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$$\begin{aligned}
 &= \lim_{v \rightarrow \infty} \left| \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) \right. \\
 &\quad - \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) (1 + O(\gamma) \omega(\delta)) \\
 &\quad \left. + \sum_{r=1}^v (p_j^k(a_r) - p_j^k(b_r)) f_j^k(q_j^k(a_r)) h_1([b_r, b_{r+1}]) \right| + O(\gamma) m \delta \\
 &= \lim_{v \rightarrow \infty} \left| \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) O(\gamma) \omega(\delta) \right. \\
 &\quad \left. + \sum_{r=1}^v f_j^k(q_j^k(a_r)) h_1([b_r, b_{r+1}]) O(\gamma) \omega(\delta) \right| + O(\gamma) m \delta \\
 &= O(\gamma) m \alpha \omega(\delta) + O(\gamma) m \alpha \omega(\delta) + O(\gamma) m \delta = O(\gamma) m (\delta + \alpha \omega(\delta)).
 \end{aligned}$$

Then

$$\begin{aligned}
 &\left| \int_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right. \\
 &\leq \left| \int_{s \in [a', a'']} \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right| \\
 &\quad + \left| \sum_{j \neq i} \int_{s \in [a', a'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\
 &\leq C_4(\gamma) \alpha \varepsilon + n (\max_{j \neq i} m_j) C_5(\gamma) m (\delta + \alpha \omega(\delta)) \\
 &\leq C_3(\gamma) (\alpha \varepsilon + m \delta + m \alpha \omega(\delta)).
 \end{aligned}$$

This proves the lemma.

### § 3. Deletion of dependent terms

On a bounded closed set  $D$  we consider the space of linear superpositions of the form  $\sum_{k=1}^m p_k(x, y) f_k(q(x, y))$ ,  $(x, y) \in D$ . Here the functions  $\{p_k(x, y)\}$  and  $q(x, y)$  are continuous and fixed, and  $\{f_k(t)\}$  are arbitrary continuous functions of one variable. We assume that the function  $q(x, y)$  is such that for any sequence  $t_n \in q(D) \rightarrow t \in q(D)$  we have  $\rho[e(q, t_n) \cap D, e(q, t) \cap D] \rightarrow 0$ . We put

$$\lambda(t, D, q, p_1, \dots, p_m) = \inf_{\{c_k\}} \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k p_k(x, y) \right|,$$

where inf is taken over all sets of numbers  $\{c_k\}$  for which  $\max_k |c_k| = 1$ . The function  $\lambda(t, D, q, \{p_k\})$ , as a function of  $t$ , is defined only on the set  $q(D)$ .

LEMMA 4.3.1. *The function  $\lambda(t, D, q, \{p_k\})$  depends continuously on  $t$ .*

*Proof.* The linear combinations  $\sum_{k=1}^m c_k p_k(x, y)$  for all possible systems of numbers  $\{c_k\}$  for which  $\max_k |c_k| \leq 1$ , form an equicontinuous set of functions, considered on the bounded closed set  $D$ . Consequently, for any  $\varepsilon > 0$  there is a  $\delta > 0$  such that if  $|t_1 - t_2| < \delta$ , then

$$\left| \sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^m c_k p_k(x, y) \right| - \sup_{(x, y) \in e(q, t_2)} \left| \sum_{k=1}^m c_k p_k(x, y) \right| \right| < \varepsilon$$

simultaneously for all systems of numbers  $\{c_k\}$  such that  $\max_k |c_k| \leq 1$ .

For definiteness, suppose that  $\lambda(t_2, D, q, \{p_k\}) \geq \lambda(t_1, D, q, \{p_k\})$ .

Since the expression  $\sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^m c_k p_k(x, y) \right|$  depends continuously on the coefficients  $\{c_k\}$ , there exists a system of numbers  $\{c_k^1\}$  such that  $\max_k |c_k^1| = 1$  and

$$\lambda(t_1, D, q, \{p_k\}) = \sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^m c_k^1 p_k(x, y) \right|.$$

Since

$$\lambda(t_2, D, q, \{p_k\}) \leq \sup_{(x, y) \in e(q, t_2)} \left| \sum_{k=1}^m c_k^1 p_k(x, y) \right|,$$

we have

$$0 \leq \lambda(t_2) - \lambda(t_1) \leq \sup_{(x, y) \in e(q, t_2)} \left| \sum_{k=1}^m c_k^1 p_k(x, y) \right|$$

$$- \sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^m c_k^1 p_k(x, y) \right| < \varepsilon.$$

This proves the lemma.

LEMMA 4.3.2. *The function  $\lambda(t, D, q, \{p_k\})$  depends continuously on  $D$  in the sense that there exists a function  $\mu(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , having the property: if the set  $D_\varepsilon \subset D$  is such that, for any  $t$ ,  $D_\varepsilon \cap e(q, t)$  forms an  $\varepsilon$ -net in the set  $e(q, t) \cap D$ , then*

$$\max_{t \in q(D)} \left| \lambda(t, D, q, \{p_k\}) - \lambda(t, D_\varepsilon, q, \{p_k\}) \right| \leq \mu(\varepsilon).$$

*Proof.* Using the equicontinuity of the set of functions  $\sum_{k=1}^n c_k p_k(x, y)$  where  $\max_k |c_k| \leq 1$ , we conclude that there exists a function  $\mu(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  such that the inequality

$$0 \leq \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k p_k(x, y) \right| - \sup_{(x, y) \in e(q, t) \cap D_\varepsilon} \left| \sum_{k=1}^m c_k p_k(x, y) \right| \leq \mu(\varepsilon).$$

uniformly over all  $t \in q(D)$  and over all systems of numbers  $\{c_k\}$  for which  $\max_k |c_k| \leq 1$ . For any  $\varepsilon > 0$  there exists a system of numbers  $\{c_k^\varepsilon\}$  such that  $\max_k |c_k^\varepsilon| = 1$  and

$$\lambda(t, D_\varepsilon, q, \{p_k\}) = \sup_{(x, y) \in e(q, t) \cap D_\varepsilon} \left| \sum_{k=1}^m c_k^\varepsilon p_k(x, y) \right|.$$

Since for any  $\varepsilon$

$$\lambda(t, D, q, \{p_k\}) \leq \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k^\varepsilon p_k(x, y) \right|$$

and, on the other hand,  $\lambda(t, D, q, \{p_k\}) \geq \lambda(t, D_\varepsilon, q, \{p_k\})$  (we recall that  $D_\varepsilon \subset D$ ), we have

$$0 \leq \lambda(t, D, q, \{p_k\}) - \lambda(t, D_\varepsilon, q, \{p_k\}) \leq \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k^\varepsilon p_k(x, y) \right| - \sup_{(x, y) \in e(q, t) \cap D_\varepsilon} \left| \sum_{k=1}^m c_k^\varepsilon p_k(x, y) \right| < \mu(\varepsilon).$$

This proves the lemma.

LEMMA 4.3.3. Let  $F$  be a closed set on the  $t$ -axis;  $F \subset q(D)$ . For every  $t \in F$ , suppose that there exists one and only one system of numbers  $\{C_k\}$  ( $\max_k |C_k| = 1$ ) such that  $\sum_{k=1}^m C_k p_k(x, y) \equiv 0$  on the set  $e(q, t) \cap D$ . Then each of the functions  $\{C_k(t)\}$  depends continuously on  $t$  on the set  $F$ .

*Proof.* Suppose that  $t_n \in F$ ,  $t \in F$  and  $t_n \rightarrow t$ . We put  $\lim_{n \rightarrow \infty} C_k(t_n) = \tilde{C}_k$  and  $\lim_{n \rightarrow \infty} C_k(t_n) = \tilde{C}_k$ . Since  $\sum_{k=1}^m C_k(t_n) p_k(x, y) \equiv 0$  on the set  $e(q, t_n) \cap D$  and  $\rho[e(q, t) \cap D, e(q, t_n) \cap D] \rightarrow 0$  as  $n \rightarrow \infty$ , we have  $\sum_{k=1}^m \tilde{C}_k p_k(x, y)$

$\equiv 0 \equiv \sum_{k=1}^m \tilde{C}_k p_k(x, y)$  on the set  $e(q, t) \cap D$ . Consequently, by the condition of the lemma,  $\tilde{C}_k = \tilde{C}_k = C_k(t)$ . This proves the lemma.

LEMMA 4.3.4. *Suppose that  $\lambda(t, D, q, \{p_k\}) \equiv 0$  on some non-empty portion  $\delta$  of the set  $q(D)$ . Then there is a non-empty portion  $\delta^* \subset \delta$  and an index  $l$  such that for any continuous functions  $\{f_k(t)\}$  there are continuous functions  $\{f_k^*(t)\}$  such that*

$$\sum_{k \neq l} f_k^*(q(x, y)) p_k(x, y) = \sum_{k=1}^m f_k(q(x, y)) p_k(x, y)$$

on the set  $q^{-1}(\delta^*) \cap D$ .

We recall that a portion  $\delta$  of a set  $E$  is that part of it which lies in the interval  $\delta$ .

*Proof.* We prove the lemma by induction on  $m$ . For  $m = 1$  the assertion of the lemma is obvious. We denote by  $\delta_k$  the set of all points  $t$  of the portion  $\delta$  for which  $\lambda(t, D, q, p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_m) = 0$ . By Lemma 4.3.1, the set is closed. Two cases are possible.

1) For some  $k$  the set  $\delta_k$  contains a non-empty portion  $\delta'_k$  of the set  $q(D)$ . Since  $\lambda(t, D, q, p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_m) = 0$  for every  $t \in \delta'_k$ , then by the inductive hypothesis there is a non-empty portion  $\delta^* \subset \delta'_k$  and an index  $l \neq k$  such that for any continuous functions  $f_1(t), \dots, f_{k-1}(t), f_{k+1}(t), \dots, f_m(t)$  there are continuous functions  $f_1^*(t), \dots, f_{k-1}^*(t), f_{k+1}^*(t), \dots, f_m^*(t)$  such that

$$\sum_{i \neq k} f_i(q(x, y)) p_i(x, y) = \sum_{i \neq k, l} f_i^*(q(x, y)) p_i(x, y).$$

on the set  $q^{-1}(\delta^*) \cap D$ . Putting  $f_k^*(t) = f_k(t)$ , we obtain

$$\sum_{i=1}^m f_i(q(x, y)) p_i(x, y) = \sum_{i \neq l} f_i^*(q(x, y)) p_i(x, y).$$

So in case 1) the lemma is proved.

2) None of the sets  $\delta_k$  contains non-empty portions of the set  $q(D)$ , that is,  $\bigcup_{k=1}^m \delta_k$  is nowhere dense in  $q(D)$ . Therefore there exists a non-empty portion  $\delta^* \subset \delta \setminus \bigcup_{k=1}^m \delta_k$ . Since  $\lambda(t, D, q, \{p_k\}) \equiv 0$  on  $\delta^*$ , for every  $t \in \delta^*$  there are numbers  $\{C_k(t)\}$  ( $\max_k |C_k(t)| = 1$ ) such that  $\sum_{k=1}^m C_k$

$(q(x, y)) p_k(x, y) \equiv 0$  on  $e(q, t) \cap D$ . If we had  $C_k(t) = 0$  for some  $k$ , then it would turn out that  $t \in \delta_k$ . Consequently,  $C_k(t) \neq 0$  for any  $k$ . We show that for every  $t \in \delta^*$  the numbers  $\{C_k(t)\}$  are uniquely determined. Assume the contrary. Then there are numbers  $\{C'_k(t)\}$  ( $\max |C'_k(t)| = 1$ ) such that  $\sum_{k=1}^m C'_k(q(x, y)) p_k(x, y) = 0$  on  $e(q, t) \cap D$  and  $C_k \neq C'_k$  for some  $k$ . Then

$$\sum_{k \neq 1} [C_k(t) C'_1(t) - C'_k(t) C_1(t)] p_k(x, y) = \sum_{k \neq 1} C'_k(t) p_k(x, y) \equiv 0$$

on  $e(q, t) \cap D$  and in addition,  $C'_k \neq 0$  for some  $k$ . Consequently,  $t \in \delta_1$ . So we have obtained a contradiction, and the uniqueness of the choice of the numbers  $C_k(t)$  is proved. Further, we may regard  $\{C_k(t)\}$  as single-valued functions of  $t$  on the portion  $\delta^*$ . By Lemma 4.3.3, the functions  $C_k(t)$  are continuous and, as noted above,  $C_k(t) \neq 0$  for any  $t \in \delta^*$ . Then

$$p_1(x, y) = \sum_{k=2}^m -\frac{C_k(q(x, y))}{C_1(q(x, y))} p_k(x, y), \quad (x, y) \in q^{-1}(\delta^*) \cap D.$$

Putting  $f(t) = f_k(t) - \frac{C_k(t)}{C_1(t)} f_1(t)$ ,  $t \in \delta^*$ , we have  $\sum_{k=2}^m f_k^*(q(x, y)) p_k(x, y)$

$$\begin{aligned} &= \sum_{k=1}^m f_k(q) p_k(x, y) - \sum_{k=2}^m \frac{C_k(q)}{C_1(q)} p_k(x, y) \\ &= \sum_{k=2}^m f_k(q) p_k(x, y) + f_1(q) p_1(x, y) \\ &= \sum_{k=1}^m f_k(q(x, y)) p_k(x, y), \quad (x, y) \in q^{-1}(\delta^*) \cap D. \end{aligned}$$

This proves the lemma.

#### § 4. *Reduction of linear superpositions to a form with independent terms*

We fix the continuous functions  $p_i^k(x, y)$  and continuously differentiable functions  $q_i(x, y)$  ( $i=0, 1, 2, \dots, n; k=1, 2, \dots, m_i$ )  $n \geq 2$ , where  $\{q_i(x, y)\}$  satisfy in  $D$  conditions (1) and (3) of Lemma 4.2.2, and we consider in  $D$  superpositions of the form

$$\sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i(x, y)),$$

where  $\{f_i^k(t)\}$  are arbitrary continuous functions of one variable.