t'_l)| \leq C_1 m$, to write the numbers $v_i(t_k, t'_l)$ (i, k, l fixed) $\log_2(C_1 m/\alpha)$ binary digits are sufficient. Since

$$|v_i(t_{k+1}, t'_l) - v_i(t_k, t'_l)| \leq c_8 \frac{1}{\delta^2} \left(\int_{-1}^1 \frac{\delta m d\tau}{\sqrt{1-\tau^2}} \right) \delta \frac{\alpha}{m} = c_9(\gamma) \alpha$$

(here we again use the mean value theorem), to store the numbers $v_i(t_{k+1}, t'_l) - v_i(t_k, t'_l)$ to within α , $\log_2 C_9$ binary digits are sufficient. Therefore to write the numbers $v_i(t_k, t'_l)$ (i, l fixed; k any admissible number)

$C_{10}(\gamma) \left[\log_2 \frac{m}{\alpha} + (b_i - a_i) \frac{m}{\delta \alpha} \right] = \mathcal{H}_{i,l}$ binary digits are sufficient. Consequently the total number of digits sufficient to store all the numbers $v_i(t_k, t'_l)$ to within α , that is, to store the functions $f_\delta(z)$ to within ε , is

$$\mathcal{H} = \sum_{i,l} \mathcal{H}_{i,l} \leq N c_{10}(\gamma) \left[\log_2 \frac{m}{\alpha} + (b_i - a_i) \frac{m}{\delta \alpha} \right] \frac{1}{\gamma} \frac{m}{\alpha} \leq \frac{B(\gamma, N, D)}{\delta} \left(\frac{m}{\varepsilon} \right)^2.$$

This proves the theorem.

§ 3. Functional "dimension" of the space of linear superpositions

Suppose that continuous functions $p_i(x, y)$ and continuously differentiable functions $q_i(x, y)$ ($i=1, 2, \dots, N$) are fixed. Let G be a closed region of the x, y plane. We denote by $F = F(G, \{p_i\}, \{q_i\})$ the set of superpositions of the form $f(x, y) = \sum_{i=1}^N p_i(x, y) f_i(q_i(x, y))$, where $(x, y) \in G$ and $\{f_i(t)\}$ are arbitrary continuous functions of one variable. We are interested in the functional dimension of the set F .

THEOREM 5.3.1. *In every region D of the x, y plane there exists a closed subregion $G \subset D$ such that*

$$r(F(G, \{p_i\}, \{q_i\})) \leq 1.$$

Proof. By Theorem 4.5.1, in D there exists a closed subregion $G^* \subset D$ such that the set of superpositions $F(G^*, \{p_i\}, \{q_i\})$ is closed (in the uniform metric) in $C(G^*)$, and the functions $\{q_i(x, y)\}$ satisfy the condition: for any i , either $\text{grad}[q_i(x, y)] \neq 0$ on G^* or $q_i(x, y) \equiv \text{const}$ on G^* . We show that $r(F(G^*, \{p_i\}, \{q_i\})) \leq 1$. By Banach's open mapping theorem, there exists a constant K such that for any superposition

$\sum_{i=1}^N p_i(x, y) f_i(q_i(x, y)) = f(x, y) \in F(G^*, \{p_i\}, \{q_i\})$ there are con-