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## CHAPTER 5. — DIMENSION OF THE SPACE OF LINEAR SUPERPOSITIONS

In this chapter we present a calculation of the functional dimension of the space of functions representable by means of linear superpositions and prove that a representation of analytic functions by means superpositions of smooth functions can not be stable.

### § 1. $(\varepsilon, \delta)$ -entropy and the “dimension” of function spaces

Let  $G_n$  be a closed region of  $n$ -dimensional euclidean space, and  $C(G_n)$  the space of all functions continuous in  $G_n$ . Two functions  $f_1(x), f_2(x) \in C(G_n)$  are called  $(\varepsilon, \delta)$ -distinguishable if there exists an  $n$ -dimensional closed sphere  $S_\delta \subset G_n$  of radius  $\delta$  such that

$$\min_{x \in S_\delta} |f_1(x) - f_2(x)| \geq \varepsilon.$$

Let  $F \subset C(G_n)$  be a set of continuous functions. A subset  $K \subset F$  is called  $(\varepsilon, \delta)$ -distinguishable if any two of its elements are  $(\varepsilon, \delta)$ -distinguishable. We denote by  $N_{\varepsilon, \delta}(F)$  the maximum number of elements in an  $(\varepsilon, \delta)$ -distinguishable subset of  $F$ .

*Definition 5.1.1.* The number  $H_{\varepsilon, \delta}(F) = \log_2 N_{\varepsilon, \delta}(F)$ , by analogy with the definition of  $\varepsilon$ -entropy, is called the  $(\varepsilon, \delta)$ -entropy of  $F$ .

Let  $f_0 \in F$ . We denote by  $F_{\lambda \varepsilon}(f_0)$  the set of functions  $f \in F$  such that  $|f(x) - f_0(x)| \leq \lambda \varepsilon$ . It follows immediately from the definition that the expression  $\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 H_{\varepsilon, \delta}(F_{\lambda \varepsilon}(f_0))}{\log_2 \delta}$  as a function of  $\lambda$  does not decrease as  $\lambda \rightarrow \infty$ .

*Definition 5.1.2.* The number

$$r(F, f_0) = \lim_{\lambda \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 H_{\varepsilon, \delta}(F_{\lambda \varepsilon}(f_0))}{\log_2 \delta}$$

is called the functional “dimension” of  $F$  at  $f_0$ . The number  $r(F) = \sup (F, f_0)$  is called the functional “dimension” of  $F$ .

The functional “dimension”  $r(F)$  of a set of functions  $F \subset C(G_n)$  has the following properties.

5.1.1. Let  $\Phi \subset F$  be a set of functions. Then  $r(\Phi) \leq r(F)$ . Moreover, if  $\Phi$  is everywhere dense in  $F$  in the uniform metric, then  $r(\Phi) = r(F)$ .

*Proof.* The first part of the assertion follows immediately from the definition. For a proof of the second part it is sufficient to show that  $r(\Phi, \varphi_0) \geq r(F, \varphi_0)$  for any element  $\varphi_0 \in \Phi$ . Suppose that the functions  $f_1, \dots, f_N$  from a  $(2\varepsilon, \delta)$ -distinguishable subset of  $F_{\lambda\varepsilon}(\varphi_0)$ . Since  $\Phi$  is everywhere dense in  $F$ , there exist functions  $\varphi_1, \dots, \varphi_N \in \Phi$  such that  $\max_{x \in G_n} |f_i(x) - \varphi_i(x)| \leq \min\left(\frac{\varepsilon}{2}, \lambda\varepsilon\right)$  ( $i = 1, 2, \dots, N$ ). These functions form an  $(\varepsilon, \delta)$ -distinguishable subset of  $F_{2\lambda\varepsilon}(\varphi_0)$ . Consequently  $N_{\varepsilon, \delta}(\Phi_{2\lambda\varepsilon}(\varphi_0)) \geq N_{2\varepsilon, \delta}(F_{\lambda\varepsilon}(\varphi_0))$ . Hence  $r(\Phi, \varphi_0) \geq r(F, \varphi_0)$ .

5.1.2. For any set  $F \subset C(G_n)$  we have  $r(F) \leq n$ .

*Proof.* Suppose that  $f_0 \in F$  and  $f_1, f_2, \dots, f_p$  is a maximal set (with respect to  $p$ ) of pairwise  $(\varepsilon, \delta)$ -distinguishable functions of  $F_{\lambda\varepsilon}(f_0)$ . Let  $\sigma_1, \sigma_2, \dots, \sigma_q$  be a maximal set (with respect to  $q$ ) of spheres of radius  $\delta/3$  in  $G_n$ , such that no two of them have common interior points. Then any pair of functions  $f_i(x)$  and  $f_j(x)$  of the given set satisfies on at least one of the spheres  $\sigma_l$  the inequality  $\min_{x \in \sigma_l} |f_i(x) - f_j(x)| \geq \varepsilon$ . For the functions  $f_i(x)$  and  $f_j(x)$  satisfy on some sphere  $S_\delta \subset G_n$  the inequality  $\min_{x \in S_\delta} |f_i(x) - f_j(x)| \geq \varepsilon$ . Since  $q$  is maximal, it follows that one of the spheres  $\sigma_l \subset S_\delta$ . Consequently on this sphere the inequality we need is satisfied. We denote by  $a_l$  the centre of the sphere  $\sigma_l$  ( $l = 1, 2, \dots, q$ ). Every set of functions  $f_{i_1}, f_{i_2}, \dots, f_{i_r}$  each pair of which has values differing by not less than  $\varepsilon$  at one and the same point consists of a number  $r \leq 2\lambda + 1$  of functions. (All functions are taken from the set indicated above.) Since every pair of functions  $f_i(x)$  and  $f_j(x)$  has values differing by not less than  $\varepsilon$  at one of the points  $a_l$  at least, we have  $p \leq 2\lambda + 1$ . But since the spheres  $\{\sigma_i\}$  do not intersect,  $q \leq C/\delta^n$ , where  $C$  is a constant depending only on  $n$ . Consequently,

$$r(F, f_0) \leq \lim_{\lambda \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 \log_2 (2\lambda + 1)^{\frac{C}{\delta^n}}}{\log_2 \delta} = n.$$

5.1.3. If  $F$  is everywhere dense (in the uniform metric) in the space  $C(G_n)$ , then  $r(F) = n$ . In particular  $r(C(G_n)) = n$ .

*Proof.* By 5.1.1 and 5.1.2 it is sufficient to show that  $r(C(G_n)) \geq n$ . We denote by  $C_\varepsilon(G_n)$  the set of all  $f(x) \in C(G_n)$  for which  $\max_{x \in G_n} |f(x)| \leq \varepsilon$ . Let  $\theta > 0$  be a constant such that for any  $\delta > 0$  we can find  $H = [\theta/\delta^n]$  closed and pairwise non-intersecting spheres  $\sigma_1, \sigma_2, \dots, \sigma_H$  of radius  $\delta$  in  $G_n$ . For any system of numbers  $\{\alpha_i\}$  ( $\alpha_i = \pm 1, i = 1, 2, \dots, H$ ) we construct a function  $f_{\{\alpha_i\}}(x) \in C_\varepsilon(G_n)$  such that  $f_{\{\alpha_i\}}(x) = \alpha_i \varepsilon$  for  $x \in \sigma_i$  ( $i = 1, 2, \dots, H$ ). These functions are obviously pairwise  $(\varepsilon, \delta)$ -distinguishable. The number of functions  $f_{\{\alpha_i\}}(x)$  for all possible sets  $\{\alpha_i\}$  is equal to  $2^H$ . Consequently  $H_{\varepsilon, \delta}(C_\varepsilon(G_n)) \geq H = [\theta/\delta^n]$ . Hence  $r(C(G)) \geq n$ .

**COROLLARY 5.1.1.** *The space of all polynomials in  $n$  variables has functional "dimension"  $n$ .*

In the same way, the following properties are easily proved.

5.1.4. Let  $G_n^1$  and  $G_n^2$  be two non-intersecting closed regions in  $n$ -dimensional space, and  $F(G_n^1 \cup G_n^2)$  a space of functions, defined and continuous on  $G_n^1 \cup G_n^2$ . Denote by  $F(G_n^1)$  the space of all functions  $\varphi(x)$ , defined on the set  $G_n^1$ , for which there exists a function  $\Phi(x) \in F(G_n^1 \cup G_n^2)$  such that  $\varphi(x) \equiv \Phi(x)$  for  $x \in G_n^1$ . The space  $F(G_n^2)$  is defined similarly. Then

$$r(F(G_n^1 \cup G_n^2)) = \max \{ r(F(G_n^1)); r(F(G_n^2)) \}.$$

5.1.5. If  $F$  is a linear space, then  $r(F) = r(F, f_0)$  for any function  $f_0 \in F$ . If  $F$  is a finite-dimensional linear space, then  $r(F) = 0$ .

5.1.6. Let  $F$  be a linear metric space with metric  $\rho(\varphi, \psi)$  between a pair of functions  $\varphi, \psi \in F$ . We denote by  $F(\rho_0)$  the set of all those functions  $\varphi \in F$  for which  $\rho(\varphi, 0) \leq \rho_0$ . Then  $r(F) = r(F(\rho_0))$ .

**COROLLARY 5.1.2.** *The set of all polynomials in  $n$  variables whose partial derivatives of order  $p$ , for any  $p = 1, 2, \dots$ , are bounded by a constant  $0 < K_p < \infty$  has functional "dimension"  $n$ .*

5.1.7. Let  $F$  be a complete linear metric space and  $F = \bigcup_{i=1}^{\infty} F_i$ , where  $\{F_i\}$  are sets of continuous functions. Then  $r(F) = \max_i r(F_i)$ .

We now write down the main result on the functional "dimension" of a set of linear superpositions.

5.1.8. Let  $q_i = q_i(x_1, x_2, \dots, x_n)$  be continuously differentiable functions of  $n$  variables, and  $p_i = p_i(x_1, x_2, \dots, x_n)$  continuous functions of  $n$  variables ( $i = 1, 2, \dots, N$ ). We denote by  $F(G_n, \{p_i\}, \{q_i\})$  the set of super-

positions of the form  $\sum_{i=1}^N p_i(x_1, x_2, \dots, x_n) f_i(q_i(x_1, x_2, \dots, x_n))$ , where  $(x_1, x_2, \dots, x_n) \in G_n$ , and  $\{f_i(t)\}$  are arbitrary continuous functions of one variable. Then in any region  $D_n$  there exists a closed subregion  $G_n \subset D_n$  such that

$$r(F(G_n, \{p_i\}, \{q_i\})) \leq 1.$$

For ease of presentation we limit the proof to the case  $n = 2$  (§ 3). It is interesting to compare the result 5.1.8 with the following proposition.

$$5.1.9. \text{ Let } \alpha_i(x_1, x_2, \dots, x_n) = \sum_{j=1}^n \alpha_{ij}(x_j) \quad (i = 1, 2, \dots, 2n+1)$$

be the continuous functions involved in Kolmogorov's formula (I). We denote by  $\psi(G_n, \alpha_i)$  the space of all functions of the form  $\psi(\alpha_i(x_1, x_2, \dots, x_n))$ , where  $\psi(t)$  is an arbitrary continuous function of one variable and  $(x_1, x_2, \dots, x_n) \in G_n$ . Then for any  $i$  and every region  $G_n$ ,  $r(\psi(G_n, \alpha_i)) = n$  (see 5.1.7).

Let  $p_i(x_1, x_2, \dots, x_n)$  be fixed continuous functions of  $n$  variables,  $q_{1,i}(x_1, x_2, \dots, x_n), q_{2,i}(x_1, x_2, \dots, x_n), \dots, q_{k,i}(x_1, x_2, \dots, x_n)$  fixed continuously differentiable functions of  $n$  variables, and  $f_i(t_1, t_2, \dots, t_k)$  arbitrary continuous functions of  $k$  variables,  $k < n$  ( $i = 1, 2, \dots, N$ ). One would expect that the set of superpositions of the form (V) (see Chapter I) has functional "dimension" not greater than  $k$ . However, in this direction, only the following partial result has so far been proved.

5.1.10. Denote by  $F(\lambda, G_n, \{p_i\}, \{q_{1,i}\}, \dots, \{q_{k,i}\})$  the set of all those continuous functions  $\varphi(x_1, x_2, \dots, x_n)$  for which there exist continuous functions  $\{f_i(t_1, t_2, \dots, t_k)\}$  such that in  $G_n$ .

$$\begin{aligned} & \varphi(x_1, x_2, \dots, x_n) \\ &= \sum_{i=1}^N p_i(x_1, x_2, \dots, x_n) f_i(q_{1,i}(x_1, x_2, \dots, x_n), \dots, q_{k,i}(x_1, x_2, \dots, x_n)) \end{aligned}$$

and

$$\max_i \sup_{(t_1, t_2, \dots, t_k)} |f_i(t_1, t_2, \dots, t_k)| \leq \lambda \sup_{(x_1, x_2, \dots, x_n) \in G_n} |\varphi(x_1, x_2, \dots, x_n)|$$

Then, for any  $\lambda < \infty$ , in any region  $D_n$  there exists a closed subregion  $G_n \subset D_n$  such that

$$r(F(\lambda, G_n, \{p_i\}, \{q_{1,i}\}, \dots, \{q_{k,i}\}), 0) \leq k.$$

From the last result and Banach's open mapping theorem there follows

COROLLARY 5.1.3. For any continuous functions  $p_i$  and continuously differentiable functions  $q_{1,i}, q_{2,i}, \dots, q_{k,i}, k < n$  ( $i = 1, 2, \dots, N$ ) and every region  $G_n$  there exists a continuous function that is not equal in  $G_n$  to any superposition of the form (V).

## § 2. $(\varepsilon, \delta)$ -entropy of the set of linear superpositions

We denote by  $S(\delta, z)$  the disc of radius  $\delta$  with centre at  $z$ . Let  $p(z) = p(x, y)$  and  $q(z) = q(x, y)$  be functions defined in a closed region  $G$  of the  $x, y$ -plane and having the properties:

a)  $p(x, y), \frac{\partial q(x, y)}{\partial x}, \frac{\partial q(x, y)}{\partial y}$  are continuous in  $G$  and have modulus of continuity  $\omega(\delta)$ ,

b) the inequalities  $0 < \gamma \leq |\text{grad}[q(r)]| \leq \frac{1}{\gamma}$  and  $|p(z)| \leq \frac{1}{\gamma}$ , where  $\gamma$  is some constant, are satisfied everywhere in  $G$ .

LEMMA 5.2.1. Let  $S(\delta, z) \subset G$  and let  $\mu_q(t)$  be the function equal to  $2 \sqrt{\delta^2 - (t - q(z))^2} |\text{grad}[q(z)]|^{-2}$  on

$$q(z) - \delta |\text{grad}[q(z)]| \leq t \leq q(z) + \delta |\text{grad}[q(z)]|$$

and equal to zero elsewhere. Then

$$\int_{-\infty}^{\infty} |\mu_q(t) - h_1(e(q, t) \cap S(\delta, z))| dt \leq c_1(\gamma) \omega(\delta) \delta^2,$$

where  $c_1(\gamma)$  is a constant depending only on  $\gamma$ .

*Proof.* Let  $[a, b] \subset e(q, t) \cap S(\delta, z)$  be the segment of the level curve  $e(q, t)$ , endpoints  $a$  and  $b$ , lying on the boundary of  $S(\delta, z)$ ;  $[z, a]$  and  $[z, b]$  the vectors with origin at  $z$  and endpoints at  $a$  and  $b$ , respectively;

$$\alpha_1 = \gamma(\overrightarrow{[z, a]}, \text{grad}[q(z)]), \alpha_2 = \gamma(\overrightarrow{[z, b]}, \text{grad}[q(z)]).$$

We have

$$\begin{aligned} |t - q(z)| &= |q(a) - q(z)| = \left| \int_{s \in [z, a]} \frac{\partial q}{\partial s} ds \right| \\ &= \delta \cos \alpha_1 |\text{grad}[q(z)]| (1 + o(1) \omega(\delta)) \end{aligned}$$