**Zeitschrift:** L'Enseignement Mathématique

**Herausgeber:** Commission Internationale de l'Enseignement Mathématique

**Band:** 23 (1977)

**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** QUADRATIC FORMS IN AN ADELIC SETTING

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**Kapitel:** 2. The Mean Value Formula

**DOI:** https://doi.org/10.5169/seals-48916

# QUADRATIC FORMS IN AN ADELIC SETTING [1]

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## 1. INTRODUCTION

The connection between Siegel's main theorem in the analytical theory of quadratic forms and the determination of the Tamagawa number of the orthogonal group has been discussed in expository articles by Kneser [1] and Tamagawa [5]. Both of these papers, however, consider only a special case of Siegel's general theorem, namely the number of representations of a quadratic form by itself. In the present paper we consider the problem of representing a positive definite form by another positive definite form. Siegel's main theorem is derived from an adelic integral formula, Ono's "mean value theorem", which is the analogue for the adelized orthogonal group of Siegel's mean value theorem in the geometry of numbers.

## 2. THE MEAN VALUE FORMULA

The adelic mean value formula generalizes Siegel's mean value theorem in the geometry of numbers [3]. We first describe Siegel's theorem as reformulated by Weil [8].

Let $\Phi$ be a continuous function of $\mathbf{R}^n$ $(n \geqslant 2)$ with compact support. Then

$$\Phi(g) = \sum_{X \in \mathbf{Z}^n - \{0\}} \Phi(gx)$$

defines a function on $SL_n(\mathbf{R})$, right invariant by $SL_n(\mathbf{Z})$. According to Siegel's theorem, $SL_n(\mathbf{R}) / SL_n(\mathbf{Z})$ has finite measure, $\Phi$ is integrable on this space, and

$$\frac{\int\limits_{SL_n(\mathbf{R})/SL_n(\mathbf{Z})} \Phi(g)\, dg}{\int\limits_{SL_n(\mathbf{R})/SL_n(\mathbf{Z})} dg} = \int\limits_{\mathbf{R}^n} \Phi(x)\, dx\,.$$

In Siegel's original formulation, $\Phi$ is taken to be the characteristic function

[1]) The author would like to express his appreciation to Professor T. Ono for his valuable advice.

of a compact set $K$. The right-hand side of the above formula reduces to the volume of $K$, while the left-hand side gives the mean value of

$$\text{card}\,(L - \{0\} \cap K)\,,$$

as $L$ varies over all $\mathbf{Z}$-lattices in $\mathbf{R}^n$ with volume 1.

We turn now to the adelic mean value formula. Let $G$ be a linear algebraic group defined over $\mathbf{Q}$, and let $X$ be an algebraic homogeneous space for $G$, defined over $\mathbf{Q}$. For $\xi \in X$, let $G_\xi = \{g \in G : g\xi = \xi\}$. We assume that

a) $X$ has at least one rational point

b) for any $\xi \in X_{\mathbf{C}}$, both $G_{\mathbf{C}}$ and $(G_\xi)_{\mathbf{C}}$ have finite fundamental groups

c) for any extension field $K$ of $\mathbf{Q}$, $G_K$ acts transitively on $X_K$.

We then have the following result.

THEOREM (Ono [2]). There are canonical measures on the adele spaces $G_A$ and $X_A$ such that, given any continuous function $\Phi$ on $X_A$ with compact support,

$$(A) \qquad \frac{\int_{G_A/G_{\mathbf{Q}}} \sum_{x \in X_{\mathbf{Q}}} \Phi(gx)\, dg}{\tau(G_\xi)} = \int_{X_A} \Phi(x)\, dx\,,$$

where $\xi$ is any element of $X_{\mathbf{Q}}$, and $\tau(G_\xi)$ = the invariant measure of $(G_\xi)_A \,/\, (G_\xi)_{\mathbf{Q}}$. The analogy to the previous mean value theorem is clear in the cases when $\tau(G) = \tau(G_\xi)$.

## 3. FORMULATION OF SIEGEL'S THEOREM

Let $S$ and $T$ be square matrices with integral entries of size $m$ and $n$, respectively. We assume that both are positive definite. For any matrix $x$, denote $S[x] = {}^t xSx$ (when defined). Let $A(S, T)$ = the number of integral $m \times n$ matrices $x$ such that $S[x] = T$. For each positive integer $q$, let $A_q(S, T)$ = the number of integral $m \times n$ matrices $x$, mod $q$, such that $S[x] \equiv T \pmod{q}$.

A positive definite integral matrix $S'$ is said to be in the same class as $S$ if $S' = S[U]$, for some $U \varepsilon SL(m, \mathbf{Z})$. $S'$ is in the same genus as $S$ if for each $q$, there exists $U \varepsilon SL(m, \mathbf{Z})$ such that $S' \equiv S[U] \pmod{q}$. Let $S_1, \ldots, S_h$ be the representatives of the classes in genus $(S)$. Let $E(S_i)$ = the finite group consisting of all $U \varepsilon SL(m, \mathbf{Z})$ such that $S_i[U] = S_i$, and put