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Kapitel: §5. The set of linear superpositions in the space of continuous functions is closed

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$$\lambda_i = \min_{j \leqq i} \inf_{t \in q_j(G_i)} \lambda(t, G_i, q_j, \{p_i^k\}) > \frac{1}{2}\min\left\{\frac{\delta}{2}, \min_{j<i} \lambda_j\right\}.$$

Thus, the regular regions $G_1, G_2, \ldots, G_n$ can be constructed. The regular region $G = G_n$ satisfies all the requirements of our lemma ($\lambda = \lambda_n$), which is now proved.

§ 5. *The set of linear superpositions in the space of continuous functions is closed*

THEOREM 4.5.1. *Suppose that continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ $(m = 1, 2, \ldots, N)$ are fixed. Then in any region D of the plane of the variables x, y. there exists a closed subregion $G \subset D$ such that the set of superpositions of the form*

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is closed (in the uniform metric) in the set of all functions continuous on the set G.

By Lemma 4.2.2 and 4.4.3 we can find a subset $G \subset D$, determine constants $\gamma > 0$ and $\lambda > 0$, and renumber the functions $\{p_m(x, y)\}$ and $\{q_m(x, y)\}$ with two indices so that the functions obtained after the renumbering, $\{p_i^k(x, y)\}$ and $\{q_i^k(x, y)\}$ $(i = 0, 1, 2, \ldots, n;\ k = 1, 2, \ldots, m_i;\ \sum_{i=0}^{n} m_i \leqslant N)$ that is, some functions may be omitted in the renumbering) satisfy conditions (1), (2), (3) of Lemma 4.2.2, and also the conditions:

(4′) for any continuous functions $\{f_m(t)\}$ there exists continuous functions $\{f_i^k(t)\}$ such that on G

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)) = \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y));$$

(5′) for every i and $t \in q_i^1(G)$ and for any functions $\{f_i^k(t)\}$

$$\max_{(x,y) \in e(q_i^1, t) \cap G} \left| \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) \right| \leqslant \lambda \max_k |f_i^k(t)|;$$

(6′) G is a regular region with respect to the functions $\{q_i^k(x, y)\}$.

LEMMA 4.5.1. *In the sets* $\{ q_i^1 (G) \}$ *we can select subsets consisting of a finite number of points* $t_{i,j} \in q_i^1 (G)$ $(i=0, 1, 2, ..., n;\ j=1, 2, ..., s_i)$ *such that for any continuous functions* $\{ f_i^k (t) \}$

$$\max_{i,k} \ \max_{t \in q_i^1 (G)} |f_i^k (t)| \leqslant c \left(\max_{(x,y) \in G} \Big| \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k (x,y) f_i^k (q_i^1 (x,y)) \Big| \right.$$
$$\left. + \max_k |f_i^k (t_{i,j})| \right),$$

where C *is a constant not depending on the functions* $\{ f_i^k (t) \}$.

Proof. Since G is polyhedral, for each i we can choose in $q_i (G)$ a finite set of points $\{ t_{i,j} \}$ so dense that the components of the level curves $e (q_i^1, t_{i,j}) \cap G$ form a δ-net in the set of all components of the level curves $e (q_i^1, t) \cap G$, $t \in q_i^1 (G)$. A sufficiently small δ, not depending on the functions $\{ f_i^k (t) \}$, will be chosen below. We put

$$\mu = \max_{i,k} \ \max_{(x,y) \in G} |f_i^k (q_i^1 (x,y))| ;$$

$$\varepsilon_1 = \max_{(x,y) \in G} \Big| \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k (x,y) f_i^k (q_i^1 (x,y)) \Big| ; \ \varepsilon_2 = \max_{k,i,j} |f_i^k (t_{i,j})| .$$

For definiteness, let $f_1^1 (q_1^1 (a)) = \mu$ at the point $a \in G$. By (5′) there exists a point $a' \in G$ such that $\Big| \sum_{k=1}^{m_i} p_1^k (a') f_1^k (q_1^1 (a')) \Big| \geqslant \lambda\mu$. Let $[a', a^*]$ be a segment of the level curve of the function $q_1^1 (x,y)$ with end-points at a' and a^* such that $h_1 ([a', a^*]) \geqslant \gamma G/2$ (see the definition of a regular region in § 4). On the arc $[a', a^*]$ we fix a point a'' such that $\omega (\alpha) \leqslant \dfrac{\lambda}{2m_1}$, where $\alpha = h_1 ([a', a''])$. Then on the segment $[a', a'']$ the function $\varphi_1 (x,y) = \sum_{k=1}^{m_1} p_1^k (x,y) f_1^k (q_1^1 (x,y))$ keeps'a constant sign and satisfies the inequality $| \varphi_1 (x,y) | \geqslant \lambda\mu/2$. In fact, $| \varphi_1 (a') | \geqslant \lambda\mu$ at the point a', and for any point $s \in [a', a'']$

$$| \varphi_1 (s) - \varphi_1 (a') | = \Big| \sum_{k=1}^{m_1} (p_1^k (s) - p_1^k (a')) f_1^k (a') \Big| \leqslant m_1 \mu \omega (\alpha) \leqslant \frac{\lambda\mu}{2} .$$

Consequently,

$$\Bigg| \int_{s \in [a', a'']} \varphi_1 (s) \, ds \Bigg| \geqslant \frac{1}{2} \lambda\mu\alpha .$$

By construction there is an index j and a segment $[b', b'']$ of the level curve $e (q_1', t_{1,j}) \cap G$ such that $\rho ([a', a''], [b', b'']) < \delta$. We have

$$\Big| \int_{s \in [b', b'']} \varphi_1(s)\, ds \Big| \leqslant c_1 \varepsilon_2 \beta ,$$

where $\beta = h_1([b', b''])$, $C_1 = m_1 \max\limits_{k} \max\limits_{(x,y)\in G} |p_1^k(x, y)|$. And since α and β are commensurable (δ will be chosen small in comparison with α),

$$\Big| \int_{s \in [a', a'']} \varphi_1(s)\, ds - \int_{s \in [b', b'']} \varphi_1(s)\, ds \Big| \geqslant \frac{1}{2} \lambda\mu\alpha - c_1' \varepsilon_2 \alpha .$$

By Lemma 4.2.3

$$\Big| \int_{s \in [a', a'']} \varphi_1(s)\, ds - \int_{s \in [b'b'']} \varphi_1(s)\, ds \Big| \leqslant c_3 (\alpha\varepsilon_1 + \mu\alpha\omega(\delta) + \mu\delta) .$$

Thus, $c_3 (\alpha\varepsilon_1 + \mu\alpha\omega(\delta) + \mu\delta) \geqslant \lambda\mu\alpha/2 - c_1' \alpha \cdot \varepsilon_2$. If δ is taken sufficiently small in comparison with α (in order that $c_3 (\alpha\omega(\delta) + \delta) < \lambda\alpha/2$), then we have $\mu \leqslant C(\varepsilon_1 + \varepsilon_2)$. This proves the lemma.

Let B be the Banach space consisting of all systems of functions $\{ f_i^k(t) \}$, defined and continuous on the sets $\{ q_i^1(G) \}$, with the norm

$$\| \{ f_i^k(t) \} \|_B = \max_{i,k} \max_{t \in q_i^1(G)} |f_i^k(t)| \quad (i = 0, 1, 2, \ldots, n;\ k = 1, 2, \ldots, m_i) .$$

We denote by $C(G)$ the space of all functions $f(x, y)$ continuous on G with the uniform metric:

$$\| f(x, y) \|_{C(G)} = \max_{(x,y)\in G} |f(x, y)| .$$

LEMMA 4.5.2. *The linear operator* $T : B \to C(G)$ *acting by the formula*

$$T(\{ f_i^k(t) \}) = f(x, y) = \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) ,$$

maps bounded closed sets of B *onto closed sets of* $C(G)$.

Proof. Let $F \subset B$ be a closed and bounded set of elements of B. Suppose that $f_n(x, y)$ is a sequence of functions in $T(F) \subset C(G)$, and that $f(x, y) \in C(G)$, where $\| f(x, y) - f_n(x, y) \|_{C(G)} \to 0$ as $n \to \infty$. We show that then $f(x, y) \in T(F)$. Since $f_n(x, y) \in T(F)$, there exists a sequence of elements $\{ f_{i,n}^k(t) \} \in F$ such that $T(\{ f_{i,n}^k(t) \}) = f_n(x, y)$. By Lemma 4.5.1 we can select in the sets $\{ q_i^1(G) \}$ subsets consisting of a finite number of points $t_{i,j} \in q_i'(G)$ ($i = 0, 1, \ldots, n$; $j = 1, 2, \ldots, s_i$) such that for each element $\{ f_i^k(t) \} \in B$ the inequality

$$\| \{ f_i^k(t) \} \|_B \leqslant c\big(\| f(x, y) \|_{C(G)} + \max_{k,j,i} |f_i^k(t_{i,j})| \big) ,$$

is satisfied, where the constant C does not depend on the functions $\{f_i^k(t)\}$. Since F is a bounded set, there exists a subsequence of suffixes $n_1, n_2, \ldots$ such that for any $i = 0, 1, \ldots, n$; $k = 1, 2, \ldots, m_i$; $j = 1, 2, \ldots, s_i$ the numerical sequence $f_{i,n_\nu}^k \to C_{k,i,j}$ as $\nu \to \infty$. From this and the previous inequality it follows that $\{f_{i,n_\nu}^k(t)\} \in F\,(\nu = 1, 2, \ldots)$ is a Cauchy sequence, because it is known that the sequence $f_n(x, y) \in T(F)$ is Cauchy sequence. Consequently there exists an element $\{f_i^k(t)\} \in B$ such that $\| \{f_i^k(t) - f_{i,n_\nu}^k(t)\} \|_B \to 0$. Since F is a closed set, $\{f_i^k(t)\} \in F$. The operator $T\colon B \to C(G)$ is bounded. Therefore $T(\{f_i^k(t)\}) = f(x, y)$. Consequently $f(x, y) \in T(F)$. This proves the lemma.

The following lemma from the theory of linear operators [28] turns out to be useful.

LEMMA 4.5.3. *Let B_1 and B_2 be Banach spaces. If a linear operator $T\colon B_1 \to B_2$ maps bounded closed sets of B_1 onto closed sets of B_2, then its domain of values is closed.*

Proof of Theorem 4.5.1. The set of superpositions of the form $\sum_{m=1}^{N} p_m(x, y) f_m(g_m(x, y))$ coincides on G with the set of superpositions of the form $\sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y))$. By Lemma 4.5.2 and 4.5.3 the set of the latter superpositions is closed in the space $C(G)$. This proves the theorem.

§ 6. *The set of linear superpositions in the space of continuous functions is nowhere dense*

THEOREM 4.6.1. *For any continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ $(m = 1, 2, \ldots, N)$ and any region D of the plane of the variables x, y the set of superpositions of the form*

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is nowhere dense in the space of all functions continuous in D with uniform convergence.

By Lemma 4.2.2 we can find a subregion $G^* \subset D$, determine a constant $\gamma^* > 0$, and renumber the functions $\{q_m(x, y)\}$, with two indices so that