

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 23 (1977)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON REPRESENTATION OF FUNCTIONS BY MEANS OF SUPERPOSITIONS AND RELATED TOPICS
Autor: Vitushkin, A. G.
Kapitel: §6. The set of linear superpositions in the space of continuous functions is nowhere dense
DOI: <https://doi.org/10.5169/seals-48931>

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is satisfied, where the constant C does not depend on the functions $\{f_i^k(t)\}$. Since F is a bounded set, there exists a subsequence of suffixes n_1, n_2, \dots such that for any $i = 0, 1, \dots, n$; $k = 1, 2, \dots, m_i$; $j = 1, 2, \dots, s_i$ the numerical sequence $f_{i,n_v}^k \rightarrow C_{k,i,j}$ as $v \rightarrow \infty$. From this and the previous inequality it follows that $\{f_{i,n_v}^k(t)\} \in F (v=1, 2, \dots)$ is a Cauchy sequence, because it is known that the sequence $f_n(x, y) \in T(F)$ is Cauchy sequence. Consequently there exists an element $\{f_i^k(t)\} \in B$ such that $\|\{f_i^k(t) - f_{i,n_v}^k(t)\}\|_B \rightarrow 0$. Since F is a closed set, $\{f_i^k(t)\} \in F$. The operator $T: B \rightarrow C(G)$ is bounded. Therefore $T(\{f_i^k(t)\}) = f(x, y)$. Consequently $f(x, y) \in T(F)$. This proves the lemma.

The following lemma from the theory of linear operators [28] turns out to be useful.

LEMMA 4.5.3. *Let B_1 and B_2 be Banach spaces. If a linear operator $T: B_1 \rightarrow B_2$ maps bounded closed sets of B_1 onto closed sets of B_2 , then its domain of values is closed.*

Proof of Theorem 4.5.1. The set of superpositions of the form $\sum_{m=1}^N p_m(x, y) f_m(g_m(x, y))$ coincides on G with the set of superpositions of the form $\sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y))$. By Lemma 4.5.2 and 4.5.3 the set of the latter superpositions is closed in the space $C(G)$. This proves the theorem.

§ 6. *The set of linear superpositions in the space of continuous functions is nowhere dense*

THEOREM 4.6.1. *For any continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ ($m=1, 2, \dots, N$) and any region D of the plane of the variables x, y the set of superpositions of the form*

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is nowhere dense in the space of all functions continuous in D with uniform convergence.

By Lemma 4.2.2 we can find a subregion $G^* \subset D$, determine a constant $\gamma^* > 0$, and renumber the functions $\{q_m(x, y)\}$, with two indices so that

the functions $\tilde{q}_i^k(x, y)$ ($i=0, 1, 2, \dots, \tilde{n}; k=1, 2, \dots, \tilde{m}_i; \sum_{i=0}^{\tilde{n}} \tilde{m}_i = N$) obtained after the renumbering satisfy conditions (1), (2), (3) of Lemma 4.2.2. We now fix the point $(x_0, y_0) \in G^*$ and the number v so that the line $(y - y_0) + v(x - x_0) = 0$ does not touch at any of the level curves of the functions $\tilde{q}_i^k(x, y)$ ($i=1, 2, \dots, \tilde{n}$) that pass through (x_0, y_0) . Let $G^{**} \subset G^*$ be a disc with centre at (x_0, y_0) and radius small enough so that the $\{\tilde{q}_i^k(x, y)\}$ and $q_{N+1}(x, y) = y + vx$ satisfy condition (3) of Lemma 4.2.2 with some constant $\gamma^{**} > 0$. We put $p_{N+1}(x, y) = 1$. By Lemma 4.4.3 we can find a set $G \subset G^{**}$, determine a constant $\lambda > 0$, and again renumber the functions $p_m(x, y)$ and $q_m(x, y)$ ($m=1, 2, \dots, N+1$) with two indices so that the functions $p_i^k(x, y)$ and

$$q_i^k(x, y) \quad (i=0, 1, 2, \dots, n+1; k=1, 2, \dots, m_i; \sum_{i=0}^{n+1} m_i \leq N+1)$$

that is, some functions may be omitted in the renumbering) obtained after the renumbering satisfy conditions (1)-(3) of Lemma 4.2.2, conditions (4')-(6') of § 5, and the condition

$$7 \quad m_{n+1} = 1, \quad p_{N+1}^1 = p_{N+1}(x, y) = 1, \quad q_{N+1}^1 = q_{N+1}(x, y) = y + vx.$$

Let L be the linear space consisting of all system of functions $\{f_i^k(t)\}$ defined and continuous on the sets $\{q_i^1(G)\}$ and satisfying the condition

$$\sum_{i=0}^{n+1} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) \equiv 0 \quad \text{in } G.$$

LEMMA 4.6.1. L is a finite-dimensional linear space.

Proof. By Lemma 4.5.1, in the sets $\{q_i^1(G)\}$ we can select a subset consisting of a finite number of points $\{t_{i,j}\}$ such that, if $\{f_i^k(t)\} \in L$ and $f_i^k(t_{i,j}) = 0$ for all k, i, j then $f_i^k(t) \equiv 0$ on $q_i^1(G)$ for all i, k . Thus, the set of functions $\{f_i^k(t)\}$ is completely determined by a finite set of parameters $\{f_i^k(t_{i,j})\}$. Consequently the dimension of the space L is finite. This proves the lemma.

LEMMA 4.6.2. There exists a natural number μ such that in D the polynomial $(y + vx)^\mu = Q(x, y)$ is not equal to any superposition of the form

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)),$$
 where $\{f_m(t)\}$ are arbitrary continuous functions.

Proof. We denote by Φ the space of functions of the form $f(y + vx) = f_{n+1}^1(q_{n+1}^1(x, y))$ that are representable on G by superpositions of the form $[\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y))]$. Or, what comes to the same thing

(see properties (4') and (7)), of the form $[\sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y))]$.

Thus, functions of Φ satisfy the relation $\sum_{i=0}^{n+1} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y)) \equiv 0$ in G . Consequently the linear space Φ is naturally embedded in L . Since L is finite-dimensional (Lemma 4.6.1), Φ is also finite-dimensional. Let l be the dimension of Φ . Since the polynomials $(y + vx), (y + vx)^2, \dots, (y + vx)^{l+1}$ are linearly independent, at least one of them $Q(x, y) = (y + vx)^\mu$ is not equal to any superposition of the form under discussion on G or, consequently, in D . This proves the lemma.

Proof of Theorem 4.6.1. By Lemma 4.6.2 the set of superpositions of the form given in Theorem 4.6.1 does not exhaust all continuous functions on G . Consequently, by Theorem 4.5.1, the set of these superpositions is a closed linear subspace of $C(G)$. Hence we conclude that the set of superpositions under discussion is nowhere dense in $C(G)$, nor consequently in $C(D)$. This proves the theorem.

COROLLARY 4.6.1. *For any continuous functions $p_m(x_1, x_2, \dots, x_n)$ and continuously differentiable functions $q_m(x_1, x_2, \dots, x_n)$ ($m=1, 2, \dots, N$) and any region D of the space of the variables (x_1, x_2, \dots, x_n) the set of superpositions of the form*

$$\sum_{m=1}^N p_m(x_1, x_2, \dots, x_n) f_m(q_m(x_1, x_2, \dots, x_n), x_2, x_3, \dots, x_{n-1}),$$

where $\{f_m(t, x_2, x_3, \dots, x_{n-1})\}$ are arbitrary continuous functions of $(n-1)$ variables, is nowhere dense in the space of all functions continuous in D with uniform convergence.