Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	23 (1977)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
Artikel:	ON REPRESENTATION OF FUNCTIONS BY MEANS OF SUPERPOSITIONS AND RELATED TOPICS
Autor:	Vitushkin, A. G.
Kapitel:	§5. The set of linear superpositions in the space of continuous functions is closed
DOI:	https://doi.org/10.5169/seals-48931

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. <u>Mehr erfahren</u>

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. <u>En savoir plus</u>

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. <u>Find out more</u>

Download PDF: 09.08.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

$$\lambda_i = \min_{j \leq i} \inf_{t \in q_j(G_i)} \lambda(t, G_i, q_j, \{p_i^k\}) > \frac{1}{2} \min \left\{ \frac{\delta}{2}, \min_{j < i} \lambda_j \right\}.$$

Thus, the regular regions $G_1, G_2, ..., G_n$ can be constructed. The regular region $G = G_n$ satisfies all the requirements of our lemma $(\lambda = \lambda_n)$, which is now proved.

§ 5. The set of linear superpositions in the space of continuous functions is closed

THEOREM 4.5.1. Suppose that continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ (m=1, 2, ..., N) are fixed. Then in any region D of the plane of the variables x, y. there exists a closed subregion $G \subset D$ such that the set of superpositions of the form

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}\$ are arbitrary continuous functions, is closed (in the uniform metric) in the set of all functions continuous on the set G.

By Lemma 4.2.2 and 4.4.3 we can find a subset $G \subset D$, determine constants $\gamma > 0$ and $\lambda > 0$, and renumber the functions $\{p_m(x, y)\}$ and $\{q_m(x, y)\}$ with two indices so that the functions obtained after the renumbering, $\{p_i^k(x, y)\}$ and $\{q_i^k(x, y)\}$ $(i=0, 1, 2, ..., n; k=1, 2, ..., m_i; \sum_{i=0}^{n} m_i \leq N)$ that is, some functions may be omitted in the renumbering) satisfy conditions (1), (2), (3) of Lemma 4.2.2, and also the conditions:

(4') for any continuous functions $\{f_m(t)\}\$ there exists continuous functions $\{f_i^k(t)\}\$ such that on G

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)) = \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y));$$

(5') for every *i* and $t \in q_i^1(G)$ and for any functions $\{f_i^k(t)\}$

$$\max_{(x,y)\in e\left(q\frac{1}{i},t\right)\cap G}\left|\sum_{k=1}^{m_{i}}p_{i}^{k}(x,y)f_{i}^{k}\left(q_{i}^{1}(x,y)\right)\right|\ll\lambda\max_{k}\left|f_{i}^{k}(t)\right|;$$

(6') G is a regular region with respect to the functions $\{q_i^k(x, y)\}$.

- 297 -

LEMMA 4.5.1. In the sets $\{q_i^1(G)\}\$ we can select subsets consisting of a finite number of points $t_{i,j} \in q_i^1(G)$ $(i=0, 1, 2, ..., n; j=1, 2, ..., s_i)$ such that for any continuous functions $\{f_i^k(t)\}\$

$$\max_{i,k} \max_{t \in q} \frac{1}{i}(G) | \leq c \left(\max_{(x,y) \in G} \left| \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x,y) f_i^k(q_i^1(x,y)) \right| \right. \\ + \max_k \left| f_i^k(t_{i,j}) \right| \right),$$

where C is a constant not depending on the functions $\{f_i^k(t)\}$.

Proof. Since G is polyhedral, for each *i* we can choose in $q_i(G)$ a finite set of points $\{t_{i,j}\}$ so dense that the components of the level curves $e(q_i^1, t_{i,j}) \cap G$ form a δ -net in the set of all components of the level curves $e(q_i^1, t) \cap G, t \in q_i^1(G)$. A sufficiently small δ , not depending on the functions $\{f_i^k(t)\}$, will be chosen below. We put

$$\mu = \max_{i,k} \max_{(x,y)\in G} \left| f_i^k \left(q_i^1 \left(x, y \right) \right) \right|;$$

$$\varepsilon_1 = \max_{(x,y)\in G} \left| \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k \left(x, y \right) f_i^k \left(q_i^1 \left(x, y \right) \right) \right|; \quad \varepsilon_2 = \max_{k,i,j} \left| f_i^k \left(t_{i,j} \right) \right|.$$

For definiteness, let $f_1^1 \left(q_1^1 \left(a \right) \right) = \mu$ at the point $a \in G$. By (5') there exists
a point $a' \in G$ such that $\left| \sum_{k=1}^{m_i} p_1^k \left(a' \right) f_1^k \left(q_1^1 \left(a' \right) \right) \right| \ge \lambda \mu.$ Let $[a', a^*]$ be a
segment of the level curve of the function $q_1^1 \left(x, y \right)$ with end-points at a'
and a^* such that $h_1 \left([a', a^*] \right) \ge \gamma G/2$ (see the definition of a regular region
in § 4). On the arc $[a', a^*]$ we fix a point a'' such that $\omega \left(\alpha \right) \le \frac{\lambda}{2m_1}$, where
 $\alpha = h_1 \left([a', a''] \right)$. Then on the segment $[a', a'']$ the function $\varphi_1 \left(x, y \right) =$
 $\sum_{k=1}^{m_1} p_1^k \left(x, y \right) f_1^k \left(q_1^1 \left(x, y \right) \right)$ keeps'a constant sign and satisfies the inequality
 $\left| \varphi_1 \left(x, y \right) \right| \ge \lambda \mu/2$. In fact, $\left| \varphi_1 \left(a' \right) \right| \ge \lambda \mu$ at the point a' , and for any
point $s \in [a', a'']$

Consequently,

$$\int_{s \in [a', a'']} \varphi_1(s) \, ds \, \bigg| \geq \frac{1}{2} \, \lambda \mu \alpha \, .$$

By construction there is an index j and a segment [b', b''] of the level curve $e(q'_1, t_{1,j}) \cap G$ such that $\rho([a', a''], [b', b'']) < \delta$. We have

L'Enseignement mathém., t. XXIII, fasc. 3-4.

$$\left|\int_{s\in [b', b'']} \varphi_1(s) \, ds \right| \leqslant c_1 \varepsilon_2 \beta \, ,$$

where $\beta = h_1([b', b''])$, $C_1 = m_1 \max_{k} \max_{(x,y) \in G} |p_1^k(x, y)|$. And since α and β are commensurable (δ will be chosen small in comparison with α),

$$\Big|\int_{s\in [a', a'']} \varphi_1(s) ds - \int_{s\in [b', b'']} \varphi_1(s) ds \Big| \ge \frac{1}{2} \lambda \mu \alpha - c_1' \varepsilon_2 \alpha.$$

By Lemma 4.2.3

$$\left|\int_{s\in [a', a'']} \varphi_1(s) \, ds - \int_{s\in [b'b'']} \varphi_1(s) \, ds \right| \leq c_3 \left(\alpha \varepsilon_1 + \mu \alpha \omega(\delta) + \mu \delta\right).$$

Thus, $c_3 (\alpha \varepsilon_1 + \mu \alpha \omega (\delta) + \mu \delta) \ge \lambda \mu \alpha/2 - c'_1 \alpha \cdot \varepsilon_2$. If δ is taken sufficiently small in comparison with α (in order that $c_3 (\alpha \omega (\delta) + \delta) < \lambda \alpha/2$), then we have $\mu \le C (\varepsilon_1 + \varepsilon_2)$. This proves the lemma.

Let *B* be the Banach space consisting of all systems of functions $\{f_i^k(t)\}$, defined and continuous on the sets $\{q_i^1(G)\}$, with the norm

$$\|\{f_{i}^{k}(t)\}\|_{B} = \max_{i,k} \max_{t \in q} \max_{i}^{1}(G) | f_{i}^{k}(t)| \quad (i = 0, 1, 2, ..., n; k = 1, 2, ..., m_{i}).$$

We denote by C(G) the space of all functions f(x, y) continuous on G with the uniform metric:

$$||f(x, y)||_{C(G)} = \max_{(x, y) \in G} |f(x, y)|.$$

LEMMA 4.5.2. The linear operator $T: B \to C(G)$ acting by the formula

$$T(\{f_{i}^{k}(t)\}) = f(x, y) = \sum_{i=0}^{n} \sum_{k=1}^{m_{i}} p_{i}^{k}(x, y) f_{i}^{k}(q_{i}^{1}(x, y)),$$

maps bounded closed sets of B onto closed sets of C(G).

Proof. Let $F \subset B$ be a closed and bounded set of elements of B. Suppose that $f_n(x, y)$ is a sequence of functions in $T(F) \subset C(G)$, and that $f(x, y) \in C(G)$, where $|| f(x, y) - f_n(x, y) ||_{C(G)} \to 0$ as $n \to \infty$. We show that then $f(x, y) \in T(F)$. Since $f_n(x, y) \in T(F)$, there exists a sequence of elements $\{f_{i,n}^k(t)\} \in F$ such that $T(\{f_{i,n}^k(t)\}) = f_n(x, y)$. By Lemma 4.5.1 we can select in the sets $\{q_i^1(G)\}$ subsets consisting of a finite number of points $t_{i,j} \in q'_i(G)$ $(i=0, 1, ..., n; j=1, 2, ..., s_i)$ such that for each element $\{f_i^k(t)\} \in B$ the inequality

$$\| \{f_{i}^{k}(t)\} \|_{B} \leq c \left(\|f(x, y)\|_{C(G)} + \max_{k, j, i} \|f_{i}^{k}(t_{i,j})\| \right),$$

is satisfied, where the constant C does not depend on the functions $\{f_i^k(t)\}$. Since F is a bounded set, there exists a subsequence of suffixes $n_1, n_2, ...$ such that for any i = 0, 1, ..., n; $k = 1, 2, ..., m_i$; $j = 1, 2, ..., s_i$ the numerical sequence $f_{i,n_v}^k \to C_{k,i,j}$ as $v \to \infty$. From this and the previous inequality it follows that $\{f_{i,n_v}^k(t)\} \in F(v=1, 2, ...)$ is a Cauchy sequence, because it is known that the sequence $f_n(x, y) \in T(F)$ is Cauchy sequence. Consequently there exists an element $\{f_i^k(t)\} \in B$ such that $\|\{f_i^k(t)\} - f_{i,n_v}^k(t)\}\|_B \to 0$. Since F is a closed set, $\{f_i^k(t)\} \in F$. The operator $T: B \to C(G)$ is bounded. Therefore $T(\{f_i^k(t)\}) = f(x, y)$. Consequently $f(x, y) \in T(F)$. This proves the lemma.

The following lemma from the theory of linear operators [28] turns out to be useful.

LEMMA 4.5.3. Let B_1 and B_2 be Banach spaces. If a linear operator $T: B_1 \rightarrow B_2$ maps bounded closed sets of B_1 onto closed sets of B_2 , then its domain of values is closed.

Proof of Theorem 4.5.1. The set of superpositions of the form $\sum_{m=1}^{N} p_m(x, y) f_m(g_m(x, y))$ coincides on G with the set of superpositions of the form $\sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y))$. By Lemma 4.5.2 and 4.5.3 the set of the latter superpositions is closed in the space C (G). This proves the theorem.

§ 6. The set of linear superpositions in the space of continuous functions is nowhere dense

THEOREM 4.6.1. For any continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ (m = 1, 2, ..., N) and any region D of the plane of the variables x, y the set of superpositions of the form

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is nowhere dense in the space of all functions continuous in D with uniform convergence.

By Lemma 4.2.2 we can find a subregion $G^* \subset D$, determine a constant $\gamma^* > 0$, and renumber the functions $\{q_m(x, y)\}$, with two indices so that