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$$\lambda_i = \min_{j \leq i} \inf_{t \in q_j(G_i)} \lambda(t, G_i, q_j, \{p_i^k\}) > \frac{1}{2} \min \left\{ \frac{\delta}{2}, \min_{j < i} \lambda_j \right\}.$$

Thus, the regular regions G_1, G_2, \dots, G_n can be constructed. The regular region $G = G_n$ satisfies all the requirements of our lemma ($\lambda = \lambda_n$), which is now proved.

§ 5. *The set of linear superpositions in the space of continuous functions is closed*

THEOREM 4.5.1. *Suppose that continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ ($m=1, 2, \dots, N$) are fixed. Then in any region D of the plane of the variables x, y , there exists a closed subregion $G \subset D$ such that the set of superpositions of the form*

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is closed (in the uniform metric) in the set of all functions continuous on the set G .

By Lemma 4.2.2 and 4.4.3 we can find a subset $G \subset D$, determine constants $\gamma > 0$ and $\lambda > 0$, and renumber the functions $\{p_m(x, y)\}$ and $\{q_m(x, y)\}$ with two indices so that the functions obtained after the renumbering, $\{p_i^k(x, y)\}$ and $\{q_i^k(x, y)\}$ ($i=0, 1, 2, \dots, n; k=1, 2, \dots, m_i; \sum_{i=0}^n m_i \leq N$) that is, some functions may be omitted in the renumbering) satisfy conditions (1), (2), (3) of Lemma 4.2.2, and also the conditions:

(4') for any continuous functions $\{f_m(t)\}$ there exists continuous functions $\{f_i^k(t)\}$ such that on G

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)) = \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y));$$

(5') for every i and $t \in q_i^1(G)$ and for any functions $\{f_i^k(t)\}$

$$\max_{(x, y) \in e(q_i^1, t) \cap G} \left| \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) \right| \leq \lambda \max_k |f_i^k(t)|;$$

(6') G is a regular region with respect to the functions $\{q_i^k(x, y)\}$.

LEMMA 4.5.1. In the sets $\{q_i^1(G)\}$ we can select subsets consisting of a finite number of points $t_{i,j} \in q_i^1(G)$ ($i=0, 1, 2, \dots, n; j=1, 2, \dots, s_i$) such that for any continuous functions $\{f_i^k(t)\}$

$$\max_{i,k} \max_{t \in q_i^1(G)} |f_i^k(t)| \leq C \left(\max_{(x,y) \in G} \left| \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x,y) f_i^k(q_i^1(x,y)) \right| + \max_k |f_i^k(t_{i,j})| \right),$$

where C is a constant not depending on the functions $\{f_i^k(t)\}$.

Proof. Since G is polyhedral, for each i we can choose in $q_i(G)$ a finite set of points $\{t_{i,j}\}$ so dense that the components of the level curves $e(q_i^1, t_{i,j}) \cap G$ form a δ -net in the set of all components of the level curves $e(q_i^1, t) \cap G$, $t \in q_i^1(G)$. A sufficiently small δ , not depending on the functions $\{f_i^k(t)\}$, will be chosen below. We put

$$\mu = \max_{i,k} \max_{(x,y) \in G} |f_i^k(q_i^1(x,y))|;$$

$$\varepsilon_1 = \max_{(x,y) \in G} \left| \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x,y) f_i^k(q_i^1(x,y)) \right|; \quad \varepsilon_2 = \max_{k,i,j} |f_i^k(t_{i,j})|.$$

For definiteness, let $f_1^1(q_1^1(a)) = \mu$ at the point $a \in G$. By (5') there exists a point $a' \in G$ such that $\left| \sum_{k=1}^{m_1} p_1^k(a') f_1^k(q_1^1(a')) \right| \geq \lambda \mu$. Let $[a', a^*]$ be a segment of the level curve of the function $q_1^1(x,y)$ with end-points at a' and a^* such that $h_1([a', a^*]) \geq \gamma G/2$ (see the definition of a regular region in § 4). On the arc $[a', a^*]$ we fix a point a'' such that $\omega(\alpha) \leq \frac{\lambda}{2m_1}$, where $\alpha = h_1([a', a''])$. Then on the segment $[a', a'']$ the function $\varphi_1(x,y) = \sum_{k=1}^{m_1} p_1^k(x,y) f_1^k(q_1^1(x,y))$ keeps a constant sign and satisfies the inequality $|\varphi_1(x,y)| \geq \lambda \mu/2$. In fact, $|\varphi_1(a')| \geq \lambda \mu$ at the point a' , and for any point $s \in [a', a'']$

$$|\varphi_1(s) - \varphi_1(a')| = \left| \sum_{k=1}^{m_1} (p_1^k(s) - p_1^k(a')) f_1^k(a') \right| \leq m_1 \mu \omega(\alpha) \leq \frac{\lambda \mu}{2}.$$

Consequently,

$$\left| \int_{s \in [a', a'']} \varphi_1(s) ds \right| \geq \frac{1}{2} \lambda \mu \alpha.$$

By construction there is an index j and a segment $[b', b'']$ of the level curve $e(q_1^1, t_{1,j}) \cap G$ such that $\rho([a', a''], [b', b'']) < \delta$. We have

$$\left| \int_{s \in [b', b'']} \varphi_1(s) ds \right| \leq c_1 \varepsilon_2 \beta,$$

where $\beta = h_1([b', b''])$, $C_1 = m_1 \max_k \max_{(x, y) \in G} |p_1^k(x, y)|$. And since α and β are commensurable (δ will be chosen small in comparison with α),

$$\left| \int_{s \in [a', a'']} \varphi_1(s) ds - \int_{s \in [b', b'']} \varphi_1(s) ds \right| \geq \frac{1}{2} \lambda \mu \alpha - c'_1 \varepsilon_2 \alpha.$$

By Lemma 4.2.3

$$\left| \int_{s \in [a', a'']} \varphi_1(s) ds - \int_{s \in [b', b'']} \varphi_1(s) ds \right| \leq c_3 (\alpha \varepsilon_1 + \mu \alpha \omega(\delta) + \mu \delta).$$

Thus, $c_3 (\alpha \varepsilon_1 + \mu \alpha \omega(\delta) + \mu \delta) \geq \lambda \mu \alpha / 2 - c'_1 \alpha \cdot \varepsilon_2$. If δ is taken sufficiently small in comparison with α (in order that $c_3 (\alpha \omega(\delta) + \delta) < \lambda \alpha / 2$), then we have $\mu \leq C (\varepsilon_1 + \varepsilon_2)$. This proves the lemma.

Let B be the Banach space consisting of all systems of functions $\{f_i^k(t)\}$, defined and continuous on the sets $\{q_i^1(G)\}$, with the norm

$$\|\{f_i^k(t)\}\|_B = \max_{i, k} \max_{t \in q_i^1(G)} |f_i^k(t)| \quad (i=0, 1, 2, \dots, n; k=1, 2, \dots, m_i).$$

We denote by $C(G)$ the space of all functions $f(x, y)$ continuous on G with the uniform metric:

$$\|f(x, y)\|_{C(G)} = \max_{(x, y) \in G} |f(x, y)|.$$

LEMMA 4.5.2. *The linear operator $T: B \rightarrow C(G)$ acting by the formula*

$$T(\{f_i^k(t)\}) = f(x, y) = \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)),$$

maps bounded closed sets of B onto closed sets of $C(G)$.

Proof. Let $F \subset B$ be a closed and bounded set of elements of B . Suppose that $f_n(x, y)$ is a sequence of functions in $T(F) \subset C(G)$, and that $f(x, y) \in C(G)$, where $\|f(x, y) - f_n(x, y)\|_{C(G)} \rightarrow 0$ as $n \rightarrow \infty$. We show that then $f(x, y) \in T(F)$. Since $f_n(x, y) \in T(F)$, there exists a sequence of elements $\{f_{i,n}^k(t)\} \in F$ such that $T(\{f_{i,n}^k(t)\}) = f_n(x, y)$. By Lemma 4.5.1 we can select in the sets $\{q_i^1(G)\}$ subsets consisting of a finite number of points $t_{i,j} \in q_i^1(G)$ ($i=0, 1, \dots, n; j=1, 2, \dots, s_i$) such that for each element $\{f_i^k(t)\} \in B$ the inequality

$$\|\{f_i^k(t)\}\|_B \leq c (\|f(x, y)\|_{C(G)} + \max_{k, j, i} |f_i^k(t_{i,j})|),$$

is satisfied, where the constant C does not depend on the functions $\{f_i^k(t)\}$. Since F is a bounded set, there exists a subsequence of suffixes n_1, n_2, \dots such that for any $i = 0, 1, \dots, n$; $k = 1, 2, \dots, m_i$; $j = 1, 2, \dots, s_i$ the numerical sequence $f_{i,n_v}^k \rightarrow C_{k,i,j}$ as $v \rightarrow \infty$. From this and the previous inequality it follows that $\{f_{i,n_v}^k(t)\} \in F (v=1, 2, \dots)$ is a Cauchy sequence, because it is known that the sequence $f_n(x, y) \in T(F)$ is Cauchy sequence. Consequently there exists an element $\{f_i^k(t)\} \in B$ such that $\|\{f_i^k(t) - f_{i,n_v}^k(t)\}\|_B \rightarrow 0$. Since F is a closed set, $\{f_i^k(t)\} \in F$. The operator $T: B \rightarrow C(G)$ is bounded. Therefore $T(\{f_i^k(t)\}) = f(x, y)$. Consequently $f(x, y) \in T(F)$. This proves the lemma.

The following lemma from the theory of linear operators [28] turns out to be useful.

LEMMA 4.5.3. *Let B_1 and B_2 be Banach spaces. If a linear operator $T: B_1 \rightarrow B_2$ maps bounded closed sets of B_1 onto closed sets of B_2 , then its domain of values is closed.*

Proof of Theorem 4.5.1. The set of superpositions of the form $\sum_{m=1}^N p_m(x, y) f_m(g_m(x, y))$ coincides on G with the set of superpositions of the form $\sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y))$. By Lemma 4.5.2 and 4.5.3 the set of the latter superpositions is closed in the space $C(G)$. This proves the theorem.

§ 6. *The set of linear superpositions in the space of continuous functions is nowhere dense*

THEOREM 4.6.1. *For any continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ ($m=1, 2, \dots, N$) and any region D of the plane of the variables x, y the set of superpositions of the form*

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is nowhere dense in the space of all functions continuous in D with uniform convergence.

By Lemma 4.2.2 we can find a subregion $G^* \subset D$, determine a constant $\gamma^* > 0$, and renumber the functions $\{q_m(x, y)\}$, with two indices so that