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Autor: Vitushkin, A. G.
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CHAPTER 5. — DIMENSION OF THE SPACE OF LINEAR SUPERPOSITIONS

In this chapter we present a calculation of the functional dimension of the space of functions representable by means of linear superpositions and prove that a representation of analytic functions by means superpositions of smooth functions can not be stable.

§ 1. (ε, δ) -entropy and the “dimension” of function spaces

Let G_n be a closed region of n -dimensional euclidean space, and $C(G_n)$ the space of all functions continuous in G_n . Two functions $f_1(x), f_2(x) \in C(G_n)$ are called (ε, δ) -distinguishable if there exists an n -dimensional closed sphere $S_\delta \subset G_n$ of radius δ such that

$$\min_{x \in S_\delta} |f_1(x) - f_2(x)| \geq \varepsilon.$$

Let $F \subset C(G_n)$ be a set of continuous functions. A subset $K \subset F$ is called (ε, δ) -distinguishable if any two of its elements are (ε, δ) -distinguishable. We denote by $N_{\varepsilon, \delta}(F)$ the maximum number of elements in an (ε, δ) -distinguishable subset of F .

Definition 5.1.1. The number $H_{\varepsilon, \delta}(F) = \log_2 N_{\varepsilon, \delta}(F)$, by analogy with the definition of ε -entropy, is called the (ε, δ) -entropy of F .

Let $f_0 \in F$. We denote by $F_{\lambda\varepsilon}(f_0)$ the set of functions $f \in F$ such that $|f(x) - f_0(x)| \leq \lambda\varepsilon$. It follows immediately from the definition that the expression $\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 H_{\varepsilon, \delta}(F_{\lambda\varepsilon}(f_0))}{\log_2 \delta}$ as a function of λ does not decrease as $\lambda \rightarrow \infty$.

Definition 5.1.2. The number

$$r(F, f_0) = \lim_{\lambda \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 H_{\varepsilon, \delta}(F_{\lambda\varepsilon}(f_0))}{\log_2 \delta}$$

is called the functional “dimension” of F at f_0 . The number $r(F) = \sup (F, f_0)$ is called the functional “dimension” of F .

The functional “dimension” $r(F)$ of a set of functions $F \subset C(G_n)$ has the following properties.

5.1.1. Let $\Phi \subset F$ be a set of functions. Then $r(\Phi) \leq r(F)$. Moreover, if Φ is everywhere dense in F in the uniform metric, then $r(\Phi) = r(F)$.

Proof. The first part of the assertion follows immediately from the definition. For a proof of the second part it is sufficient to show that $r(\Phi, \varphi_0) \geq r(F, \varphi_0)$ for any element $\varphi_0 \in \Phi$. Suppose that the functions f_1, \dots, f_N from a $(2\varepsilon, \delta)$ -distinguishable subset of $F_{\lambda\varepsilon}(\varphi_0)$. Since Φ is everywhere dense in F , there exist functions $\varphi_1, \dots, \varphi_N \in \Phi$ such that $\max_{x \in G_n} |f_i(x) - \varphi_i(x)| \leq \min\left(\frac{\varepsilon}{2}, \lambda\varepsilon\right)$ ($i = 1, 2, \dots, N$). These functions form an (ε, δ) -distinguishable subset of $F_{2\lambda\varepsilon}(\varphi_0)$. Consequently $N_{\varepsilon, \delta}(\Phi_{2\lambda\varepsilon}(\varphi_0)) \geq N_{2\varepsilon, \delta}(F_{\lambda\varepsilon}(\varphi_0))$. Hence $r(\Phi, \varphi_0) \geq r(F, \varphi_0)$.

5.1.2. For any set $F \subset C(G_n)$ we have $r(F) \leq n$.

Proof. Suppose that $f_0 \in F$ and f_1, f_2, \dots, f_p is a maximal set (with respect to p) of pairwise (ε, δ) -distinguishable functions of $F_{\lambda\varepsilon}(f_0)$. Let $\sigma_1, \sigma_2, \dots, \sigma_q$ be a maximal set (with respect to q) of spheres of radius $\delta/3$ in G_n , such that no two of them have common interior points. Then any pair of functions $f_i(x)$ and $f_j(x)$ of the given set satisfies on at least one of the spheres σ_l the inequality $\min_{x \in \sigma_l} |f_i(x) - f_j(x)| \geq \varepsilon$. For the functions $f_i(x)$ and $f_j(x)$ satisfy on some sphere $S_\delta \subset G_n$ the inequality $\min_{x \in S_\delta} |f_i(x) - f_j(x)| \geq \varepsilon$. Since q is maximal, it follows that one of the spheres $\sigma_l \subset S_\delta$. Consequently on this sphere the inequality we need is satisfied. We denote by a_l the centre of the sphere σ_l ($l = 1, 2, \dots, q$). Every set of functions $f_{i_1}, f_{i_2}, \dots, f_{i_r}$ each pair of which has values differing by not less than ε at one and the same point consists of a number $r \leq 2\lambda + 1$ of functions. (All functions are taken from the set indicated above.) Since every pair of functions $f_i(x)$ and $f_j(x)$ has values differing by not less than ε at one of the points a_l at least, we have $p \leq 2\lambda + 1$. But since the spheres $\{\sigma_i\}$ do not intersect, $q \leq C/\delta^n$, where C is a constant depending only on n . Consequently,

$$r(F, f_0) \leq \lim_{\lambda \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 \log_2 (2\lambda + 1)^{\frac{C}{\delta^n}}}{\log_2 \delta} = n.$$

5.1.3. If F is everywhere dense (in the uniform metric) in the space $C(G_n)$, then $r(F) = n$. In particular $r(C(G_n)) = n$.

Proof. By 5.1.1 and 5.1.2 it is sufficient to show that $r(C(G_n)) \geq n$. We denote by $C_\varepsilon(G_n)$ the set of all $f(x) \in C(G_n)$ for which $\max_{x \in G_n} |f(x)| \leq \varepsilon$. Let $\theta > 0$ be a constant such that for any $\delta > 0$ we can find $H = [\theta/\delta^n]$ closed and pairwise non-intersecting spheres $\sigma_1, \sigma_2, \dots, \sigma_H$ of radius δ in G_n . For any system of numbers $\{\alpha_i\}$ ($\alpha_i = \pm 1, i = 1, 2, \dots, H$) we construct a function $f_{\{\alpha_i\}}(x) \in C_\varepsilon(G_n)$ such that $f_{\{\alpha_i\}}(x) = \alpha_i \varepsilon$ for $x \in \sigma_i$ ($i = 1, 2, \dots, H$). These functions are obviously pairwise (ε, δ) -distinguishable. The number of functions $f_{\{\alpha_i\}}(x)$ for all possible sets $\{\alpha_i\}$ is equal to 2^H . Consequently $H_{\varepsilon, \delta}(C_\varepsilon(G_n)) \geq H = [\theta/\delta^n]$. Hence $r(C(G)) \geq n$.

COROLLARY 5.1.1. *The space of all polynomials in n variables has functional "dimension" n .*

In the same way, the following properties are easily proved.

5.1.4. Let G_n^1 and G_n^2 be two non-intersecting closed regions in n -dimensional space, and $F(G_n^1 \cup G_n^2)$ a space of functions, defined and continuous on $G_n^1 \cup G_n^2$. Denote by $F(G_n^1)$ the space of all functions $\varphi(x)$, defined on the set G_n^1 , for which there exists a function $\Phi(x) \in F(G_n^1 \cup G_n^2)$ such that $\varphi(x) \equiv \Phi(x)$ for $x \in G_n^1$. The space $F(G_n^2)$ is defined similarly. Then

$$r(F(G_n^1 \cup G_n^2)) = \max \{ r(F(G_n^1)); r(F(G_n^2)) \}.$$

5.1.5. If F is a linear space, then $r(F) = r(F, f_0)$ for any function $f_0 \in F$. If F is a finite-dimensional linear space, then $r(F) = 0$.

5.1.6. Let F be a linear metric space with metric $\rho(\varphi, \psi)$ between a pair of functions $\varphi, \psi \in F$. We denote by $F(\rho_0)$ the set of all those functions $\varphi \in F$ for which $\rho(\varphi, 0) \leq \rho_0$. Then $r(F) = r(F(\rho_0))$.

COROLLARY 5.1.2. *The set of all polynomials in n variables whose partial derivatives of order p , for any $p = 1, 2, \dots$, are bounded by a constant $0 < K_p < \infty$ has functional "dimension" n .*

5.1.7. Let F be a complete linear metric space and $F = \bigcup_{i=1}^{\infty} F_i$, where $\{F_i\}$ are sets of continuous functions. Then $r(F) = \max_i r(F_i)$.

We now write down the main result on the functional "dimension" of a set of linear superpositions.

5.1.8. Let $q_i = q_i(x_1, x_2, \dots, x_n)$ be continuously differentiable functions of n variables, and $p_i = p_i(x_1, x_2, \dots, x_n)$ continuous functions of n variables ($i = 1, 2, \dots, N$). We denote by $F(G_n, \{p_i\}, \{q_i\})$ the set of super-

positions of the form $\sum_{i=1}^N p_i(x_1, x_2, \dots, x_n) f_i(q_i(x_1, x_2, \dots, x_n))$, where $(x_1, x_2, \dots, x_n) \in G_n$, and $\{f_i(t)\}$ are arbitrary continuous functions of one variable. Then in any region D_n there exists a closed subregion $G_n \subset D_n$ such that

$$r(F(G_n, \{p_i\}, \{q_i\})) \leq 1.$$

For ease of presentation we limit the proof to the case $n = 2$ (§ 3). It is interesting to compare the result 5.1.8 with the following proposition.

5.1.9. Let $\alpha_i(x_1, x_2, \dots, x_n) = \sum_{j=1}^n \alpha_{ij}(x_j) \quad (i = 1, 2, \dots, 2n + 1)$

be the continuous functions involved in Kolmogorov's formula (I). We denote by $\psi(G_n, \alpha_i)$ the space of all functions of the form $\psi(\alpha_i(x_1, x_2, \dots, x_n))$, where $\psi(t)$ is an arbitrary continuous function of one variable and $(x_1, x_2, \dots, x_n) \in G_n$. Then for any i and every region G_n , $r(\psi(G_n, \alpha_i)) = n$ (see 5.1.7).

Let $p_i(x_1, x_2, \dots, x_n)$ be fixed continuous functions of n variables, $q_{1,i}(x_1, x_2, \dots, x_n), q_{2,i}(x_1, x_2, \dots, x_n), \dots, q_{k,i}(x_1, x_2, \dots, x_n)$ fixed continuously differentiable functions of n variables, and $f_i(t_1, t_2, \dots, t_k)$ arbitrary continuous functions of k variables, $k < n$ ($i = 1, 2, \dots, N$). One would expect that the set of superpositions of the form (V) (see Chapter I) has functional "dimension" not greater than k . However, in this direction, only the following partial result has so far been proved.

5.1.10. Denote by $F(\lambda, G_n, \{p_i\}, \{q_{1,i}\}, \dots, \{q_{k,i}\})$ the set of all those continuous functions $\varphi(x_1, x_2, \dots, x_n)$ for which there exist continuous functions $\{f_i(t_1, t_2, \dots, t_k)\}$ such that in G_n .

$$\begin{aligned} & \varphi(x_1, x_2, \dots, x_n) \\ &= \sum_{i=1}^N p_i(x_1, x_2, \dots, x_n) f_i(q_{1,i}(x_1, x_2, \dots, x_n), \dots, q_{k,i}(x_1, x_2, \dots, x_n)) \end{aligned}$$

and

$$\max_i \sup_{(t_1, t_2, \dots, t_k)} |f_i(t_1, t_2, \dots, t_k)| \leq \lambda \sup_{(x_1, x_2, \dots, x_n) \in G_n} |\varphi(x_1, x_2, \dots, x_n)|$$

Then, for any $\lambda < \infty$, in any region D_n there exists a closed subregion $G_n \subset D_n$ such that

$$r(F(\lambda, G_n, \{p_i\}, \{q_{1,i}\}, \dots, \{q_{k,i}\}), 0) \leq k.$$

From the last result and Banach's open mapping theorem there follows

COROLLARY 5.1.3. For any continuous functions p_i and continuously differentiable functions $q_{1,i}, q_{2,i}, \dots, q_{k,i}, k < n$ ($i = 1, 2, \dots, N$) and every region G_n there exists a continuous function that is not equal in G_n to any superposition of the form (V).

§ 2. (ε, δ) -entropy of the set of linear superpositions

We denote by $S(\delta, z)$ the disc of radius δ with centre at z . Let $p(z) = p(x, y)$ and $q(z) = q(x, y)$ be functions defined in a closed region G of the x, y -plane and having the properties:

a) $p(x, y), \frac{\partial q(x, y)}{\partial x}, \frac{\partial q(x, y)}{\partial y}$ are continuous in G and have modulus of continuity $\omega(\delta)$,

b) the inequalities $0 < \gamma \leq |\text{grad}[q(r)]| \leq \frac{1}{\gamma}$ and $|p(z)| \leq \frac{1}{\gamma}$, where γ is some constant, are satisfied everywhere in G .

LEMMA 5.2.1. Let $S(\delta, z) \subset G$ and let $\mu_q(t)$ be the function equal to $2 \sqrt{\delta^2 - (t - q(z))^2} |\text{grad}[q(z)]|^{-2}$ on

$$q(z) - \delta |\text{grad}[q(z)]| \leq t \leq q(z) + \delta |\text{grad}[q(z)]|$$

and equal to zero elsewhere. Then

$$\int_{-\infty}^{\infty} |\mu_q(t) - h_1(e(q, t) \cap S(\delta, z))| dt \leq c_1(\gamma) \omega(\delta) \delta^2,$$

where $c_1(\gamma)$ is a constant depending only on γ .

Proof. Let $[a, b] \subset e(q, t) \cap S(\delta, z)$ be the segment of the level curve $e(q, t)$, endpoints a and b , lying on the boundary of $S(\delta, z)$; $[z, a]$ and $[z, b]$ the vectors with origin at z and endpoints at a and b , respectively;

$$\alpha_1 = \gamma(\overrightarrow{[z, a]}, \text{grad}[q(z)]), \alpha_2 = \gamma(\overrightarrow{[z, b]}, \text{grad}[q(z)]).$$

We have

$$\begin{aligned} |t - q(z)| &= |q(a) - q(z)| = \left| \int_{s \in [z, a]} \frac{\partial q}{\partial s} ds \right| \\ &= \delta \cos \alpha_1 |\text{grad}[q(z)]| (1 + o(1) \omega(\delta)) \end{aligned}$$