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CHAPTER 5. — DIMENSION OF THE SPACE OF LINEAR SUPERPOSITIONS

In this chapter we present a calculation of the functional dimension of the space of functions representable by means of linear superpositions and prove that a representation of analytic functions by means superpositions of smooth functions can not be stable.

§ 1.  $(\varepsilon, \delta)$ -entropy and the “dimension” of function spaces

Let  $G_n$  be a closed region of  $n$ -dimensional euclidean space, and  $C(G_n)$  the space of all functions continuous in  $G_n$ . Two functions  $f_1(x), f_2(x) \in C(G_n)$  are called  $(\varepsilon, \delta)$ -distinguishable if there exists an  $n$ -dimensional closed sphere  $S_\delta \subset G_n$  of radius  $\delta$  such that

$$\min_{x \in S_\delta} |f_1(x) - f_2(x)| \geq \varepsilon.$$

Let  $F \subset C(G_n)$  be a set of continuous functions. A subset  $K \subset F$  is called  $(\varepsilon, \delta)$ -distinguishable if any two of its elements are  $(\varepsilon, \delta)$ -distinguishable. We denote by  $N_{\varepsilon, \delta}(F)$  the maximum number of elements in an  $(\varepsilon, \delta)$ -distinguishable subset of  $F$ .

*Definition 5.1.1.* The number  $H_{\varepsilon, \delta}(F) = \log_2 N_{\varepsilon, \delta}(F)$ , by analogy with the definition of  $\varepsilon$ -entropy, is called the  $(\varepsilon, \delta)$ -entropy of  $F$ .

Let  $f_0 \in F$ . We denote by  $F_{\lambda\varepsilon}(f_0)$  the set of functions  $f \in F$  such that  $|f(x) - f_0(x)| \leq \lambda\varepsilon$ . It follows immediately from the definition that the expression  $\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 H_{\varepsilon, \delta}(F_{\lambda\varepsilon}(f_0))}{\log_2 \delta}$  as a function of  $\lambda$  does not decrease as  $\lambda \rightarrow \infty$ .

*Definition 5.1.2.* The number

$$r(F, f_0) = \lim_{\lambda \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 H_{\varepsilon, \delta}(F_{\lambda\varepsilon}(f_0))}{\log_2 \delta}$$

is called the functional “dimension” of  $F$  at  $f_0$ . The number  $r(F) = \sup (F, f_0)$  is called the functional “dimension” of  $F$ .

The functional “dimension”  $r(F)$  of a set of functions  $F \subset C(G_n)$  has the following properties.

5.1.1. Let  $\Phi \subset F$  be a set of functions. Then  $r(\Phi) \leq r(F)$ . Moreover, if  $\Phi$  is everywhere dense in  $F$  in the uniform metric, then  $r(\Phi) = r(F)$ .

*Proof.* The first part of the assertion follows immediately from the definition. For a proof of the second part it is sufficient to show that  $r(\Phi, \varphi_0) \geq r(F, \varphi_0)$  for any element  $\varphi_0 \in \Phi$ . Suppose that the functions  $f_1, \dots, f_N$  from a  $(2\varepsilon, \delta)$ -distinguishable subset of  $F_{\lambda\varepsilon}(\varphi_0)$ . Since  $\Phi$  is everywhere dense in  $F$ , there exist functions  $\varphi_1, \dots, \varphi_N \in \Phi$  such that  $\max_{x \in G_n} |f_i(x) - \varphi_i(x)| \leq \min\left(\frac{\varepsilon}{2}, \lambda\varepsilon\right)$  ( $i = 1, 2, \dots, N$ ). These functions form an  $(\varepsilon, \delta)$ -distinguishable subset of  $F_{2\lambda\varepsilon}(\varphi_0)$ . Consequently  $N_{\varepsilon, \delta}(\Phi_{2\lambda\varepsilon}(\varphi_0)) \geq N_{2\varepsilon, \delta}(F_{\lambda\varepsilon}(\varphi_0))$ . Hence  $r(\Phi, \varphi_0) \geq r(F, \varphi_0)$ .

5.1.2. For any set  $F \subset C(G_n)$  we have  $r(F) \leq n$ .

*Proof.* Suppose that  $f_0 \in F$  and  $f_1, f_2, \dots, f_p$  is a maximal set (with respect to  $p$ ) of pairwise  $(\varepsilon, \delta)$ -distinguishable functions of  $F_{\lambda\varepsilon}(f_0)$ . Let  $\sigma_1, \sigma_2, \dots, \sigma_q$  be a maximal set (with respect to  $q$ ) of spheres of radius  $\delta/3$  in  $G_n$ , such that no two of them have common interior points. Then any pair of functions  $f_i(x)$  and  $f_j(x)$  of the given set satisfies on at least one of the spheres  $\sigma_l$  the inequality  $\min_{x \in \sigma_l} |f_i(x) - f_j(x)| \geq \varepsilon$ . For the functions  $f_i(x)$  and  $f_j(x)$  satisfy on some sphere  $S_\delta \subset G_n$  the inequality  $\min_{x \in S_\delta} |f_i(x) - f_j(x)| \geq \varepsilon$ . Since  $q$  is maximal, it follows that one of the spheres  $\sigma_l \subset S_\delta$ . Consequently on this sphere the inequality we need is satisfied. We denote by  $a_l$  the centre of the sphere  $\sigma_l$  ( $l = 1, 2, \dots, q$ ). Every set of functions  $f_{i_1}, f_{i_2}, \dots, f_{i_r}$  each pair of which has values differing by not less than  $\varepsilon$  at one and the same point consists of a number  $r \leq 2\lambda + 1$  of functions. (All functions are taken from the set indicated above.) Since every pair of functions  $f_i(x)$  and  $f_j(x)$  has values differing by not less than  $\varepsilon$  at one of the points  $a_l$  at least, we have  $p \leq 2\lambda + 1$ . But since the spheres  $\{\sigma_i\}$  do not intersect,  $q \leq C/\delta^n$ , where  $C$  is a constant depending only on  $n$ . Consequently,

$$r(F, f_0) \leq \lim_{\lambda \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 \log_2 (2\lambda + 1)^{\frac{C}{\delta^n}}}{\log_2 \delta} = n.$$

5.1.3. If  $F$  is everywhere dense (in the uniform metric) in the space  $C(G_n)$ , then  $r(F) = n$ . In particular  $r(C(G_n)) = n$ .

*Proof.* By 5.1.1 and 5.1.2 it is sufficient to show that  $r(C(G_n)) \geq n$ . We denote by  $C_\varepsilon(G_n)$  the set of all  $f(x) \in C(G_n)$  for which  $\max_{x \in G_n} |f(x)| \leq \varepsilon$ . Let  $\theta > 0$  be a constant such that for any  $\delta > 0$  we can find  $H = [\theta/\delta^n]$  closed and pairwise non-intersecting spheres  $\sigma_1, \sigma_2, \dots, \sigma_H$  of radius  $\delta$  in  $G_n$ . For any system of numbers  $\{\alpha_i\}$  ( $\alpha_i = \pm 1, i = 1, 2, \dots, H$ ) we construct a function  $f_{\{\alpha_i\}}(x) \in C_\varepsilon(G_n)$  such that  $f_{\{\alpha_i\}}(x) = \alpha_i \varepsilon$  for  $x \in \sigma_i$  ( $i = 1, 2, \dots, H$ ). These functions are obviously pairwise  $(\varepsilon, \delta)$ -distinguishable. The number of functions  $f_{\{\alpha_i\}}(x)$  for all possible sets  $\{\alpha_i\}$  is equal to  $2^H$ . Consequently  $H_{\varepsilon, \delta}(C_\varepsilon(G_n)) \geq H = [\theta/\delta^n]$ . Hence  $r(C(G)) \geq n$ .

**COROLLARY 5.1.1.** *The space of all polynomials in  $n$  variables has functional "dimension"  $n$ .*

In the same way, the following properties are easily proved.

5.1.4. Let  $G_n^1$  and  $G_n^2$  be two non-intersecting closed regions in  $n$ -dimensional space, and  $F(G_n^1 \cup G_n^2)$  a space of functions, defined and continuous on  $G_n^1 \cup G_n^2$ . Denote by  $F(G_n^1)$  the space of all functions  $\varphi(x)$ , defined on the set  $G_n^1$ , for which there exists a function  $\Phi(x) \in F(G_n^1 \cup G_n^2)$  such that  $\varphi(x) \equiv \Phi(x)$  for  $x \in G_n^1$ . The space  $F(G_n^2)$  is defined similarly. Then

$$r(F(G_n^1 \cup G_n^2)) = \max \{ r(F(G_n^1)); r(F(G_n^2)) \}.$$

5.1.5. If  $F$  is a linear space, then  $r(F) = r(F, f_0)$  for any function  $f_0 \in F$ . If  $F$  is a finite-dimensional linear space, then  $r(F) = 0$ .

5.1.6. Let  $F$  be a linear metric space with metric  $\rho(\varphi, \psi)$  between a pair of functions  $\varphi, \psi \in F$ . We denote by  $F(\rho_0)$  the set of all those functions  $\varphi \in F$  for which  $\rho(\varphi, 0) \leq \rho_0$ . Then  $r(F) = r(F(\rho_0))$ .

**COROLLARY 5.1.2.** *The set of all polynomials in  $n$  variables whose partial derivatives of order  $p$ , for any  $p = 1, 2, \dots$ , are bounded by a constant  $0 < K_p < \infty$  has functional "dimension"  $n$ .*

5.1.7. Let  $F$  be a complete linear metric space and  $F = \bigcup_{i=1}^{\infty} F_i$ , where  $\{F_i\}$  are sets of continuous functions. Then  $r(F) = \max_i r(F_i)$ .

We now write down the main result on the functional "dimension" of a set of linear superpositions.

5.1.8. Let  $q_i = q_i(x_1, x_2, \dots, x_n)$  be continuously differentiable functions of  $n$  variables, and  $p_i = p_i(x_1, x_2, \dots, x_n)$  continuous functions of  $n$  variables ( $i = 1, 2, \dots, N$ ). We denote by  $F(G_n, \{p_i\}, \{q_i\})$  the set of super-

positions of the form  $\sum_{i=1}^N p_i(x_1, x_2, \dots, x_n) f_i(q_i(x_1, x_2, \dots, x_n))$ , where  $(x_1, x_2, \dots, x_n) \in G_n$ , and  $\{f_i(t)\}$  are arbitrary continuous functions of one variable. Then in any region  $D_n$  there exists a closed subregion  $G_n \subset D_n$  such that

$$r(F(G_n, \{p_i\}, \{q_i\})) \leq 1.$$

For ease of presentation we limit the proof to the case  $n = 2$  (§ 3). It is interesting to compare the result 5.1.8 with the following proposition.

5.1.9. Let  $\alpha_i(x_1, x_2, \dots, x_n) = \sum_{j=1}^n \alpha_{ij}(x_j) \quad (i = 1, 2, \dots, 2n + 1)$

be the continuous functions involved in Kolmogorov's formula (I). We denote by  $\psi(G_n, \alpha_i)$  the space of all functions of the form  $\psi(\alpha_i(x_1, x_2, \dots, x_n))$ , where  $\psi(t)$  is an arbitrary continuous function of one variable and  $(x_1, x_2, \dots, x_n) \in G_n$ . Then for any  $i$  and every region  $G_n$ ,  $r(\psi(G_n, \alpha_i)) = n$  (see 5.1.7).

Let  $p_i(x_1, x_2, \dots, x_n)$  be fixed continuous functions of  $n$  variables,  $q_{1,i}(x_1, x_2, \dots, x_n), q_{2,i}(x_1, x_2, \dots, x_n), \dots, q_{k,i}(x_1, x_2, \dots, x_n)$  fixed continuously differentiable functions of  $n$  variables, and  $f_i(t_1, t_2, \dots, t_k)$  arbitrary continuous functions of  $k$  variables,  $k < n$  ( $i = 1, 2, \dots, N$ ). One would expect that the set of superpositions of the form (V) (see Chapter I) has functional "dimension" not greater than  $k$ . However, in this direction, only the following partial result has so far been proved.

5.1.10. Denote by  $F(\lambda, G_n, \{p_i\}, \{q_{1,i}\}, \dots, \{q_{k,i}\})$  the set of all those continuous functions  $\varphi(x_1, x_2, \dots, x_n)$  for which there exist continuous functions  $\{f_i(t_1, t_2, \dots, t_k)\}$  such that in  $G_n$ .

$$\begin{aligned} & \varphi(x_1, x_2, \dots, x_n) \\ &= \sum_{i=1}^N p_i(x_1, x_2, \dots, x_n) f_i(q_{1,i}(x_1, x_2, \dots, x_n), \dots, q_{k,i}(x_1, x_2, \dots, x_n)) \end{aligned}$$

and

$$\max_i \sup_{(t_1, t_2, \dots, t_k)} |f_i(t_1, t_2, \dots, t_k)| \leq \lambda \sup_{(x_1, x_2, \dots, x_n) \in G_n} |\varphi(x_1, x_2, \dots, x_n)|$$

Then, for any  $\lambda < \infty$ , in any region  $D_n$  there exists a closed subregion  $G_n \subset D_n$  such that

$$r(F(\lambda, G_n, \{p_i\}, \{q_{1,i}\}, \dots, \{q_{k,i}\}), 0) \leq k.$$

From the last result and Banach's open mapping theorem there follows

COROLLARY 5.1.3. For any continuous functions  $p_i$  and continuously differentiable functions  $q_{1,i}, q_{2,i}, \dots, q_{k,i}, k < n$  ( $i = 1, 2, \dots, N$ ) and every region  $G_n$  there exists a continuous function that is not equal in  $G_n$  to any superposition of the form (V).

§ 2.  $(\varepsilon, \delta)$ -entropy of the set of linear superpositions

We denote by  $S(\delta, z)$  the disc of radius  $\delta$  with centre at  $z$ . Let  $p(z) = p(x, y)$  and  $q(z) = q(x, y)$  be functions defined in a closed region  $G$  of the  $x, y$ -plane and having the properties:

a)  $p(x, y), \frac{\partial q(x, y)}{\partial x}, \frac{\partial q(x, y)}{\partial y}$  are continuous in  $G$  and have modulus of continuity  $\omega(\delta)$ ,

b) the inequalities  $0 < \gamma \leq |\text{grad}[q(r)]| \leq \frac{1}{\gamma}$  and  $|p(z)| \leq \frac{1}{\gamma}$ , where  $\gamma$  is some constant, are satisfied everywhere in  $G$ .

LEMMA 5.2.1. Let  $S(\delta, z) \subset G$  and let  $\mu_q(t)$  be the function equal to  $2 \sqrt{\delta^2 - (t - q(z))^2} |\text{grad}[q(z)]|^{-2}$  on

$$q(z) - \delta |\text{grad}[q(z)]| \leq t \leq q(z) + \delta |\text{grad}[q(z)]|$$

and equal to zero elsewhere. Then

$$\int_{-\infty}^{\infty} |\mu_q(t) - h_1(e(q, t) \cap S(\delta, z))| dt \leq c_1(\gamma) \omega(\delta) \delta^2,$$

where  $c_1(\gamma)$  is a constant depending only on  $\gamma$ .

*Proof.* Let  $[a, b] \subset e(q, t) \cap S(\delta, z)$  be the segment of the level curve  $e(q, t)$ , endpoints  $a$  and  $b$ , lying on the boundary of  $S(\delta, z)$ ;  $[z, a]$  and  $[z, b]$  the vectors with origin at  $z$  and endpoints at  $a$  and  $b$ , respectively;

$$\alpha_1 = \gamma(\overrightarrow{[z, a]}, \text{grad}[q(z)]), \alpha_2 = \gamma(\overrightarrow{[z, b]}, \text{grad}[q(z)]).$$

We have

$$\begin{aligned} |t - q(z)| &= |q(a) - q(z)| = \left| \int_{s \in [z, a]} \frac{\partial q}{\partial s} ds \right| \\ &= \delta \cos \alpha_1 |\text{grad}[q(z)]| (1 + o(1) \omega(\delta)) \end{aligned}$$

Hence

$$\delta \sin \alpha_1 = \sqrt{\delta^2 - (t - q(z) + o(\gamma) \delta \omega(\delta))^2} \left| \text{grad } [q(z)] \right|^{-2}$$

and similarly

$$\delta \sin \alpha_2 = \sqrt{\delta^2 - (t - q(z) + o(\gamma) \delta \omega(\delta))^2} \left| \text{grad } [q(z)] \right|^{-2}$$

By b) the size of the angle swept out by the tangent vector to the level curve  $e(q, t)$  on moving along  $[a, b]$  does not exceed  $C_2(\gamma) \omega(\delta)$ . Therefore

$$\begin{aligned} h_1([a, b]) &= \delta (\sin \alpha_1 + \sin \alpha_2) (1 + o(\gamma) \omega(\delta)) \\ &= 2 \sqrt{\delta^2 - (t - q(z) + o(\gamma) \delta \omega(\delta))^2} \left| \text{grad } [q(z)] \right|^{-2} + o(\gamma) \delta \omega(\delta). \end{aligned}$$

If  $\alpha_1 \geq C_3(\gamma) \omega(\delta)$  ( $C_3$  is a sufficiently large constant), then  $[a, b] = e(q, t) \cap S(\delta, z)$ . Consequently, for

$$\left| t - q(z) \right| \leq \theta = \delta \cos [C_3 \omega(\delta)] \left| \text{grad } [q(z)] \right| \times (1 + o(1) \omega(\delta))$$

we have  $h_1(e(q, t) \cap S(\delta, z)) = h_1([a, b])$ . Since for every  $t$  (by b))

$$h_1(e(q, t) \cap S(\delta, z)) \leq C_4(\gamma) \delta (1 + \omega(\delta)),$$

we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \left| h_1(e(q, t) \cap S(\delta, z)) - \mu_q(t) \right| dt = \\ &= \int_{q(z) - \theta}^{q(z) + \theta} \left| h_1(e(q, t) \cap S(\delta, z)) - \mu_q(t) \right| dt + o(\gamma) \delta^2 \omega(\delta). \end{aligned}$$

We now estimate

$$\begin{aligned} &\int_{q(z) - \theta}^{q(z) + \theta} \left| h_1(e(q, t) \cap S(\delta, z)) - \mu_q(t) \right| dt = \\ &= \int_{q(z) - \theta}^{q(z) + \theta} \left| h_1([a, b]) - \mu_q(t) \right| dt \leq \\ &\leq 2 \int_{q(z) - \theta}^{q(z) + \theta} \left( \sqrt{\delta^2 - (t - q(z) + o(\gamma) \delta \omega(\delta))^2} \left| \text{grad } [q(z)] \right|^{-2} \right. \\ &\quad \left. - \sqrt{\delta^2 - (t - q(z))^2} \left| \text{grad } [q(z)] \right|^{-2} \right) dt + o(\gamma) \delta^2 \omega(\delta) \\ &= o(\gamma) \delta^2 \omega(\delta) \int_{-1}^1 \frac{d\tau}{\sqrt{1 - \tau^2}} + o(\gamma) \delta^2 \omega(\delta) = o(\gamma) \delta^2 \omega(\delta). \end{aligned}$$

Here we have the mean value theorem. This proves the lemma.

LEMMA 5.2.2. Let  $p(z), q(z)$  satisfy conditions a) and b);  $S(\delta, z) \subset G$ ; let  $f(t)$  be an arbitrary continuous function, uniformly bounded in modulus by the constant  $m$ . Then

$$\begin{aligned} & \iint_{(u, v) \in S(\delta, z)} p(u, v) f(q(u, v)) \, dudv \\ &= p(z) \left| \operatorname{grad} [q(z)] \right|^{-1} \int_{-\infty}^{\infty} f(t) \mu_q(t) \, dt + \lambda(z) m \delta^2 \omega(\delta), \end{aligned}$$

where  $|\lambda(z)| \leq C_5(\gamma)$ .

*Proof.* Using a) and b) and Lemma 5.2.1 we have

$$\begin{aligned} & \iint_{S(\delta, z)} p(u, v) f(q(u, v)) \, dudv \\ &= p(z) \iint_{(u, v) \in S(\delta, z)} f(q(u, v)) \, dudv + O(1) m \delta^2 \omega(\delta) \\ &= p(z) \int_{-\infty}^{\infty} \left\{ f(t) \int_{s \in e(q, t) \cap S(\delta, z)} \left| \operatorname{grad} [q(s)] \right|^{-2} ds \right\} dt + O(1) m \delta^2 \omega(\delta) \\ &= p(z) \left| \operatorname{grad} [q(z)] \right|^{-1} \int_{-\infty}^{\infty} \left\{ f(t) \int_{s \in e(q, t) \cap S(\delta, z)} ds \right\} dt + O(\gamma) m \delta^2 \omega(\delta) \\ &= p(z) \left| \operatorname{grad} [q(z)] \right|^{-2} \int_{-\infty}^{\infty} f(t) h_1(e(q, t) \cap S(\delta, z)) \, dt + O(\gamma) m \delta^2 \omega(\delta) \\ &= p(z) \left| \operatorname{grad} [q(z)] \right|^{-1} \int_{-\infty}^{\infty} f(t) \mu_q(t) \, dt + O(\gamma) m \delta^2 \omega(\delta). \end{aligned}$$

This proves the lemma.

LEMMA 5.2.3. Suppose that a number  $\alpha > 0$  and functions  $p(z), q(z), f(t)$  satisfying the conditions of Lemma 5.2.2. are given. If for every integer  $k$  such that

$$\min_{z \in G} q(z) \leq t_k = k \delta \frac{\alpha}{m} \leq \max_{z \in G} q(z)$$

and any integer  $l$  such that

$$\min_{z \in G} \left| \operatorname{grad} [q(z)] \right| \leq t'_l = l \frac{\alpha}{m} \leq \max_{z \in G} \left| \operatorname{grad} [q(z)] \right|,$$

the inequality

$$\left| \int_{t_k - t'_l \delta}^{t_k + t'_l \delta} f(t) \sqrt{\delta^2 - \left( \frac{t - t_k}{t'_l} \right)^2} dt \right| \leq \alpha \delta^2$$

is satisfied, then for every disc  $S(\delta, z) \subset G$

$$\left| \iint_{(u, v) \in S(\delta, z)} p(u, v) f(q(u, v)) \, dudv \right| \leq c_6(\gamma) (\alpha\delta^2 + m\delta^2\omega(\delta)).$$

*Proof.* Suppose that a disc  $S(\delta, z) \subset G$  is given. By the condition of the lemma there are integers  $k$  and  $l$  such that  $|q(z) - t_k| \leq \delta\alpha/m$  and  $|| \text{grad}[q(z)] | - t'_l | \leq \alpha/m$ . From Lemma 5.2.2 we obtain

$$\begin{aligned} \left| \iint_{(u, v) \in S(\delta, z)} p(u, v) f(q(u, v)) \, dudv \right| &\leq \frac{|p(z)|}{|\text{grad}[q(z)]|} \left| \int_{-\infty}^{\infty} f(t) \mu_q(t) \, dt \right| \\ &+ c_5(\gamma) m\delta^2\omega(\delta) \leq \frac{2}{\gamma^2} \left| \int_{\substack{q(z) + \\ -\delta|\text{grad}[q(z)]|}}^{\substack{q(z) + \\ +\delta|\text{grad}[q(z)]|}} f(t) \sqrt{\delta^2 - \frac{(t - q(z))^2}{|\text{grad}[q(z)]|^2}} \, dt \right. \\ &\left. - \int_{t_k - t'_l\delta}^{t_k + t'_l\delta} f(t) \sqrt{\delta^2 - \left(\frac{t - t_k}{t'_l}\right)^2} \, dt \right| + \frac{2}{\gamma^2} \alpha\delta^2 + c_5(\gamma) m\delta^2\omega(\delta) \leq \end{aligned}$$

(by the mean value theorem)

$$\begin{aligned} &\leq \frac{2}{\gamma^2} \alpha\delta^2 + c_5(\gamma) m\delta^2\omega(\delta) + \frac{2}{\gamma^2} \left( \int_{-1}^1 \frac{\delta m d\tau}{\sqrt{1 - \tau^2}} \right) \delta \frac{\alpha}{m} \\ &+ \frac{2}{\gamma^2} \left( \int_{-1}^1 \frac{\delta^2 m d\tau}{\sqrt{1 - \tau^2}} \right) \frac{\alpha}{m} \leq c_6(\gamma) (\alpha\delta^2 + m\delta^2\omega(\delta)). \end{aligned}$$

This proves the lemma.

We denote by  $F_m = F_m(D; p_1, p_2, \dots, p_N; q_1, q_2, \dots, q_N)$  the set of superpositions of the form

$$f(x, y) = \sum_{i=1}^N p_i(x, y) f_i(q_i(x, y)), \text{ where } \{p_i(x, y)\}$$

and  $\{q_i(x, y)\}$  are fixed functions, defined in the closed region  $D$  of the  $x, y$  plane and satisfying conditions a) and b) with a constant  $\gamma$  not depending on  $i$  and  $\{f_i(t)\}$  are arbitrary continuous functions, defined on  $\{[a_i, b_i]\} = \{[\min_{z \in D} q_i(z); \max_{z \in D} q_i(z)]\}$  and uniformly bounded in modulus by the constant  $m$ .

**THEOREM 5.2.1.** *There exist constants  $A$  and  $B$  such that if  $\varepsilon > Am\omega(\delta)$  then for the  $(\varepsilon, \delta)$ -entropy of the set of functions  $F_m$ ,  $H_{\varepsilon, \delta}(F_m) \leq \frac{B}{\delta} \left(\frac{m}{\varepsilon}\right)^2$ , where  $A$  and  $B$  depend only on  $\gamma, N$  and  $D$ .*

*Proof.* We put

$$R(f(z), \delta) = \max_{S(\delta, z) \subset D} \left| \frac{1}{\pi\delta^2} \iint_{(u, v) \in S(\delta, z)} f(u, v) \, dudv \right|.$$

We denote by  $\mathcal{H}_{\varepsilon, \delta}(F_m)$  the  $\varepsilon$ -entropy of the space  $F_m$ , taking as the distance between the functions  $f_1(z), f_2(z) \in F_m$  the number  $R(f_1(z) - f_2(z), \delta)$ . The inequality  $H_{2\varepsilon, \delta}(F_m) \leq \mathcal{H}_{\varepsilon, \delta}(F_m)$  holds owing to the fact that if two functions  $f_1(z)$  and  $f_2(z)$  are  $(\varepsilon, \delta)$ -distinguishable, then they are  $\varepsilon$ -distinguishable also in the sense of the metric  $R(f_1(z) - f_2(z), \delta)$ . We now estimate the value of  $\mathcal{H}_{\varepsilon, \delta}(F_m)$ . Let  $k$  and  $l$  be integers such that

$$\min_{z \in D} q_i(z) \leq t_k = k\delta \frac{\alpha}{m} \leq \max_{z \in D} q_i(z)$$

and

$$\min_{z \in D} |\text{grad } [q_i(z)]| \leq t'_l = l \frac{\alpha}{m} \leq \max_{z \in D} |\text{grad } [q_i(z)]|.$$

To compute the function

$$f_\delta(z) = \frac{1}{\pi\delta^2} \iint_{(u, v) \in S(\delta, z)} f(u, v) \, dudv,$$

where  $f(x, y) \in F_m$ ,  $S(\delta, z) \subset D$  to within  $\varepsilon$ , it is sufficient by Lemma 5.2.3 to give the values of

$$v_i(t_k, t'_l) = \frac{1}{\pi\delta^2} \int_{t_k - t'_l\delta}^{t_k + t'_l\delta} f_i(t) \sqrt{\delta^2 - \left(\frac{t - t_k}{t'_l}\right)^2} \, dt$$

to within  $\alpha = \pi\varepsilon / (2NC_B(\gamma))$  and to assume that  $\delta$  is small enough so that

$$\varepsilon > \frac{2NC_B(\gamma) m\omega(\delta)}{\pi} = A(\gamma, N) m\omega(\delta).$$

Since  $|v_i(t_k, t'_l)| \leq C_1 m$ , to write the numbers  $v_i(t_k, t'_l)$  ( $i, k, l$  fixed)  $\log_2(C_1 m/\alpha)$  binary digits are sufficient. Since

$$|v_i(t_{k+1}, t'_l) - v_i(t_k, t'_l)| \leq c_8 \frac{1}{\delta^2} \left( \int_{-1}^1 \frac{\delta m d\tau}{\sqrt{1-\tau^2}} \right) \delta \frac{\alpha}{m} = c_9(\gamma) \alpha$$

(here we again use the mean value theorem), to store the numbers  $v_i(t_{k+1}, t'_l) - v_i(t_k, t'_l)$  to within  $\alpha$ ,  $\log_2 C_9$  binary digits are sufficient. Therefore to write the numbers  $v_i(t_k, t'_l)$  ( $i, l$  fixed;  $k$  any admissible number)

$C_{10}(\gamma) \left[ \log_2 \frac{m}{\alpha} + (b_i - a_i) \frac{m}{\delta \alpha} \right] = \mathcal{H}_{i,l}$  binary digits are sufficient. Consequently the total number of digits sufficient to store all the numbers  $v_i(t_k, t'_l)$  to within  $\alpha$ , that is, to store the functions  $f_\delta(z)$  to within  $\varepsilon$ , is

$$\mathcal{H} = \sum_{i,l} \mathcal{H}_{i,l} \leq N c_{10}(\gamma) \left[ \log_2 \frac{m}{\alpha} + (b_i - a_i) \frac{m}{\delta \alpha} \right] \frac{1}{\gamma} \frac{m}{\alpha} \leq \frac{B(\gamma, N, D)}{\delta} \left( \frac{m}{\varepsilon} \right)^2.$$

This proves the theorem.

### § 3. Functional "dimension" of the space of linear superpositions

Suppose that continuous functions  $p_i(x, y)$  and continuously differentiable functions  $q_i(x, y)$  ( $i=1, 2, \dots, N$ ) are fixed. Let  $G$  be a closed region of the  $x, y$  plane. We denote by  $F = F(G, \{p_i\}, \{q_i\})$  the set of superpositions of the form  $f(x, y) = \sum_{i=1}^N p_i(x, y) f_i(q_i(x, y))$ , where  $(x, y) \in G$  and  $\{f_i(t)\}$  are arbitrary continuous functions of one variable. We are interested in the functional dimension of the set  $F$ .

**THEOREM 5.3.1.** *In every region  $D$  of the  $x, y$  plane there exists a closed subregion  $G \subset D$  such that*

$$r(F(G, \{p_i\}, \{q_i\})) \leq 1.$$

*Proof.* By Theorem 4.5.1, in  $D$  there exists a closed subregion  $G^* \subset D$  such that the set of superpositions  $F(G^*, \{p_i\}, \{q_i\})$  is closed (in the uniform metric) in  $C(G^*)$ , and the functions  $\{q_i(x, y)\}$  satisfy the condition: for any  $i$ , either  $\text{grad}[q_i(x, y)] \neq 0$  on  $G^*$  or  $q_i(x, y) \equiv \text{const}$  on  $G^*$ . We show that  $r(F(G^*, \{p_i\}, \{q_i\})) \leq 1$ . By Banach's open mapping theorem, there exists a constant  $K$  such that for any superposition

$$\sum_{i=1}^N p_i(x, y) f_i(q_i(x, y)) = f(x, y) \in F(G^*, \{p_i\}, \{q_i\}) \text{ there are con-}$$

tinuous functions  $\{f_i^*(t)\}$ , defined on the sets  $\{q_i(G^*)\}$  and satisfying the conditions

$$8) \quad f(x, y) = \sum_{i=1}^N p_i(x, y) f_i^*(q_i(x, y)) \text{ for all } (x, y) \in G^* ;$$

$$9) \quad \max_i \max_{t \in q_i(G^*)} |f_i^*(t)| \geq K \max_{(x, y) \in G^*} |f(x, y)| .$$

Denote by  $F_{\lambda\varepsilon} = F_{\lambda\varepsilon}(G^*, \{p_i\}, \{q_i\})$  the set of superpositions  $f(x, y) \in F(G^*, \{p_i\}, \{q_i\})$  such that  $\max_{(x, y) \in G^*} |f(x, y)| \leq \lambda\varepsilon$ . By Theorem 5.2.1

and (8), (9), there exist constants  $A$  and  $B$  such that if  $\omega(\delta) \leq (\lambda AK)^{-1}$  then  $H_{\varepsilon, \delta}(F_{\lambda\varepsilon}) \leq B(\lambda K)^2/\delta$ . Hence the functional dimension

$$r(F_i(G^*, \{p_i\}, \{q_i\})) \leq \lim_{\lambda \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\log_2 \log_2 \frac{B(\lambda K)^2}{\delta}}{\log_2 \delta} = 1$$

This proves the theorem.

From Theorem 5.3.1 and the properties of functional dimension (§ 1) we have the following result, which is a stronger form of Theorem 4.6.1.

**COROLLARY 5.3.1.** *For any continuous functions  $\{p_i(x, y)\}$  and continuously differentiable functions  $\{q_i(x, y)\}$  and every region  $D$  the set of linear superpositions  $F(D, \{p_i\}, \{q_i\})$  is nowhere dense in any space of functions that has in every region  $G \subset D$  functional "dimension" greater than 1.*

*Remark 5.3.1.* All the results about linear superpositions of the form  $\sum_{i=1}^N p_i(x, y) f_i(q_i(x, y))$  remain valid if we assume that  $\{f_i(t)\}$  are arbitrary bounded measurable functions.

#### § 4. Variation of superpositions of smooth functions

Let  $G_n$  be a closed region of the space of the variables  $x_1, x_2, \dots, x_n$  ( $n \geq 2$ ). A function  $F(x) = F(x_1, x_2, \dots, x_n)$  is called a superposition of order  $s$  generated by the functions of  $k$  ( $k > 1$ ) variables

$$f_{\beta_1, \beta_2, \dots, \beta_\alpha}(t_1, t_2, \dots, t_k) \quad (\alpha = 0, 1, 2, \dots, s; \beta_i = 1, 2, \dots, k)$$

if it is defined in  $G$  by relations



LEMMA 5.4.1. *The inequality*

$$\sup_{x \in G} \left| \tilde{F}(x_1, x_2, \dots, x_n) - F(x_1, x_2, \dots, x_n) \right| \leq A(\mu, s) \varepsilon.$$

holds, where the constant  $A(\mu, s)$  depends only on  $\mu$  and  $s$ .

*Proof.* We proceed by induction on  $s$ . For definiteness suppose that  $k < 1$ . Having verified the statement of the lemma for  $s = 1$  and having made an appropriate inductive assumption for superpositions of order  $s - 1$ , we have

$$\begin{aligned} \sup_{x \in G} \left| \tilde{F}(x_1, x_2, \dots, x_n) - F(x_1, x_2, \dots, x_n) \right| \\ \leq \left| f(\tilde{q}_1, \dots, \tilde{q}_k) - f(q_1, \dots, q_k) \right| + \left| \varphi(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_k) \right| \\ \leq \mu \max_{\beta_1} \sup_{x \in G} \left| \tilde{q}_{\beta_1} - q_{\beta_1} \right| + \varepsilon \leq \mu \cdot A(\mu, s-1) \varepsilon + \varepsilon = A(\mu, s) \varepsilon. \end{aligned}$$

(the last by the inductive assumption). This proves the lemma.

Further, let  $\omega(\delta)$  be the common modulus of continuity of all the functions  $\left\{ \frac{\partial f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)}{\partial t_i} \right\}$  and, in addition, put

$$\varepsilon' = \max_{\alpha, \beta_1, \dots, \beta_\alpha} \sum_{i=1}^k \sup_t \left| \frac{\partial \varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)}{\partial t_i} \right|$$

LEMMA 5.4.2. *We have (for case  $k > 1$ )*

$$\begin{aligned} \tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n) = \sum_{\alpha, \beta_1, \dots, \beta_\alpha} p_{\beta_1, \dots, \beta_\alpha}(x_1, x_2, \dots, x_n) \\ \times \varphi_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}(x_1, \dots, x_n), \dots, q_{\beta_1, \dots, \beta_\alpha, k}(x_1, \dots, x_n)) \\ + R(x_1, x_2, \dots, x_n), \end{aligned}$$

where

$$\left| R(x_1, x_2, \dots, x_n) \right| \leq B(\mu, s, k) [\varepsilon' + \omega(A(\mu, s) \varepsilon)] \varepsilon,$$

$$p_{\beta_1, \dots, \beta_\alpha}(x_1, x_2, \dots, x_n) = \prod_{i=0}^{\alpha-1} \frac{\partial f_{\beta_1, \dots, \beta_i}}{\partial q_{\beta_1, \dots, \beta_{i+1}}}$$

(for  $\alpha=0$   $p(x_1, x_2, \dots, x_n) \equiv 1$ ),

$B(\mu, s, k)$  is a constant depending only on  $\mu, s, k$ . For  $k = 1$  the corresponding equation is slightly different (see Chapter I, (III)):

$$\begin{aligned} & \tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n) \\ = & \sum_{\alpha, \beta_1, \dots, \beta_\alpha} p_{\beta_1, \dots, \beta_\alpha}(x_1, x_2, \dots, x_n) \varphi_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}(x_1, \dots, x_n)) \\ & + q_{\beta_1, \dots, \beta_\alpha, 2}(x_1, \dots, x_n) + R(x_1, \dots, x_n). \end{aligned}$$

*Proof.* As in the preceding lemma we proceed by induction on  $s$ . Again for definiteness we limit ourselves to the case  $k > 1$ . For  $s = 1$  the assertion of the lemma is easily verified. We assume that it is true for superpositions of order  $s - 1$ . By Lemma 5.4.1, for superpositions of order  $s$  we have

$$\begin{aligned} \tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n) &= f(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_k) - f(q_1, q_2, \dots, q_k) \\ &+ \varphi(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_k) - \varphi(q_1, q_2, \dots, q_k) + \sum_{\beta_1=1}^k \frac{\partial f}{\partial q_{\beta_1}}(\tilde{q}_{\beta_1} - q_{\beta_1}) \\ &+ A(\mu, s) \varepsilon' \cdot \varepsilon + k \cdot A(\mu, s) \omega(A(\mu, s) \varepsilon) \varepsilon. \end{aligned}$$

Since  $\tilde{q}_{\beta_1}$  and  $q_{\beta_1}$  ( $\beta_1 = 1, 2, \dots, k$ ) are superpositions of order  $s - 1$ , by the inductive hypothesis we have

$$\begin{aligned} \tilde{q}_{\beta_1} - q_{\beta_1} &= \sum_{\substack{\alpha > 0 \\ \beta_2, \beta_3, \dots, \beta_\alpha}} \hat{p}_{\beta_1, \dots, \beta_\alpha}(x_1, x_2, \dots, x_n) \\ &\times \varphi_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}(x_1, x_2, \dots, x_n), \dots, q_{\beta_1, \dots, \beta_\alpha, k}(x_1, x_2, \dots, x_n)) \\ &+ \hat{R}(x_1, x_2, \dots, x_n), \end{aligned}$$

where

$$\begin{aligned} |\hat{R}(x_1, x_2, \dots, x_n)| &\leq B(\mu, s - 1, k) [\varepsilon' + \omega(A(\mu, s - 1) \varepsilon)] \varepsilon, \\ \hat{p}_{\beta_1, \dots, \beta_\alpha}(x_1, \dots, x_n) &= \prod_{i=1}^{\alpha-1} \frac{\partial f_{\beta_1, \beta_2, \dots, \beta_i}}{\partial q_{\beta_1, \dots, \beta_{i+1}}} \end{aligned}$$

(for  $\alpha = 1, \hat{p}_{\beta_1}(x_1, \dots, x_n) \equiv 1$ ).

When we now substitute the expressions for the differences  $\tilde{q}_{\beta_1} - q_{\beta_1}$  in the formula for  $\tilde{F} - F$  above, we obtain the required representation of the difference of two superpositions  $\tilde{F} - F$ . This proves the lemma.

§ 5. *Instability of the representation of functions  
as superpositions of smooth functions*

Let  $A$  be a set of functions of  $n$  variables and  $B$  a set of functions of  $k$  variables ( $k < n$ ). Suppose that a function  $F(x_1, \dots, x_n) \in A$  is in a region  $G_n$  of the space  $x_1, x_2, \dots, x_n$  an  $s$ -fold superposition, generated by a system of functions  $\{f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  of  $B$ .

We say that this superposition is  $(A, B)$ -stable in  $G_n$  if every function  $\tilde{F}(x_1, \dots, x_n) \in A$  can be represented in  $G_n$  as the  $s$ -fold superposition of the same form of functions  $\{\tilde{f}_{\beta_1, \dots, \beta_\alpha}(t_1, t_2, \dots, t_k)\}$  of  $B$  such that

$$\begin{aligned} \max_{\alpha; \beta_1, \dots, \beta_\alpha} \sup_t & \left| \tilde{f}_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k) - f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k) \right| \\ & \leq \lambda \sup_{x \in G_n} \left| \tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n) \right|, \end{aligned}$$

where  $\lambda$  is a constant not depending either on  $\tilde{F}$  or on the  $\{\tilde{f}_{\beta_1, \dots, \beta_\alpha}\}$ .

We denote by  $C_{\omega(\delta)}^{(1)}$  the space of all continuously differentiable functions of  $k$  variables whose partial derivatives have modulus of continuity  $\omega(\delta)$  ( $\omega(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ ).

**THEOREM 5.5.1.** *Suppose that each function  $F(x_1, \dots, x_n) \in A$  is in some region  $D_n$  of the space  $x_1, \dots, x_n$  a superposition of order  $s$  of functions of  $k$  variables  $\{f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  belonging to  $C_{\omega(\delta)}^{(1)}$  ( $k < n$ ). If for any subregion  $G_n \subset D_n$  the functional "dimension" of  $A$  at  $F(x_1, \dots, x_n) \in A$  is greater than  $k$ , then the function  $F(x_1, \dots, x_n)$  cannot be an  $(A, C_{\omega(\delta)}^{(1)})$ -stable superposition in any such region  $G \subset D_n$ .*

*Proof.* Assume the contrary, that is, in a region  $G_n \subset D_n$  the function  $F(x_1, \dots, x_n) \in A$  is an  $(A, C_{\omega(\delta)}^{(1)})$ -stable  $s$ -fold superposition of functions  $\{f_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  of  $C_{\omega(\delta)}^{(1)}$ . Then any function  $\tilde{F}(x_1, \dots, x_n) \in A$  can be represented as the superposition of the same form of functions  $\{\tilde{f}_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  of  $C_{\omega(\delta)}^{(1)}$  such that

$$\max_{\alpha; \beta_1, \dots, \beta_\alpha} \sup_t \left| \varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k) \right| \leq \lambda \sup_{x \in G_n} \left| \tilde{F} - F \right|,$$

where  $\varphi_{\beta_1, \dots, \beta_\alpha} = \tilde{f}_{\beta_1, \dots, \beta_\alpha} - f_{\beta_1, \dots, \beta_\alpha}$ . By Lemma 5.4.2 we have (for definiteness,  $k > 1$ )

$$\begin{aligned} \tilde{F} - F &= \sum_{\alpha; \beta_1, \dots, \beta_\alpha} p_{\beta_1, \dots, \beta_\alpha}(x_1, \dots, x_n) \\ &\times \varphi_{\beta_1, \dots, \beta_\alpha}(q_{\beta_1, \dots, \beta_\alpha, 1}(x_1, \dots, x_n), \dots, q_{\beta_1, \dots, \beta_\alpha, k}(x_1, \dots, x_n)) + R(x_1, \dots, x_n), \end{aligned}$$

where  $|R(x_1, \dots, x_n)| \leq \gamma(\varepsilon)\varepsilon$ ,  $\gamma(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , and

$$\begin{aligned} \varepsilon &= \max_{\alpha; \beta_1, \dots, \beta_\alpha} \sup_t |\varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)| \\ &\leq \lambda \sup_{x \in G_n} |\tilde{F}(x_1, \dots, x_n) - F(x_1, \dots, x_n)|. \end{aligned}$$

That  $\gamma(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  follows from the fact that as  $\varepsilon \rightarrow 0$  the quantity

$$\varepsilon' = \max_{\alpha; \beta_1, \dots, \beta_\alpha} \sum_{i=1}^k \sup \left| \frac{\partial \varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)}{\partial t_i} \right| \rightarrow 0,$$

provided only that the modulus of continuity of the partial derivatives of the functions  $\{\varphi_{\beta_1, \dots, \beta_\alpha}(t_1, \dots, t_k)\}$  is fixed. By 5.1.10 it follows that  $r(A, F) \leq k$  in some subregion  $G_n \subset D_n$ . So we have obtained a contradiction to the assumption that  $r(A, F) > k$  in any subregion  $G_n \subset D_n$  and this proves the theorem.

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