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$$= \lim_{\nu\to\infty} \Big| \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}])$$

$$- \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}])\,(1 + O(\gamma)\,\omega(\delta))$$

$$+ \sum_{r=1}^{\nu} (p_j^k(a_r) - p_j^k(b_r)) f_j^k(q_j^k(a_r))\, h_1([b_r, b_{r+1}]) \Big| + O(\gamma)\, m\delta$$

$$= \lim_{\nu\to\infty} \Big| \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}])\, O(\gamma)\,\omega(\delta)$$

$$+ \sum_{r=1}^{\nu} f_j^k(q_j^k(a_r))\, h_1([b_r, b_{r+1}])\, O(\gamma)\,\omega(\delta) \Big| + O(\gamma)\, m\delta$$

$$= O(\gamma)\, m\alpha\omega(\delta) + O(\gamma)\, m\alpha\omega(\delta) + O(\gamma)\, m\delta = O(\gamma)\, m(\delta + \alpha\omega(\delta)).$$

Then

$$\Big| \int\limits_{s\in[a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds - \int\limits_{s\in[b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds$$

$$\leqslant \Big| \int\limits_{s\in[a', a'']} \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds - \int\limits_{s\in[b', b'']} \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds \Big|$$

$$+ \sum_{j\neq i} \Big| \int\limits_{s\in[a', a'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s))\, ds - \int\limits_{s\in[b', b'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s))\, ds \Big|$$

$$\leqslant C_4(\gamma)\,\alpha\varepsilon + n(\max_{j\neq i} m_j)\, C_5(\gamma)\, m(\delta + \alpha\omega(\delta))$$

$$\leqslant C_3(\gamma)\,(\alpha\varepsilon + m\delta + m\alpha\omega(\delta)).$$

This proves the lemma.

§ 3. *Deletion of dependent terms*

On a bounded closed set D we consider the space of linear superpositions of the form $\sum_{k=1}^{m} p_k(x, y) f_k(q(x, y))$, $(x, y) \in D$. Here the functions $\{p_k(x, y)\}$ and $q(x, y)$ are continuous and fixed, and $\{f_k(t)\}$ are arbitrary continuous functions of one variable. We assume that the function $q(x, y)$ is such that for any sequence $t_n \in q(D) \to t \in q(D)$ we have $\rho\,[e(q, t_n) \cap D, e(q, t) \cap D] \to 0$. We put

$$\lambda(t, D, q, p_1, \ldots, p_m) = \inf_{\{c_k\}} \sup_{(x,y)\in e(q,t)\cap D} \Big| \sum_{k=1}^{m} c_k p_k(x, y) \Big|,$$

where inf is taken over all sets of numbers $\{ c_k \}$ for which $\max_k | c_k | = 1$. The function $\lambda (t, D, q, \{ p_k \})$, as a function of t, is defined only on the set $q(D)$.

LEMMA 4.3.1. *The function* $\lambda (t, D, q, \{ p_k \})$ *depends continuously on* t.

Proof. The linear combinations $\sum_{k=1}^{m} c_k p_k (x, y)$ for all possible systems of numbers $\{ c_k \}$ for which $\max_k | c_k | \leqslant 1$, form an equicontinuous set of functions, considered on the bounded closed set D. Consequently, for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $| t_1 - t_2 | < \delta$, then

$$\Big| \sup_{(x, y) \in e(q, t_1)} \Big| \sum_{k=1}^{m} c_k p_k (x, y) \Big| - \sup_{(x, y) \in e(q, t_2)} \Big| \sum_{k=1}^{m} c_k p_k (x, y) \Big\| < \varepsilon$$

simultaneously for all systems of numbers $\{ c_k \}$ such that $\max_k | c_k | \leqslant 1$. For definiteness, suppose that $\lambda (t_2, D, q, \{ p_k \}) \geqslant \lambda (t_1, D, q, \{ p_k \})$. Since the expression $\sup_{(x, y) \in e(q, t_1)} \Big| \sum_{k=1}^{m} c_k p_k (x, y) \Big|$ depends continuously on the coefficients $\{ c_k \}$, there exists a system of numbers $\{ c_k^1 \}$ such that $\max_k | c_k^1 | = 1$ and

$$\lambda (t_1, D, q, \{ p_k \}) = \sup_{(x, y) \in e(q, t_1)} \Big| \sum_{k=1}^{m} c_k^1 p_k (x, y) \Big| .$$

Since

$$\lambda (t_2, D, q, \{ p_k \}) \leqslant \sup_{(x, y) \in e(q, t_2)} \Big| \sum_{k=1}^{m} c_k^1 p_k (x, y) \Big| ,$$

we have

$$0 \leqslant \lambda (t_2) - \lambda (t_1) \leqslant \sup_{(x, y) \in e(q, t_2)} \Big| \sum_{k=1}^{m} c_k^1 p_k (x, y) \Big|$$

$$- \sup_{(x, y) \in e(q, t_1)} \Big| \sum_{k=1}^{m} c_k^1 p_k (x, y) \Big| < \varepsilon .$$

This proves the lemma.

LEMMA 4.3.2. *The function* $\lambda (t, D, q, \{ p_k \})$ *depends continuously on* D *in the sense that there exists a function* $\mu (\varepsilon) \to 0$ *as* $\varepsilon \to 0$, *having the property: if the set* $D_\varepsilon \subset D$ *is such that, for any* t, $D_\varepsilon \cap e (q, t)$ *forms an* ε*-net in the set* $e (q, t) \cap D$, *then*

$$\max_{t \in q(D)} \left| \lambda(t, D, q, \{p_k\}) - \lambda(t, D_\varepsilon, q, \{p_k\}) \right| \leqslant \mu(\varepsilon) .$$

Proof. Using the equicontinuity of the set of functions $\sum_{k=1}^{n} c_k \, p_k \, (x, y)$ where $\max_k | c_k | \leqslant 1$, we conclude that there exists a function $\mu(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that the inequality

$$0 \leqslant \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^{m} c_k p_k (x, y) \right| - \sup_{(x, y) \in e(q, t) \cap D_\varepsilon} \left| \sum_{k=1}^{m} c_k p_k (x, y) \right| \leqslant \mu(\varepsilon).$$

uniformly over all $t \in q(D)$ and over all systems of numbers $\{ c_k \}$ for which $\max_k | c_k | \leqslant 1$. For any $\varepsilon > 0$ there exists a system of numbers $\{ c_k^\varepsilon \}$ such that $\max_k | c_k^\varepsilon | = 1$ and

$$\lambda(t, D_\varepsilon, q, \{p_k\}) = \sup_{(x, y) \in e(q, t) \cap D_\varepsilon} \left| \sum_{k=1}^{m} c_k^\varepsilon p_k (x, y) \right| .$$

Since for any ε

$$\lambda(t, D, q, \{p_k\}) \leqslant \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^{m} c_k^\varepsilon p_k (x, y) \right|$$

and, on the other hand, $\lambda(t, D, q, \{p_k\}) \geqslant \lambda(t, D_\varepsilon, q, \{p_k\})$ (we recall that $D_\varepsilon \subset D$), we have

$$0 \leqslant \lambda(t, D, q, \{p_k\}) - \lambda(t, D_\varepsilon, q, \{p_k\}) \leqslant \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^{m} c_k^\varepsilon p_k (x, y) \right|$$
$$- \sup_{(x, y) \in e(q, t) \cap D_\varepsilon} \left| \sum_{k=1}^{m} c_k^\varepsilon p_k (x, y) \right| < \mu(\varepsilon) .$$

This proves the lemma.

LEMMA 4.3.3. *Let F be a closed set on the t-axis; $F \subset q(D)$. For every $t \in F$, suppose that there exists one and only one system of numbers $\{ C_k \}$ $(\max_k | C_k | = 1)$ such that $\sum_{k=1}^{m} C_k p_k (x, y) \equiv 0$ on the set $e(q, t) \cap D$. Then each of the functions $\{ C_k(t) \}$ depends continuously on t on the set F.*

Proof. Suppose that $t_n \in F$, $t \in F$ and $t_n \to t$. We put $\overline{\lim_{n \to \infty}} C_k(t_n) = \tilde{C}_k$ and $\underline{\lim_{n \to \infty}} C_k(t_n) = \tilde{\tilde{C}}_k$. Since $\sum_{k=1}^{m} C_k(t_n) \, p_k(x, y) \equiv 0$ on the set $e(q, t_n) \cap D$ and $\rho\,[e(q, t) \cap D, e(q, t_n) \cap D] \to 0$ as $n \to \infty$, we have $\sum_{k=1}^{m} \tilde{C}_k \, p_k (x, y)$

$\equiv 0 \equiv \sum_{k=1}^{m} \tilde{\tilde{C}}_k p_k(x, y)$ on the set $e(q, t) \cap D$. Consequently, by the condition of the lemma, $\tilde{C}_k = \tilde{\tilde{C}}_k = C_k(t)$. This proves the lemma.

Lemma 4.3.4. *Suppose that* $\lambda(t, D, q, \{p_k\}) \equiv 0$ *on some non-empty portion* δ *of the set* $q(D)$. *Then there is a non-empty portion* $\delta^* \subset \delta$ *and an index* l *such that for any continuous functions* $\{f_k(t)\}$ *there are continuous functions* $\{f_k^*(t)\}$ *such that*

$$\sum_{k \neq l} f_k^*(q(x, y))\, p_k(x, y) = \sum_{k=1}^{m} f_k(q(x, y))\, p_k(x, y)$$

on the set $q^{-1}(\delta^*) \cap D$.

We recall that a portion δ of a set E is that part of it which lies in the interval δ.

Proof. We prove the lemma by induction on m. For $m = 1$ the assertion of the lemma is obvious. We denote by δ_k the set of all points t of the portion δ for which $\lambda(t, D, q, p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_m) = 0$. By Lemma 4.3.1, the set is closed. Two cases are possible.

1) For some k the set δ_k contains a non-empty portion δ_k' of the set $q(D)$. Since $\lambda(t, D, q, p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_m) = 0$ for every $t \in \delta_k'$, then by the inductive hypothesis there is a non-empty portion $\delta^* \subset \delta_k'$ and an index $l \neq k$ such that for any continuous functions $f_1(t), \dots, f_{k-1}(t), f_{k+1}(t), \dots, f_m(t)$ there are continuous functions $f_1^*(t), \dots, f_{k-1}^*(t), f_{k+1}^*(t), \dots, f_m^*(t)$ such that

$$\sum_{i \neq k} f_i(q(x, y))\, p_i(x, y) = \sum_{i \neq k, l} f_i^*(q(x, y))\, p_i(x, y).$$

on the set $q^{-1}(\delta^*) \cap D$. Putting $f_k^*(t) = f_k(t)$, we obtain

$$\sum_{i=1}^{m} f_i(q(x, y))\, p_i(x, y) = \sum_{i \neq l} f_i^*(q(x, y))\, p_i(x, y).$$

So in case 1) the lemma is proved.

2) None of the sets δ_k contains non-empty portions of the set $q(D)$, that is, $\bigcup_{k=1}^{m} \delta_k$ is nowhere dense in $q(D)$. Therefore there exists a non-empty portion $\delta^* \subset \delta \setminus \bigcup_{k=1}^{m} \delta_k$. Since $\lambda(t, D, q, \{p_k\}) \equiv 0$ on δ^*, for every $t \in \delta^*$ there are numbers $\{C_k(t)\}$ $(\max_k |C_k(t)| = 1)$ such that $\sum_{k=1}^{m} C_k$

$(q(x, y)) p_k(x, y) \equiv 0$ on $e(q, t) \cap D$. If we had $C_k(t) = 0$ for some k, then it would turn out that $t \in \delta_k$. Consequently, $C_k(t) \neq 0$ for any k. We show that for every $t \in \delta^*$ the numbers $\{ C_k(t) \}$ are uniquely determined. Assume the contrary. Then there are numbers $\{ C'_k(t) \}$ $(\max | C'_k(t) | = 1)$ such that $\sum_{k=1}^{m} C'_k(q(x, y)) p_k(x, y) = 0$ on $e(q, t) \cap D$ and $C_k \neq C'_k$ for some k. Then

$$\sum_{k \neq 1} [C_k(t) C'_1(t) - C'_k(t) C_1(t)] p_k(x, y) = \sum_{k \neq 1} C''_k(t) p_k(x, y) \equiv 0$$

on $e(q, t) \cap D$ and in addition, $C''_k \neq 0$ for some k. Consequently, $t \in \delta_1$. So we have obtained a contradiction, and the uniqueness of the choice of the numbers $C_k(t)$ is proved. Further, we may regard $\{ C_k(t) \}$ as single-valued functions of t on the portion δ^*. By Lemma 4.3.3, the functions $C_k(t)$ are continuous and, as noted above, $C_k(t) \neq 0$ for any $t \in \delta^*$. Then

$$p_1(x, y) = \sum_{k=2}^{m} - \frac{C_k(q(x, y))}{C_1(q(x, y))} p_k(x, y), \quad (x, y) \in q^{-1}(\delta^*) \cap D.$$

Putting $f(t) = f_k(t) - \dfrac{C_k(t)}{C_1(t)} f_1(t)$, $t \in \delta^*$, we have $\sum_{k=2}^{m} f^*_k(q(x, y)) p_k(x, y)$

$$= \sum_{k=1}^{m} f_k(q) p_k(x, y) - \sum_{k=2}^{m} \frac{C_k(q)}{C_1(q)} p_k(x, y)$$

$$= \sum_{k=2}^{m} f_k(q) p_k(x, y) + f_1(q) p_1(x, y)$$

$$= \sum_{k=1}^{m} f_k(q(x, y)) p_k(x, y), \quad (x, y) \in q^{-1}(\delta^*) \cap D.$$

This proves the lemma.

§ 4. *Reduction of linear superpositions to a form with independent terms*

We fix the continuous functions $p_i^k(x, y)$ and continuously differentiable functions $q_i(x, y)$ $(i = 0, 1, 2, \ldots, n;\ k = 1, 2, \ldots, m_i)$ $n \geqslant 2$, where $\{ q_i(x, y) \}$ satisfy in D conditions (1) and (3) of Lemma 4.2.2, and we consider in D superpositions of the form

$$\sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i(x, y)),$$

where $\{ f_i^k(t) \}$ are arbitrary continuous functions of one variable.