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$(q(x, y)) p_k(x, y) \equiv 0$  on  $e(q, t) \cap D$ . If we had  $C_k(t) = 0$  for some  $k$ , then it would turn out that  $t \in \delta_k$ . Consequently,  $C_k(t) \neq 0$  for any  $k$ . We show that for every  $t \in \delta^*$  the numbers  $\{C_k(t)\}$  are uniquely determined. Assume the contrary. Then there are numbers  $\{C'_k(t)\}$  ( $\max |C'_k(t)| = 1$ ) such that  $\sum_{k=1}^m C'_k(q(x, y)) p_k(x, y) = 0$  on  $e(q, t) \cap D$  and  $C_k \neq C'_k$  for some  $k$ . Then

$$\sum_{k \neq 1} [C_k(t) C'_1(t) - C'_k(t) C_1(t)] p_k(x, y) = \sum_{k \neq 1} C'_k(t) p_k(x, y) \equiv 0$$

on  $e(q, t) \cap D$  and in addition,  $C''_k \neq 0$  for some  $k$ . Consequently,  $t \in \delta_1$ . So we have obtained a contradiction, and the uniqueness of the choice of the numbers  $C_k(t)$  is proved. Further, we may regard  $\{C_k(t)\}$  as single-valued functions of  $t$  on the portion  $\delta^*$ . By Lemma 4.3.3, the functions  $C_k(t)$  are continuous and, as noted above,  $C_k(t) \neq 0$  for any  $t \in \delta^*$ . Then

$$p_1(x, y) = \sum_{k=2}^m -\frac{C_k(q(x, y))}{C_1(q(x, y))} p_k(x, y), \quad (x, y) \in q^{-1}(\delta^*) \cap D.$$

$$\begin{aligned} \text{Putting } f(t) = f_k(t) - \frac{C_k(t)}{C_1(t)} f_1(t), \quad t \in \delta^*, \quad \text{we have } & \sum_{k=2}^m f_k^*(q(x, y)) p_k(x, y) \\ &= \sum_{k=1}^m f_k(q) p_k(x, y) - \sum_{k=2}^m \frac{C_k(q)}{C_1(q)} p_k(x, y) \\ &= \sum_{k=2}^m f_k(q) p_k(x, y) + f_1(q) p_1(x, y) \\ &= \sum_{k=1}^m f_k(q(x, y)) p_k(x, y), \quad (x, y) \in q^{-1}(\delta^*) \cap D. \end{aligned}$$

This proves the lemma.

#### § 4. Reduction of linear superpositions to a form with independent terms

We fix the continuous functions  $p_i^k(x, y)$  and continuously differentiable functions  $q_i(x, y)$  ( $i = 0, 1, 2, \dots, n$ ;  $k = 1, 2, \dots, m_i$ )  $n \geq 2$ , where  $\{q_i(x, y)\}$  satisfy in  $D$  conditions (1) and (3) of Lemma 4.2.2, and we consider in  $D$  superpositions of the form

$$\sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i(x, y)),$$

where  $\{f_i^k(t)\}$  are arbitrary continuous functions of one variable.

We call a bounded closed region  $G \subset D$  polyhedral if the boundary of  $G$  consists of a finite number of mutually non-intersecting simple closed contours that are unions of a finite number of segments of level curves of the functions  $q_i(x, y)$  ( $i = 1, 2, \dots, n$ ). Let  $G \subset D$  be a polyhedral region. We denote by  $\Gamma_i$  the set of those  $t \in q_i(G)$  for which the set  $e(q_i, t) \cap G$  contains a segment of a level curve belonging to the boundary of  $G$ . For any  $i$  the set  $\Gamma_i$  consists of a finite number of points. By property (1) of the functions  $\{q_i(x, y)\}$  for every  $i$  and for all points  $t_0 \in q_i(G) \setminus \Gamma_i$  there exists  $\lim_{t \rightarrow t_0} e(q_i, t) = e(q_i, t_0)$ . If  $t_0 \in \Gamma_i$ , then the last assertion need not hold, but in any case there exists  $\lim_{t \rightarrow +t_0} e(q_i, t) \subset e(q_i, t_0)$  and  $\lim_{t \rightarrow -t_0} e(q_i, t) \subset e(q_i, t_0)$  where the limit is taken over the points  $t \in q_i(G)$ . Here the limit is understood in the sense of the distance  $\rho(e(q_i, t), e(q_i, t_0))$ .

LEMMA 4.4.1. *There is a region  $G \subset D$  and a system of numbers  $\tau_i^k = 0$  or  $1$  ( $i = 0, 1, 2, \dots, n$ ;  $k = 1, 2, \dots, m_i$ ) such that*

(4) *for any  $i$  and for any continuous functions  $\{\varphi_i^k(t)\}$  there exist continuous functions  $\{f_i^k(t)\}$  such that in  $G$*

$$\sum_{k=1}^{m_i} p_i^k(x, y) \varphi_i^k(q_i(x, y)) \equiv \sum_{k=1}^{m_i} \tau_i^k p_i^k(x, y) f_i^k(q_i(x, y));$$

(5\*) *for any polyhedral region  $G^* \subset G$  and any  $i$ , the set*

$$\{t : \lambda(t, G^*, q_i, p_i^{k_1}, \dots, p_i^{k_s}) = 0\}$$

*is nowhere dense in  $q_i(G^*)$ , where*

$$k_1 = k_1(i), k_2 = k_2(i), \dots, k_s = k_s(i)$$

*is the set of all values of  $k$  for which  $\tau_i^k = 1$ .*

*Proof.* If  $i = 0$ , then by (1) the set  $q_0(D)$  consists of only one point. We choose a region  $G_0 \subset D$  and number  $\tau_0^k$  ( $k = 1, 2, \dots, m_0$ ) such that in  $G_0$  the functions  $p_0^{k_1}, \dots, p_0^{k_s}$  are a basis for the linear hull of the functions  $\{p_0^k\}$  (condition (4) for  $i = 0$ ) and in any region  $G^* \subset G_0$  these functions are linearly independent (condition (5\*) for  $i = 0$ ). Let  $G^* \subset D$  be an arbitrary polyhedral region. Then  $\lambda(t, G^*, q, \{p_i^k\})$  as a function of  $t$  has, for any  $i > 0$ , a finite number of points of discontinuity (of the first kind) on the set  $q_i(G^*)$ , which consists of a finite number of segments (see Lemma 4.3.1). Hence it follows that if the set  $\{t : \lambda(t, G^*, q_i, \{p_i^k\}) = 0\}$  is not

nowhere dense on  $q_i(G^*)$ , then the function  $\lambda(t) \equiv 0$  on some segment  $\delta \subset q_i(G^*)$  not containing points of  $\Gamma_i$ . By Lemma 4.3.4, there is a segment  $\delta^* \subset \delta$  such that in the expression  $\sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i(x, y))$  one of the terms can be deleted, without narrowing the class of the functions representable in the region  $q^{-1}(\delta^*) \cap G^*$  as superpositions of the given form. Carrying out all possible deletions we can find a region  $G \subset G_0 \subset D$  for which the assertion of the lemma is satisfied.

A region  $G \subset D$  is called regular if, firstly, it is polyhedral and, secondly, there is a number  $\gamma_G > 0$  such that for every  $i > 0$  and every  $t \in q_i(G)$  the set  $e(q_i, t) \cap G$  is the union of a finite number of simple arcs, each of which has length not less than  $\gamma_G$ . A point  $A$  of the boundary of the polyhedral region  $G$  is called a vertex if it belongs simultaneously to two segments of the level curves of  $q_i(x, y)$  and  $q_j(x, y)$  ( $i \neq j$ ) on the boundary of  $G$ . Every polyhedral region has a finite number of vertices.

**LEMMA 4.4.2.** *For every polyhedral region  $G$  and every neighbourhood  $U$  of the vertices of this region we can construct a regular region  $G^* \subset G$  such that  $G \setminus U \subset G^*$ .*

*Proof.* Let  $A_1, A_2, \dots, A_r$  be the vertices of the polyhedral region  $G$ ;  $U_1, U_2, \dots, U_r$  suitably small neighbourhoods of these vertices. Let  $k_m = k_m(A_m)$  be the number of all those functions  $\{q_i(x, y)\}$  for each of which the level curve passing through the point  $A_m$  does not contain any other points of the set  $U_m \cap G$ . Let  $q_{im}(x, y)$  be one of these functions. We put  $k(G) \in q_i(G)$ . If  $k(G) = 0$ , then for any  $i$  and any  $t \in q_i(G)$  the length of any component of the set  $e(q_i, t) \cap G$  is greater than zero and consequently the region  $G$  is regular. Suppose that  $k(G) > 0$  and  $m$  such that  $k_m \neq 0$ .

We fix  $\varepsilon > 0$  and put

$$G_{1m}^* = G \mid \{(x, y) : |q_{im}(x, y) - q(A_m)| < \varepsilon\} \cap U_m.$$

If  $U_m$  and  $\varepsilon$  are sufficiently small, then inside  $U_m$  the region  $G_{1m}^*$  has two vertices  $A'_m$  and  $A''_m$ , while the region  $G$  has only one vertex  $A_m$  there, but  $k_m(A'_m) = k_m(A''_m) = k_m(A_m) - 1$ . We now put  $G_1^* = \cap G_{1m}^*$ , where the intersection is taken over all  $m$  such that  $k_m \neq 0$ . Then  $k(G_1^*) = k(G) - 1$ . Repeating this construction  $k(G)$  times, we obtain a polyhedral region  $G^*$  for which  $G \setminus G^* \subset U$  and  $k(G^*) = 0$ . Consequently,  $G^*$  is regular. This proves the lemma.

LEMMA 4.4.3. *There exists a set  $G \subset D$ , a number  $\lambda > 0$ , and a set of numbers  $\tau_i^k = 0$  or  $1$  ( $i = 0, 1, \dots, n$ ;  $k = 1, 2, \dots, m_i$ ) such that condition (4) of Lemma 4.4.1 is satisfied, and also the conditions*

(5) *for every  $i$  and  $t \in q_i(G)$  and for any functions  $\{f_i^k(t)\}$*

$$\max_{(x,y) \in e(q_i, t) \cap G} \left| \sum_{k=1}^{m_i} \tau_i^k p_i^k(x, y) f_i^k(q_i(x, y)) \right| \geq \lambda \max_k |\tau_i^k f_i^k(t)|;$$

(6)  *$G$  is a regular region.*

*Proof.* By Lemma 4.4.1 there exists a region  $G^* \subset D$  and a set of numbers  $\tau_i^k$  such that for every polyhedral subregion  $G^{**} \subset G^*$  and for every  $i$  the set  $\{t : \lambda(t, G^{**}, q_i, p_i^{k_1}, \dots, p_i^{k_s}) = 0\}$  is nowhere dense in  $q_i(G^{**})$ , where  $k_1, k_2, \dots, k_s$  is the set of all values of  $k$  for which  $\tau_i^k = 1$ ; moreover, on the set  $G^*$ , for any  $i$  the property (4) of Lemma 4.4.1 is satisfied. In order not to change the notation unnecessarily, we assume that all  $\tau_i^k = 1$ . We now construct a system of regular regions  $G_0 \supset G_1 \supset G_2 \supset \dots \supset G_n = G$ , having the following property: for every  $j \leq i$ ,  $\inf_{t \in q_j(G_i)} \lambda(t, G_i, q_j, \{p_j^k\}) \geq \lambda_i > 0$ . For  $G_0$  we choose any regular region  $G_0 \in G^*$ . Suppose that the regular regions  $G_0, G_1, \dots, G_{i-1}$  have been constructed. We now construct the set  $G_i$ . We denote by  $\alpha_\delta$  the set  $\{t : \lambda(t, q_i, G_{i-1}, \{p_i^k\}) > \delta\}$ . Since the functions  $\lambda(t, q_i, G_{i-1}, \{p_i^k\})$ , have only finitely many points of discontinuity (of the first kind) on the set  $q_i(G_{i-1})$ , which consists of a finite number of segments (see Lemma 4.3.1), any component of  $\alpha_\delta$  is either an interval, or a half-interval, or a segment, or a point. Suppose that the set  $\alpha_\delta^N \subset \alpha_\delta$  consists of the  $N$  longest components of non-zero length of the set  $\alpha_\delta$  (if  $\alpha_\delta$  has only  $N_0$  ( $< N$ ) components of non-zero length, then let  $\alpha_\delta^N = \alpha_\delta^{N_0}$ ). We denote by  $\bar{\alpha}_\delta^N$  the closure of the set  $\alpha_\delta^N$ . We put  $G_{i-1}^* = G_{i-1} \cap q_i^{-1}(\bar{\alpha}_\delta^N)$ . We fix  $\varepsilon > 0$ . Since  $G_{i-1}$  is regular, for every  $j$  the length of any component of  $e(q_j, t) \cap G_{i-1}$  is greater than  $\gamma_G > 0$ . And since the set  $\{t : \lambda(t, q_i, G_{i-1}, \{p_i^k\}) = 0\}$  is nowhere dense in  $q_i(G_{i-1})$ , for sufficiently small  $\delta$  and sufficiently large  $N$  the set  $G_{i-1}^*$  forms a  $\varepsilon/2$ -net on every set  $e(q_j, t) \cap G_{i-1}$ ,  $j < i$ . The set  $G_{i-1}^*$  is a polyhedral region. We denote by  $U(\varepsilon)$  the set of points  $(x, y)$  each of which is at a distance of no more than  $\varepsilon/4$  from one of the vertices of the set  $G_{i-1}^*$ . By Lemma 4.4.2 there exists a regular region  $G_i \subset G_{i-1}^*$  such that  $G_{i-1}^* \setminus G_i \subset U(\varepsilon)$ . The set  $G_i$  forms an  $\varepsilon$ -net on every set  $e(q_j, t) \cap G_{i-1}$ ,  $j < i$  and forms an  $\varepsilon/2$ -net on every set  $e(q_i, t) \cap G_{i-1}^*$ . By Lemma 4.3.2, for sufficiently small  $\varepsilon$ ,

$$\lambda_i = \min_{j \leq i} \inf_{t \in q_j(G_i)} \lambda(t, G_i, q_j, \{p_i^k\}) > \frac{1}{2} \min \left\{ \frac{\delta}{2}, \min_{j < i} \lambda_j \right\}.$$

Thus, the regular regions  $G_1, G_2, \dots, G_n$  can be constructed. The regular region  $G = G_n$  satisfies all the requirements of our lemma ( $\lambda = \lambda_n$ ), which is now proved.

§ 5. *The set of linear superpositions in the space of continuous functions is closed*

**THEOREM 4.5.1.** *Suppose that continuous functions  $p_m(x, y)$  and continuously differentiable functions  $q_m(x, y)$  ( $m = 1, 2, \dots, N$ ) are fixed. Then in any region  $D$  of the plane of the variables  $x, y$ , there exists a closed subregion  $G \subset D$  such that the set of superpositions of the form*

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)),$$

*where  $\{f_m(t)\}$  are arbitrary continuous functions, is closed (in the uniform metric) in the set of all functions continuous on the set  $G$ .*

By Lemma 4.2.2 and 4.4.3 we can find a subset  $G \subset D$ , determine constants  $\gamma > 0$  and  $\lambda > 0$ , and renumber the functions  $\{p_m(x, y)\}$  and  $\{q_m(x, y)\}$  with two indices so that the functions obtained after the renumbering,  $\{p_i^k(x, y)\}$  and  $\{q_i^k(x, y)\}$  ( $i = 0, 1, 2, \dots, n$ ;  $k = 1, 2, \dots, m_i$ ;  $\sum_{i=0}^n m_i \leq N$ ) that is, some functions may be omitted in the renumbering) satisfy conditions (1), (2), (3) of Lemma 4.2.2, and also the conditions:

(4') for any continuous functions  $\{f_m(t)\}$  there exists continuous functions  $\{f_i^k(t)\}$  such that on  $G$

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)) = \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y));$$

(5') for every  $i$  and  $t \in q_i^1(G)$  and for any functions  $\{f_i^k(t)\}$

$$\max_{(x, y) \in e(q_i^1, t) \cap G} \left| \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) \right| \leq \lambda \max_k |f_i^k(t)|;$$

(6')  $G$  is a regular region with respect to the functions  $\{q_i^k(x, y)\}$ .